

Dynamical systems tutorial

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“dynamics”

- the word “dynamics”

- time-varying measures

- range of a quantity

- forces causing/accounting for movement => dynamical systems

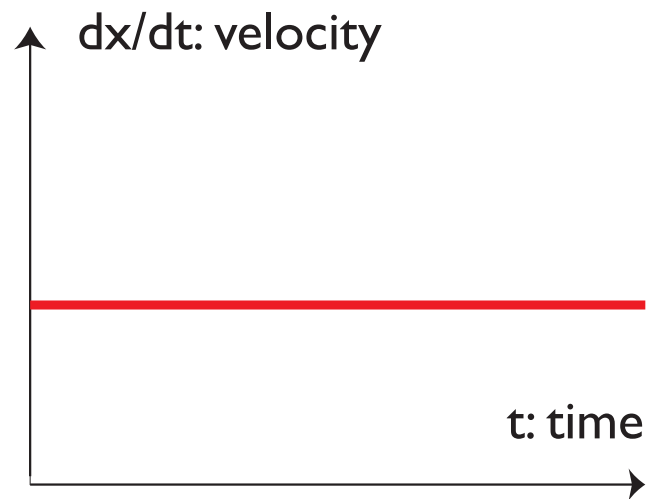
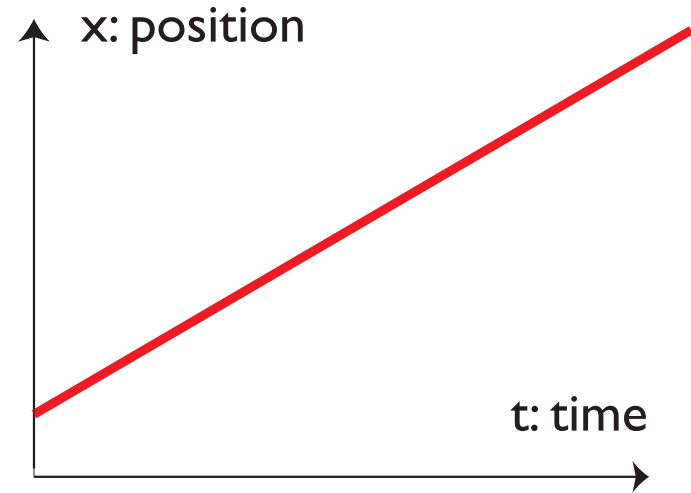
- dynamical systems are the universal language of science

- physics, engineering, chemistry, theoretical biology, economics, quantitative sociology, ...

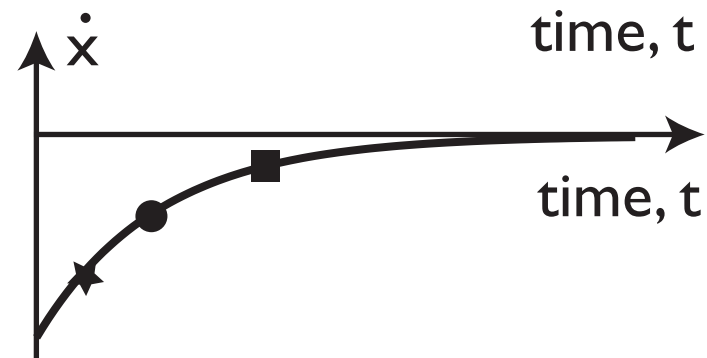
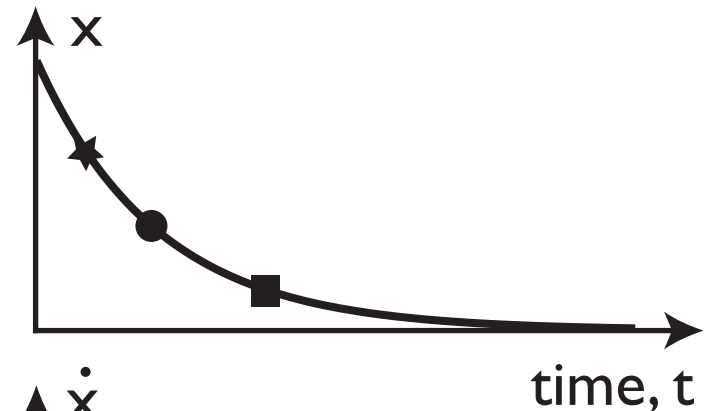
time-variation and rate of change

■ variable $x(t)$;

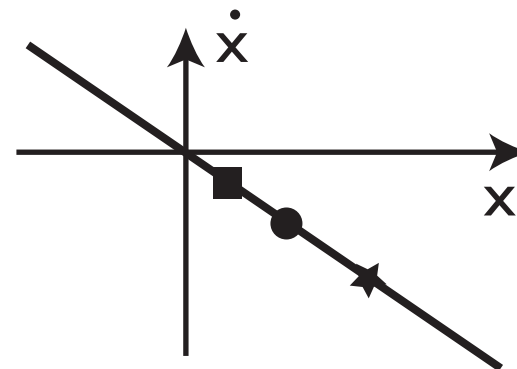
■ rate of change dx/dt



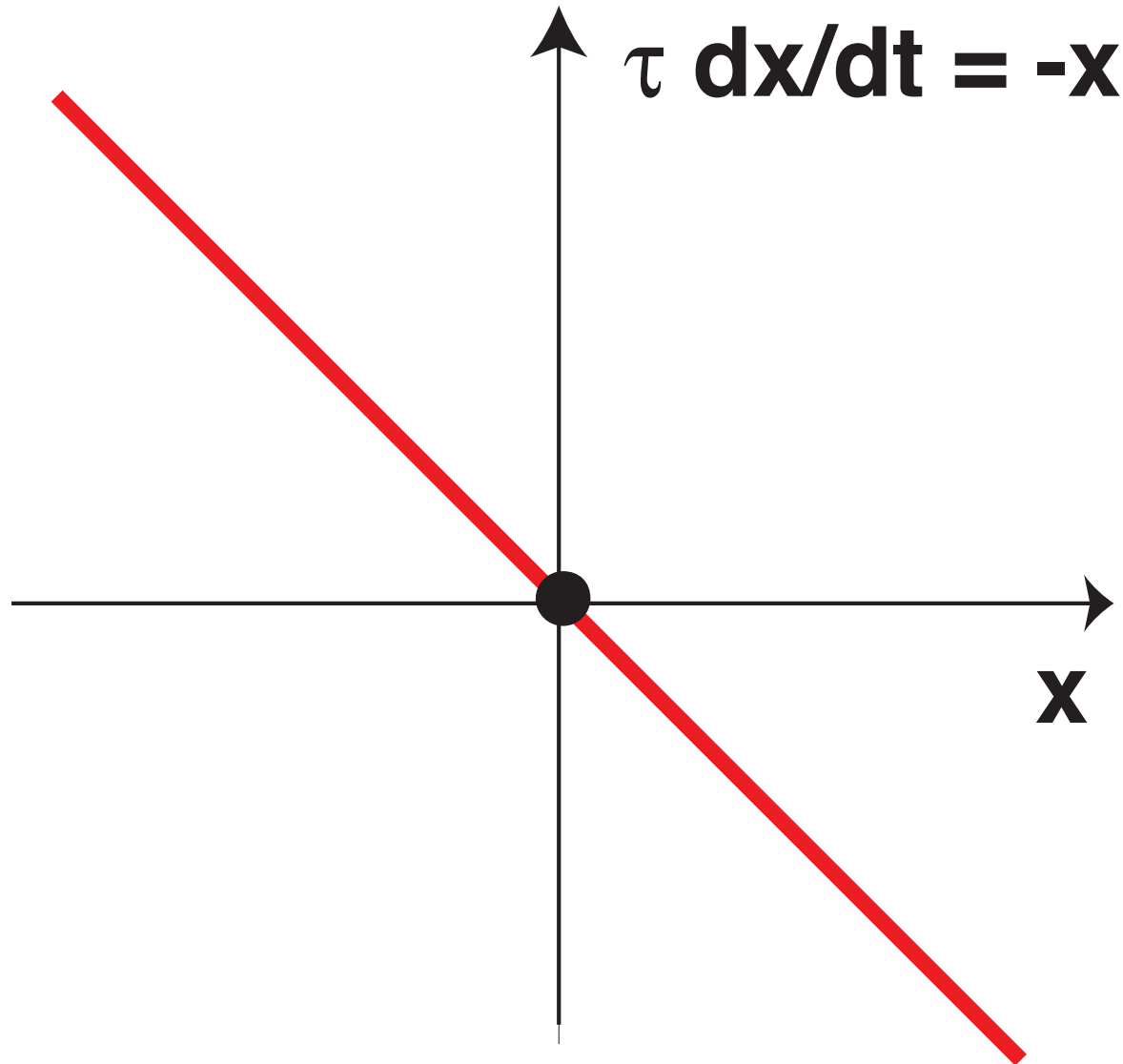
functional relationship between a variable and its rate of change



=> dynamical system



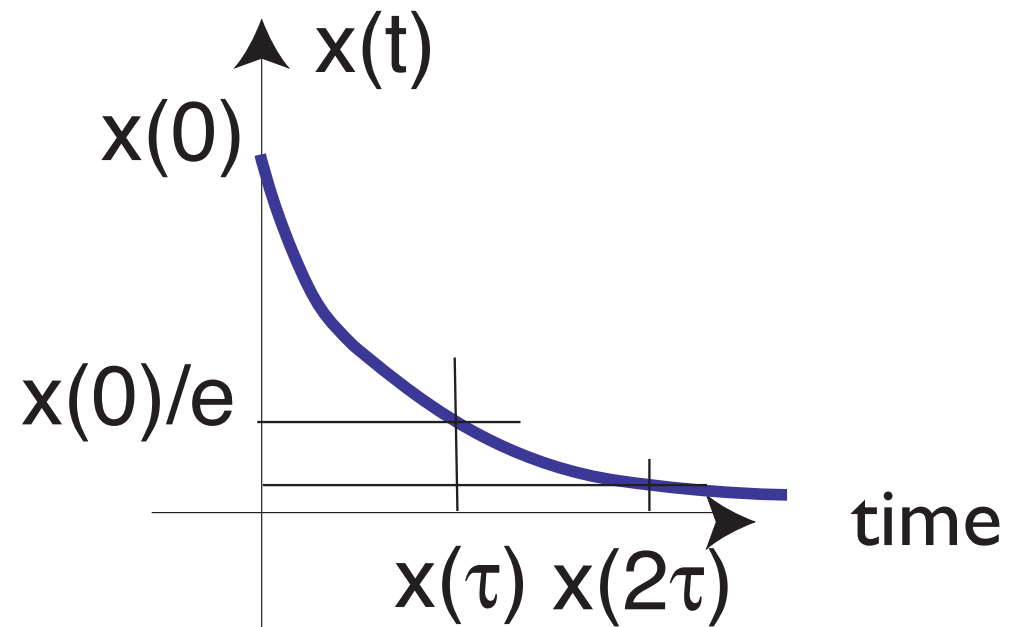
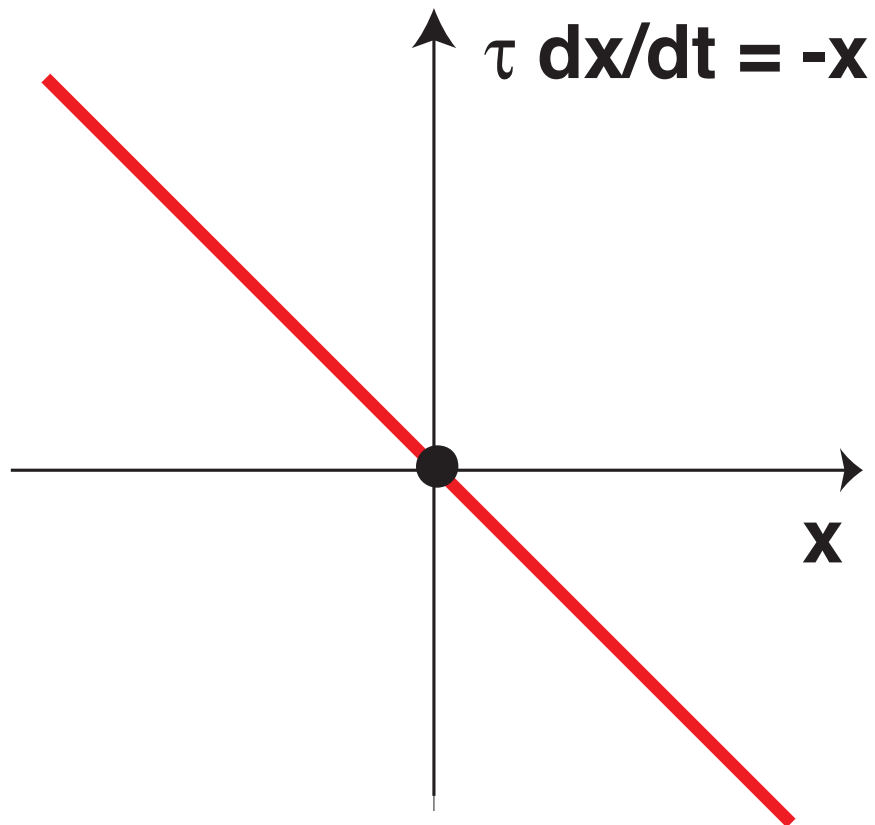
(linear) dynamical system



exponential relaxation to attractors

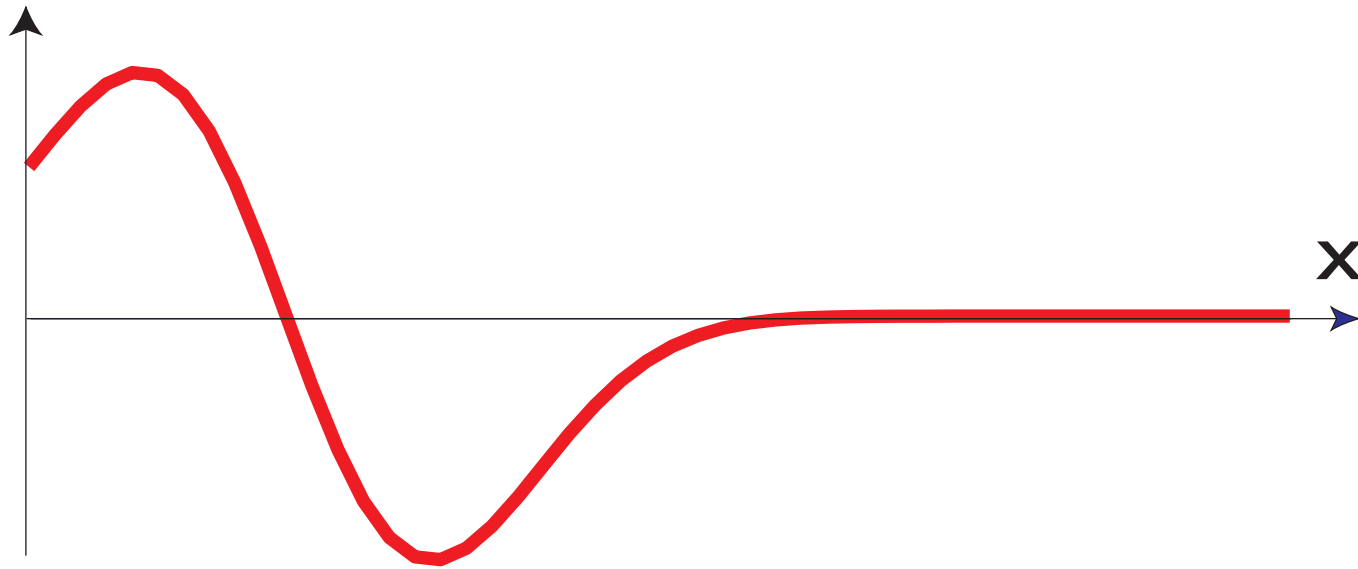
■ $\tau \dot{x} = -x \Rightarrow x(t) = x(0)\exp[-t/\tau]$ (check!)

■ \Rightarrow has a well-defined time scale



(nonlinear) dynamical system

$$dx/dt=f(x)$$



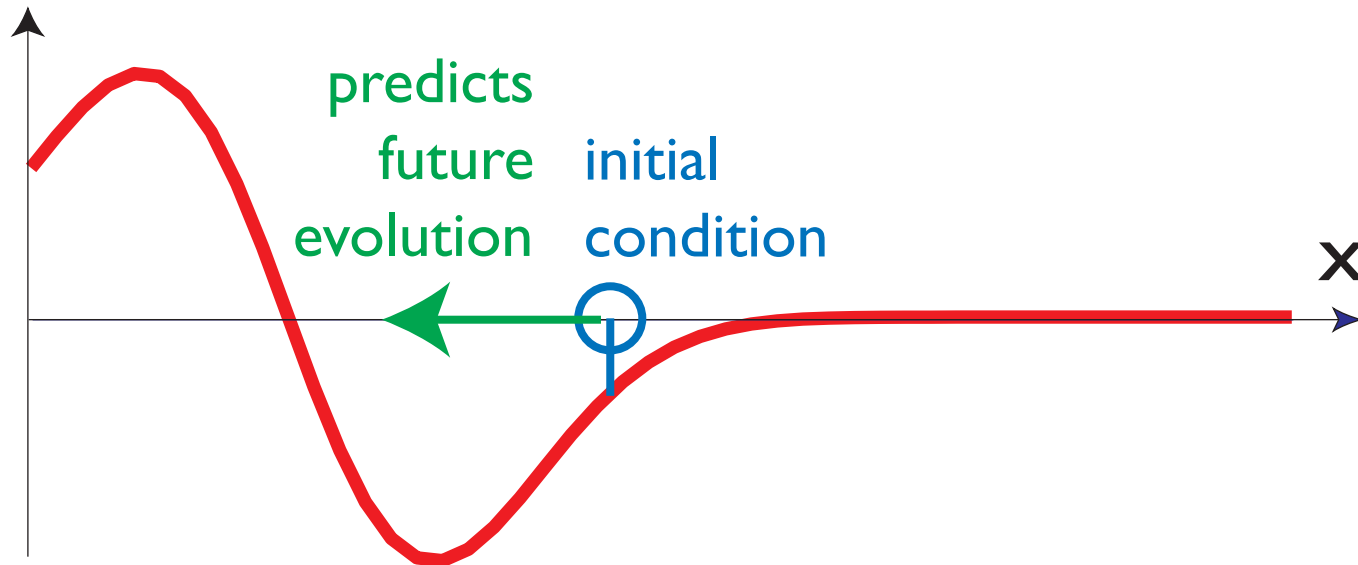
dynamical system

■ present determines the future

■ given initial condition

■ predict evolution (or predict the past)

$$dx/dt=f(x)$$



dynamical systems

- x : spans the state space (or phase space)
- $f(x)$: is the “dynamics” of x (or vector-field)
- $x(t)$ is a **solution** of the dynamical systems to the initial condition x_0
 - if its rate of change = $f(x)$
 - and $x(0)=x_0$

Dynamical systems

- differential equation $\dot{x} = f(x)$ in one dimension
- \Rightarrow an initial value of x determines the future

Dynamical systems

- system of differential equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$
- \Rightarrow a vector of initial states,
 $\mathbf{x} = (x_1, x_2, \dots, x_n)$ determines the future

Dynamical systems

- partial differential equations

$$\dot{x}(y, t) = f \left(x(y, t), \frac{\partial x(y, t)}{\partial y}, \dots \right)$$

- integro-differential equations

$$\dot{x}(y, t) = \int dy' f(y, y', x(y', t))$$

- \Rightarrow continuously many initial values = initial function $x(y)$ determine the future

Dynamical systems

■ delay differential equations $\dot{x}(t) = f(x(t - \tau))$

■ functional differential equations

$$\dot{x}(t) = \int^t dt' f(x(t'))$$

■ \Rightarrow a past piece of trajectory determines the future

Dynamical systems

- iteration equation in discrete time (map)

$$x_{n+1} = g(x_n)$$

- every dynamical system in continuous time
=> dynamical system in discrete time
(Poincaré)

- a dynamical system in discrete time can be lifted to a dynamical system in continuous time (but not uniquely)

Resources

- free online textbook by Scheinermann

- <https://github.com/scheinerman/InvitationToDynamicalSystems>

- send him a postcard (as instructed there)

- really nice book for beginners...

- focus on the time-continuous part..

numerics

- sample time discretely
- compute solution by iterating through time
- valid approximation for small time steps...

forward Euler

- $t_i = i\Delta t$ so that $x_i = x(t_i)$
- $\dot{x} = dx/dt \approx \Delta x/\Delta t$ where $\Delta x = x_{i+1} - x_i$
- $\dot{x} = f(x) \Rightarrow x_{i+1} = x_i + \Delta t f(x_i)$
- ... valid for small Δt
- is the “worst” approximation scheme
(needs smallest time step to achieve given precision...)
- but useful for real-time embedded (and for stochastic systems)

modern numerics

- Runge-Kutte: error scales with step size to a power (e.g. 4)
- adaptive step size..
- built-into numerical packages... e.g. ode45 in Matlab

 => simulation

qualitative theory of dynamical systems

■ good source:

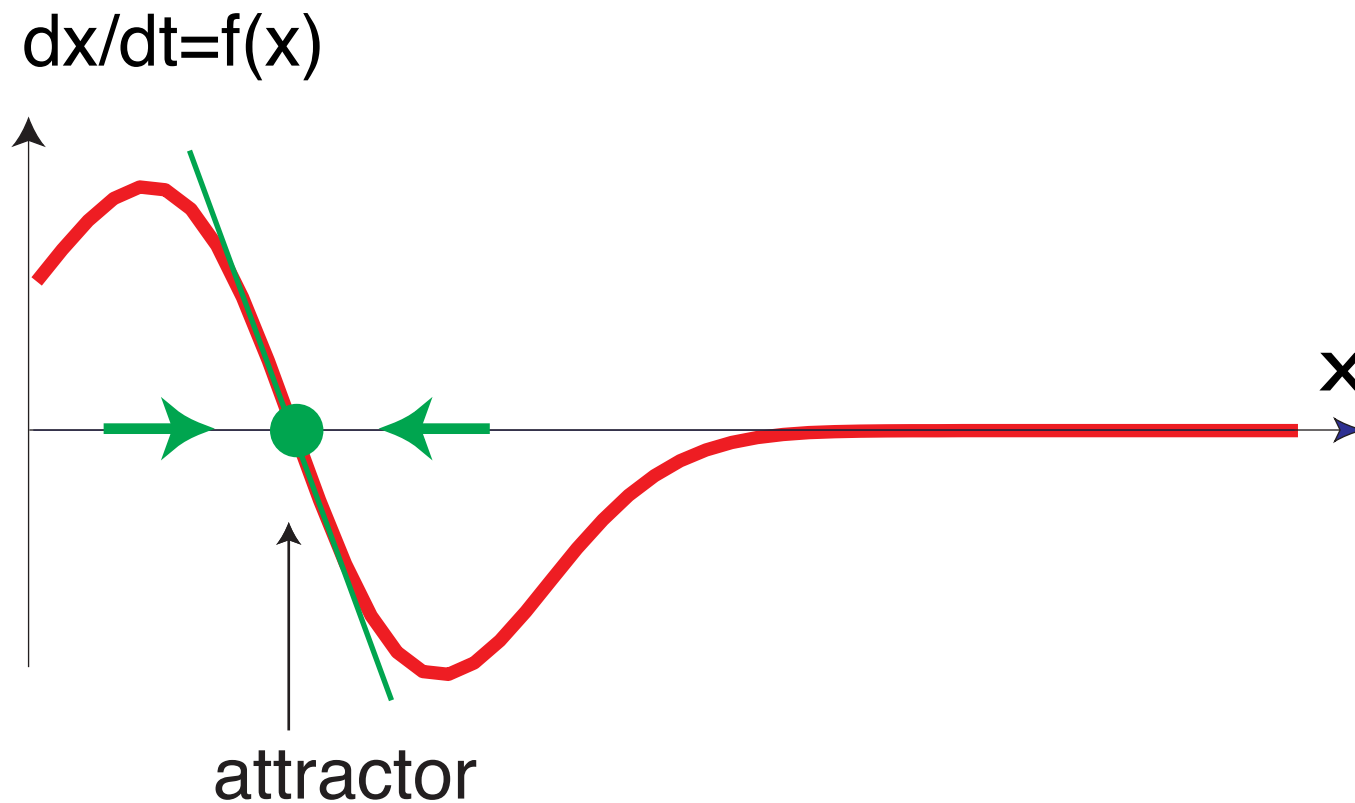
■ Lawrence Perko: Differential Equations and Dynamical Systems, Springer 2001 (4th edition)

qualitative theory of dynamical systems

- the goal is to characterize the ensemble of solutions of the dynamical system (or a family of such)
- = the flow
- ... use special invariant solutions to do that... fixed points, their stable/unstable manifolds...

attractor

- **fixed point**, to which neighboring initial conditions converge = **attractor**



fixed point

■ is a constant solution of the dynamical system

$$\dot{x} = f(x)$$

$$\dot{x} = 0 \Rightarrow f(x_0) = 0$$

stability

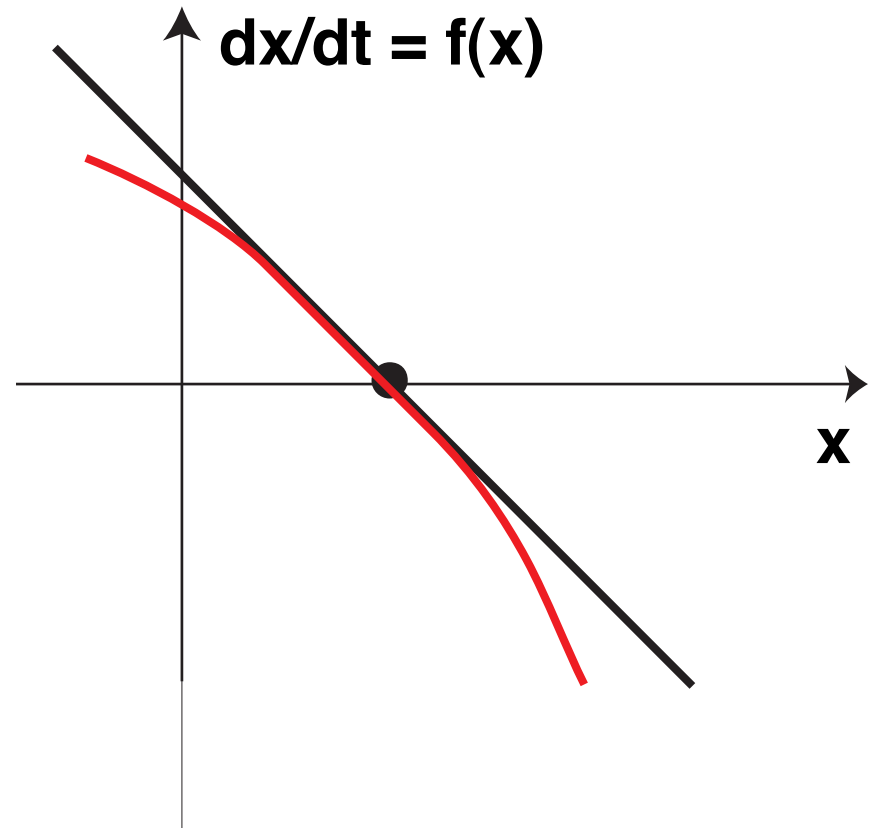
- mathematically really: **asymptotic stability**
- defined: a fixed point is asymptotically stable, when solutions of the dynamical system that start nearby converge in time to the fixed point

stability

- the mathematical concept of stability (which we do not use) requires only that nearby solutions stay nearby
- defined: a fixed point is **unstable** if it is not stable in that more general sense,
 - that is: if nearby solutions do not necessarily stay nearby (may diverge)

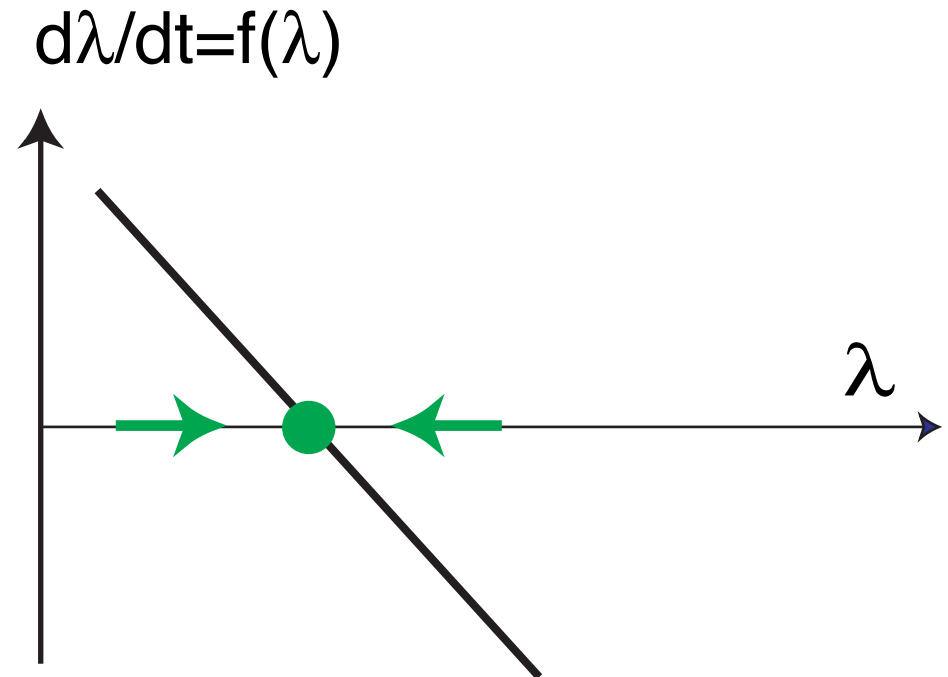
linear approximation near attractor

- non-linearity as a small perturbation/
deformation of linear system
- \Rightarrow non-essential non-linearity



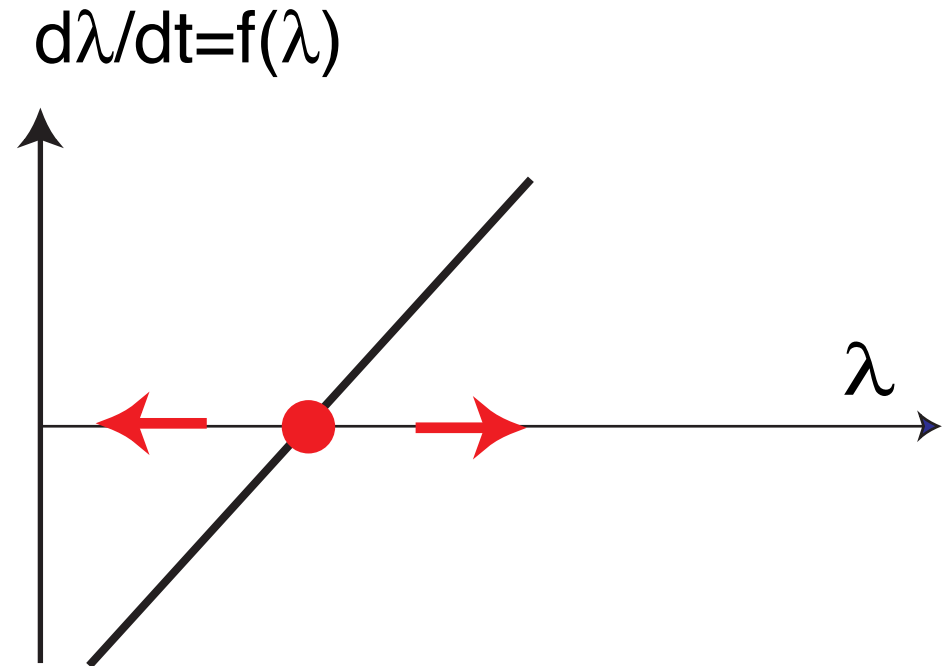
stability in a linear system

- if the slope of the linear system is negative, the fixed point is (asymptotically stable)



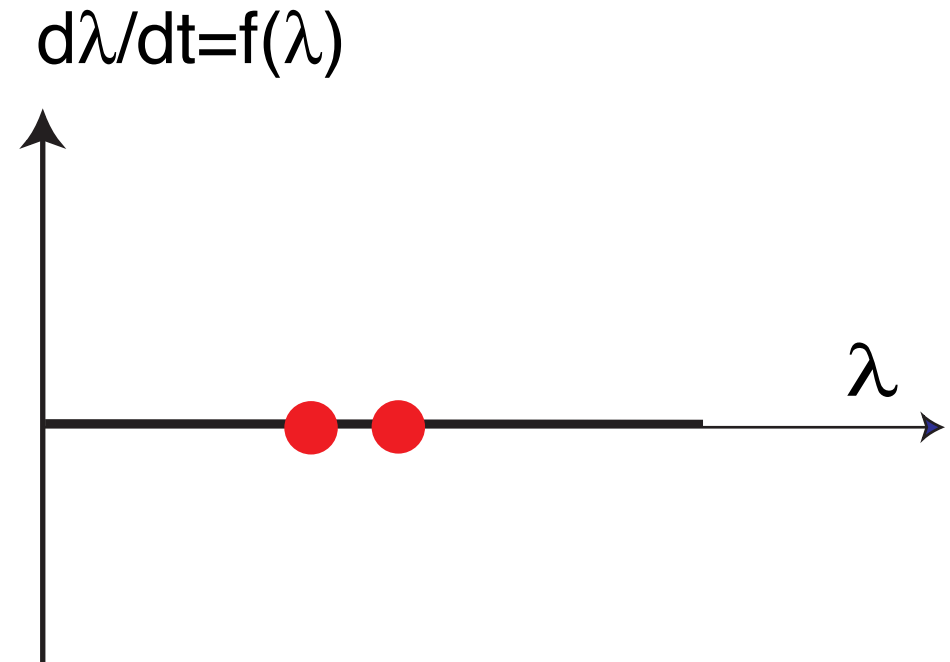
stability in a linear system

- if the slope of the linear system is positive, then the fixed point is unstable



stability in a linear system

- if the slope of the linear system is zero, then the system is indifferent (marginally stable: stable but not asymptotically stable)



stability in linear systems

■ generalization to multiple dimensions

- if the real-parts of all Eigenvalues are negative: stable
- if the real-part of any Eigenvalue is positive: unstable
- if the real-part of any Eigenvalue is zero: marginally stable in that direction (stability depends on other eigenvalues)

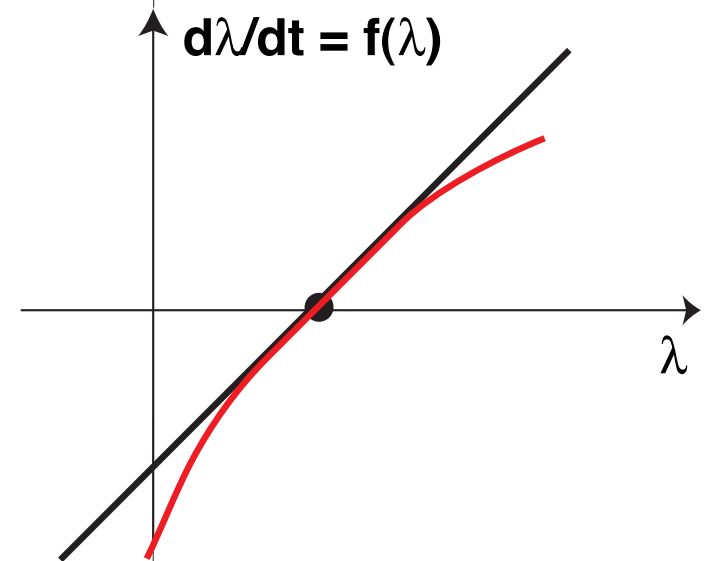
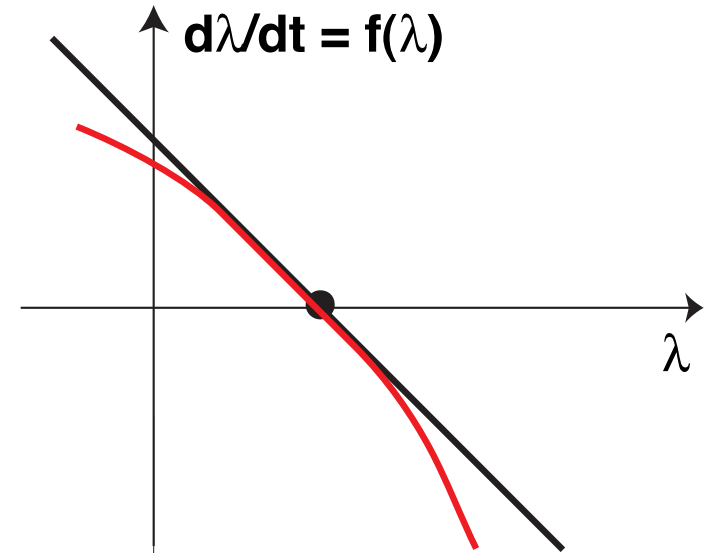
stability in nonlinear systems

- stability is a local property of the fixed point
- \Rightarrow linear stability theory
 - the eigenvalues of the linearization around the fixed point determine stability
 - all real-parts negative: stable
 - any real-part positive: unstable
 - any real-part zero: undecided: now nonlinearity decides (non-hyperbolic fixed point)

stability in nonlinear systems

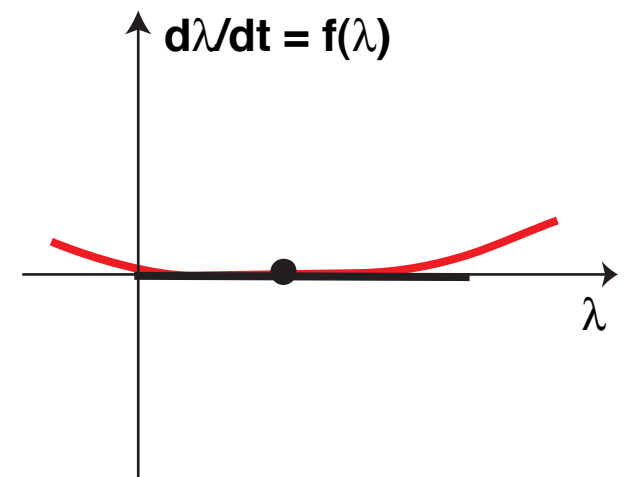
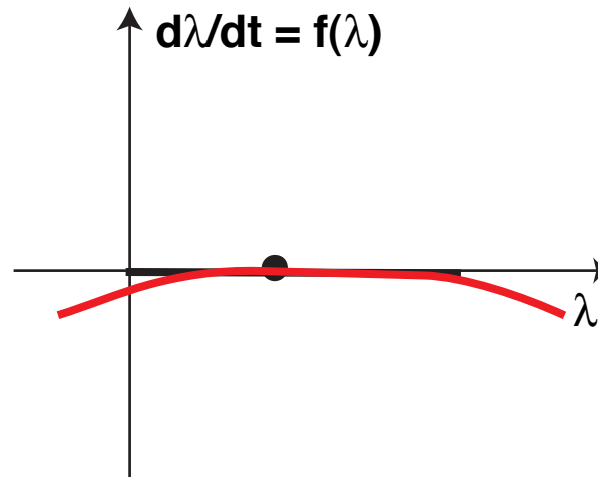
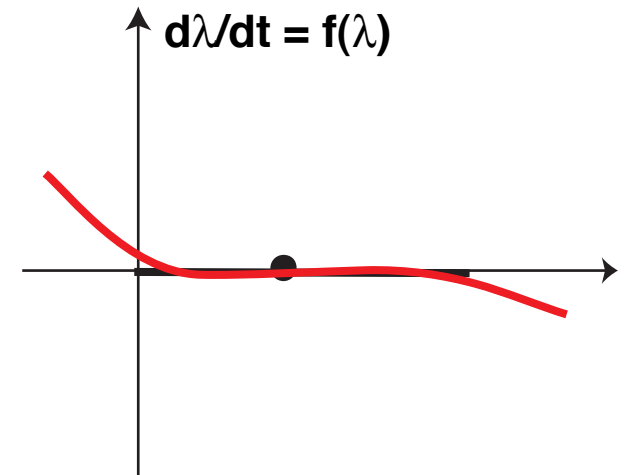
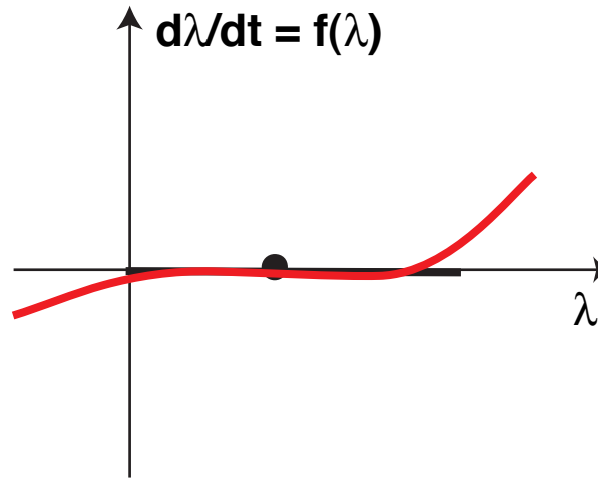
■ all real-parts negative: stable

■ any real-part positive: unstable



stability in nonlinear systems

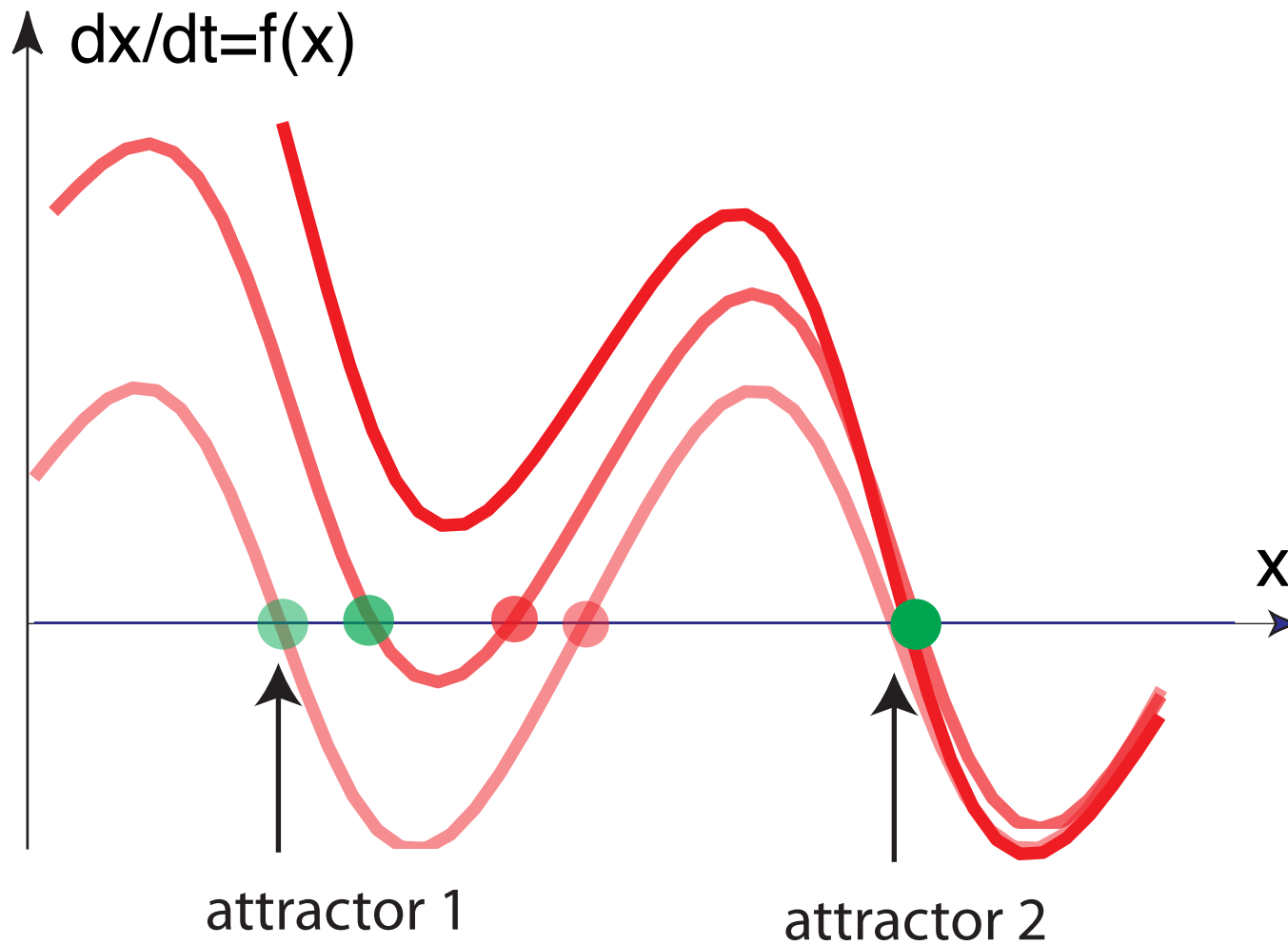
any real-part zero:
undecided: now
nonlinearity decides
(non-hyperbolic fixed
point)



bifurcations

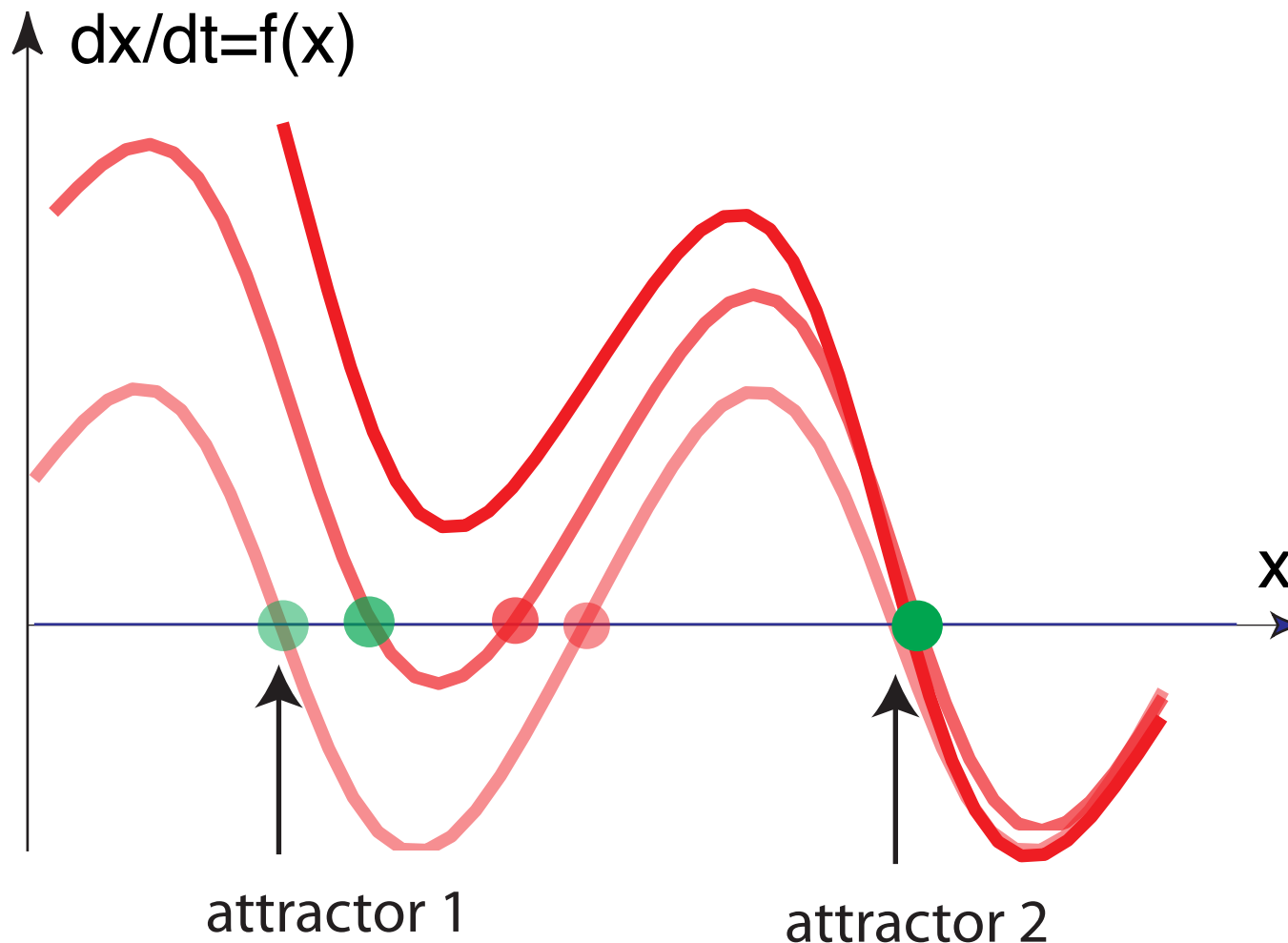
- look now at families of dynamical systems, which depend (smoothly) on parameters
- ask: as the parameters change (smoothly), how do the solutions change (smoothly?)
 - smoothly: topological equivalence of the dynamical systems at neighboring parameter values
 - bifurcation: dynamical systems NOT topological equivalent as parameter changes infinitesimally

bifurcation



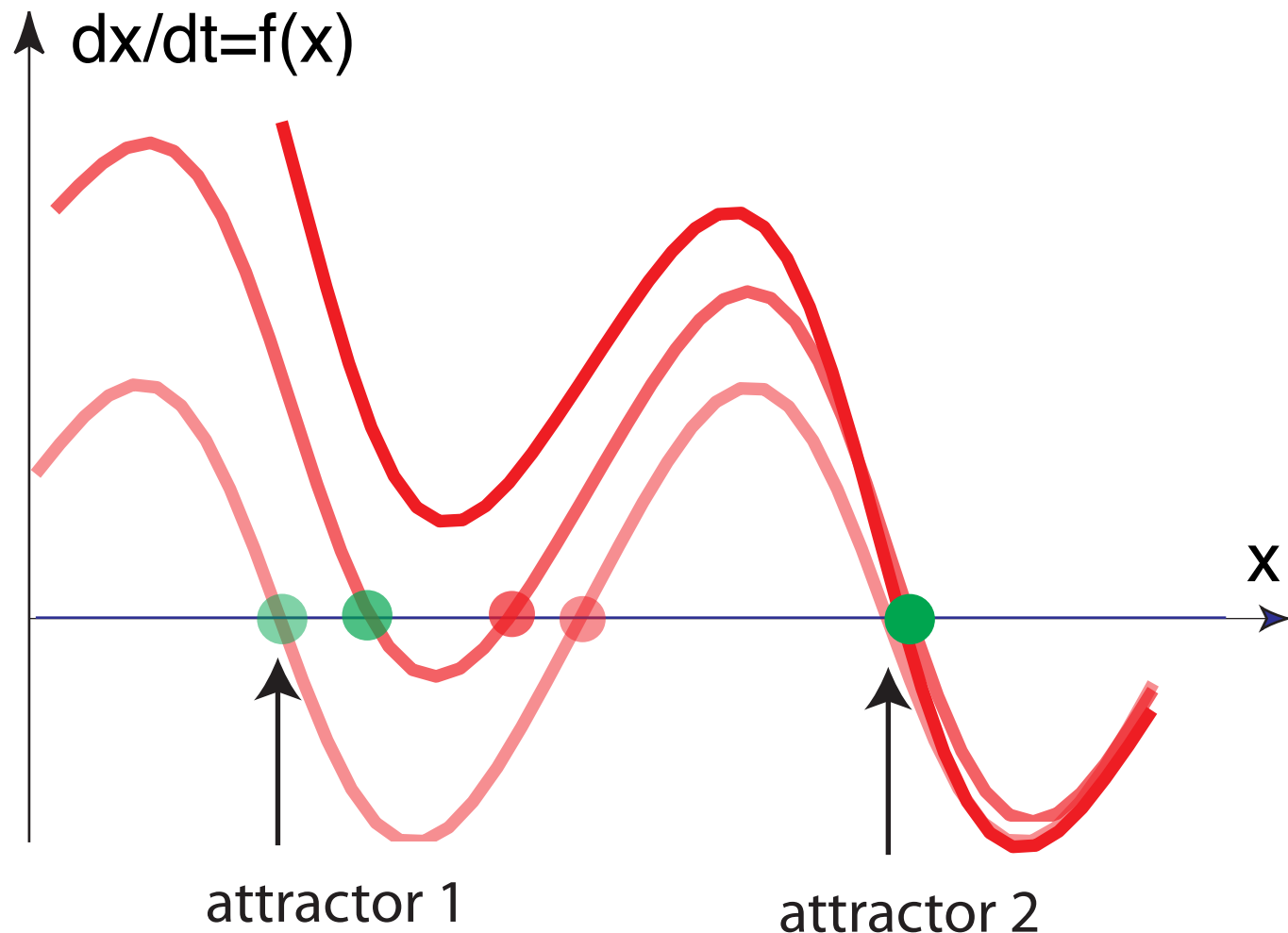
bifurcation

- bifurcation=qualitative change of dynamics (change in number, nature, or stability of fixed points) as the dynamics changes smoothly

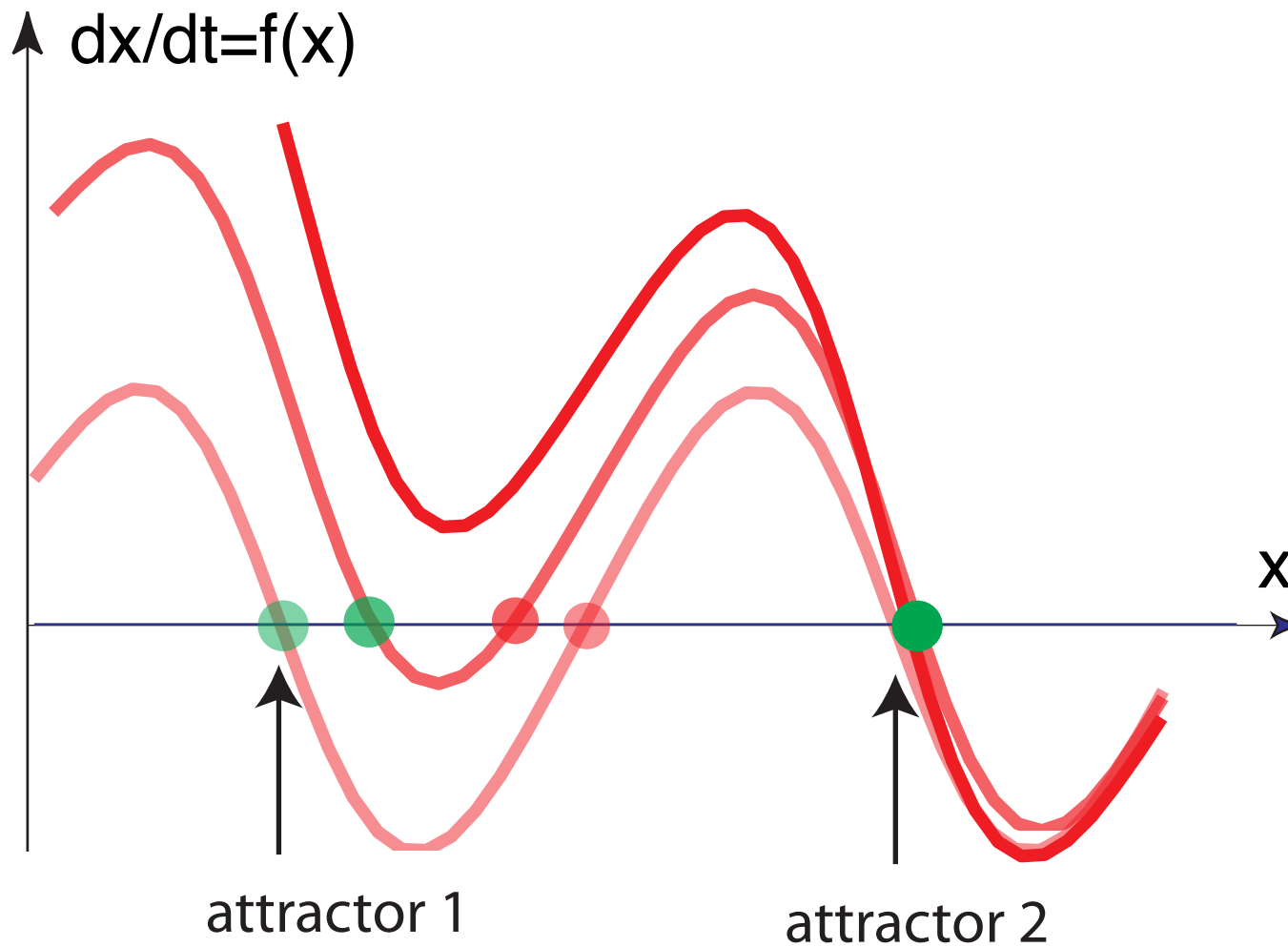


tangent bifurcation

- the simplest bifurcation (co-dimension 0): an attractor collides with a repeller and the two annihilate

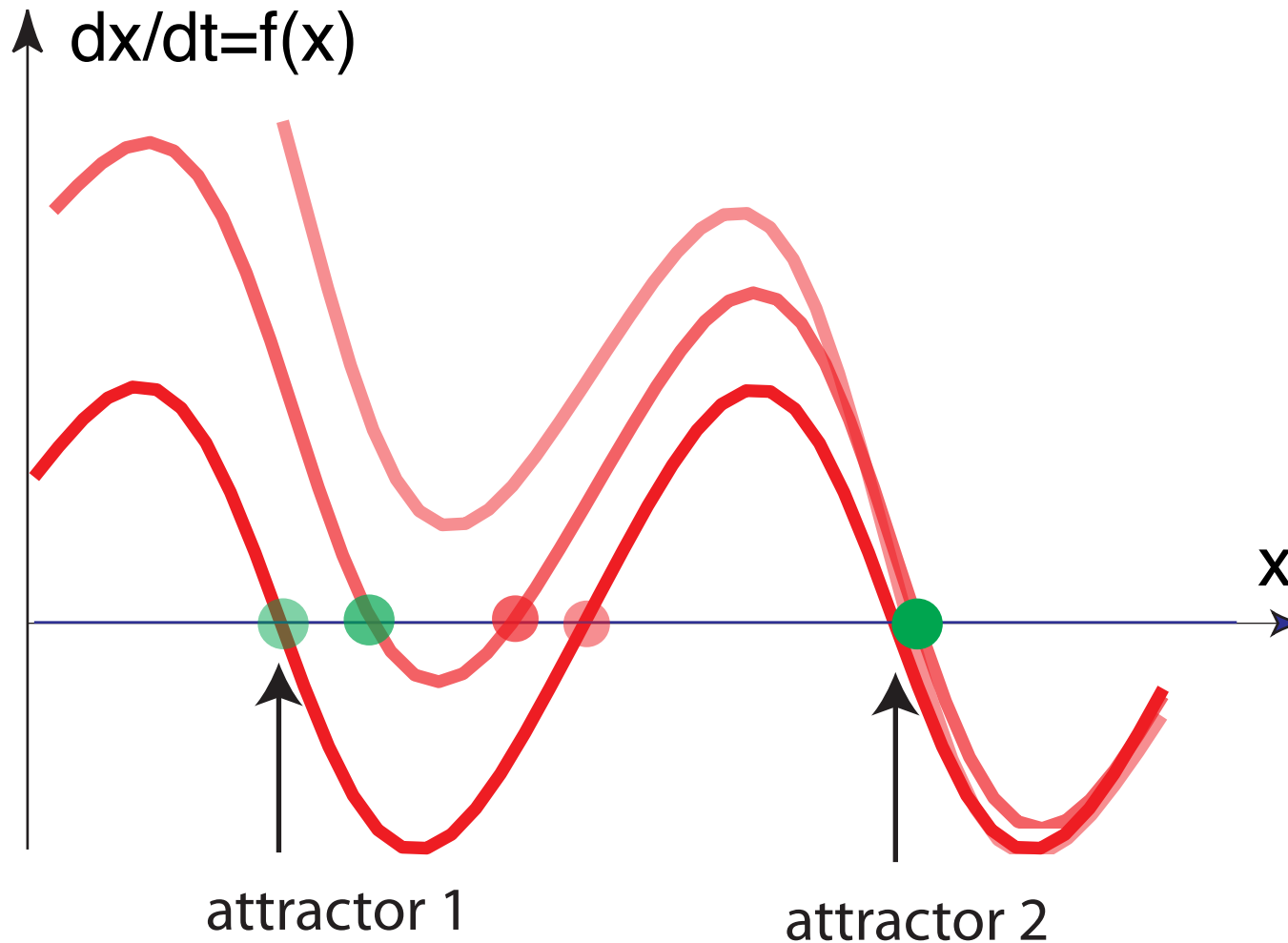


local bifurcation



reverse bifurcation

■ changing the dynamics in the opposite direction



bifurcations are instabilities

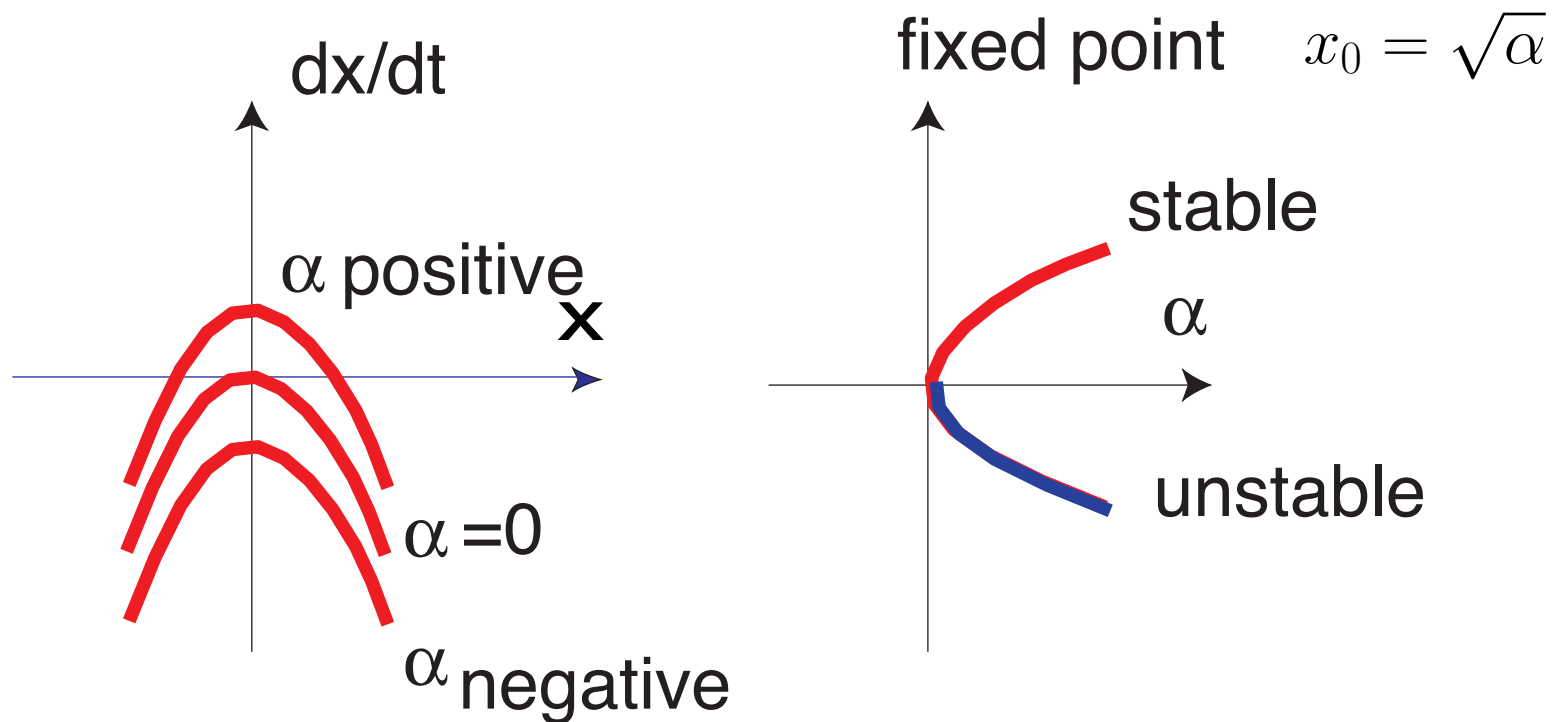
- that is, an attractor becomes unstable before disappearing
- (or the attractor appears with reduced stability)
- formally: a zero-real part is a necessary condition for a bifurcation to occur

tangent bifurcation

- normal form of tangent bifurcation

$$\dot{x} = \alpha - x^2$$

- (=simplest polynomial equation whose flow is topologically equivalent to the bifurcation)



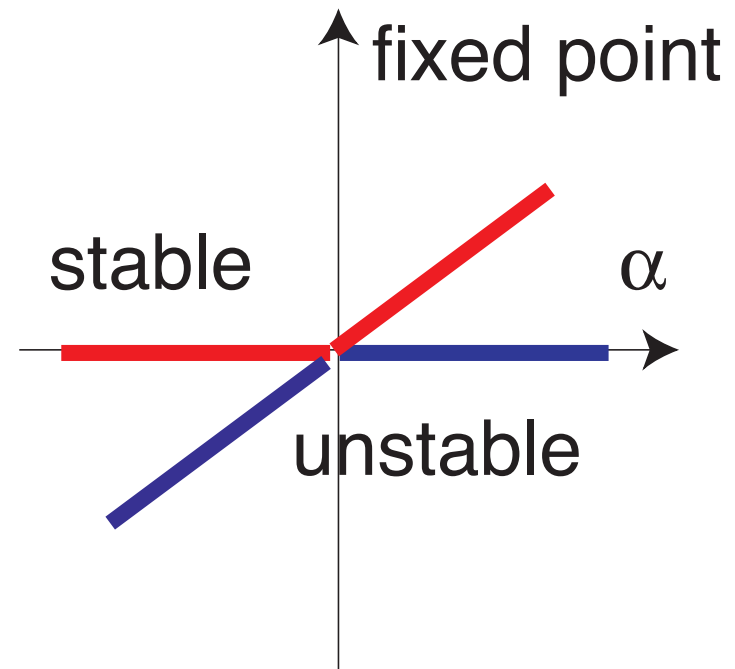
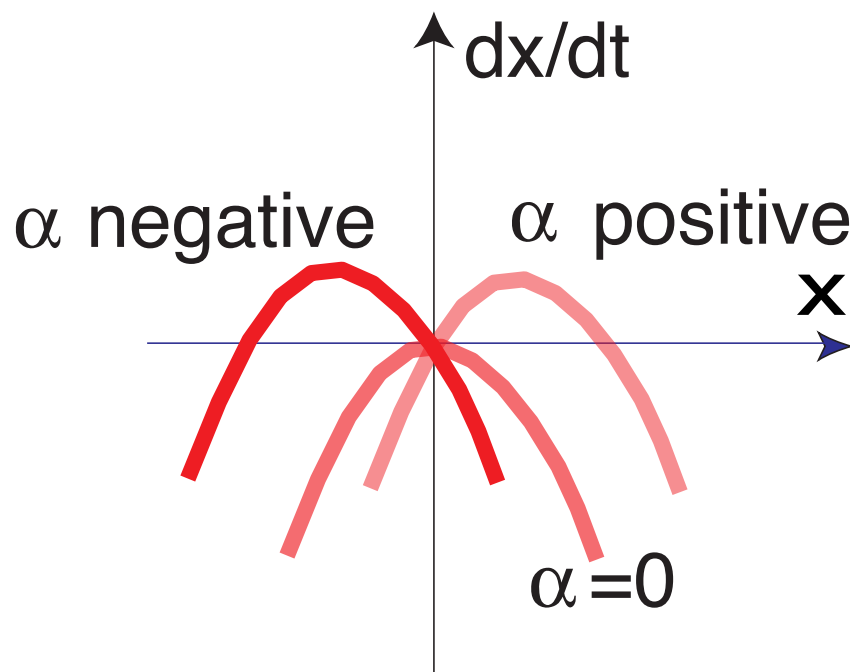
Hopf theorem

- when a single (or pair of complex conjugate) eigenvalue crosses the imaginary axis, one of four bifurcations occur
 - tangent bifurcation
 - transcritical bifurcation
 - pitchfork bifurcation
 - Hopf bifurcation

transcritical bifurcation

■ normal form

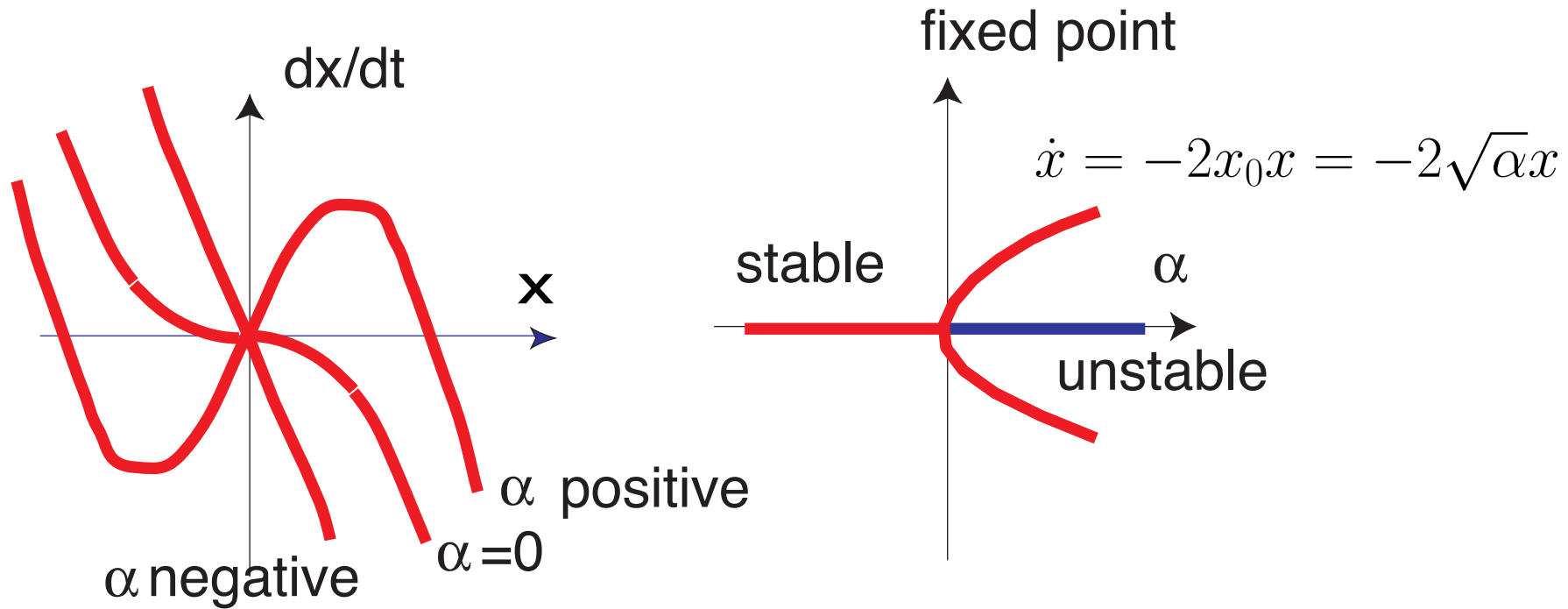
$$\dot{x} = \alpha x - x^2$$



pitchfork bifurcation

■ normal form

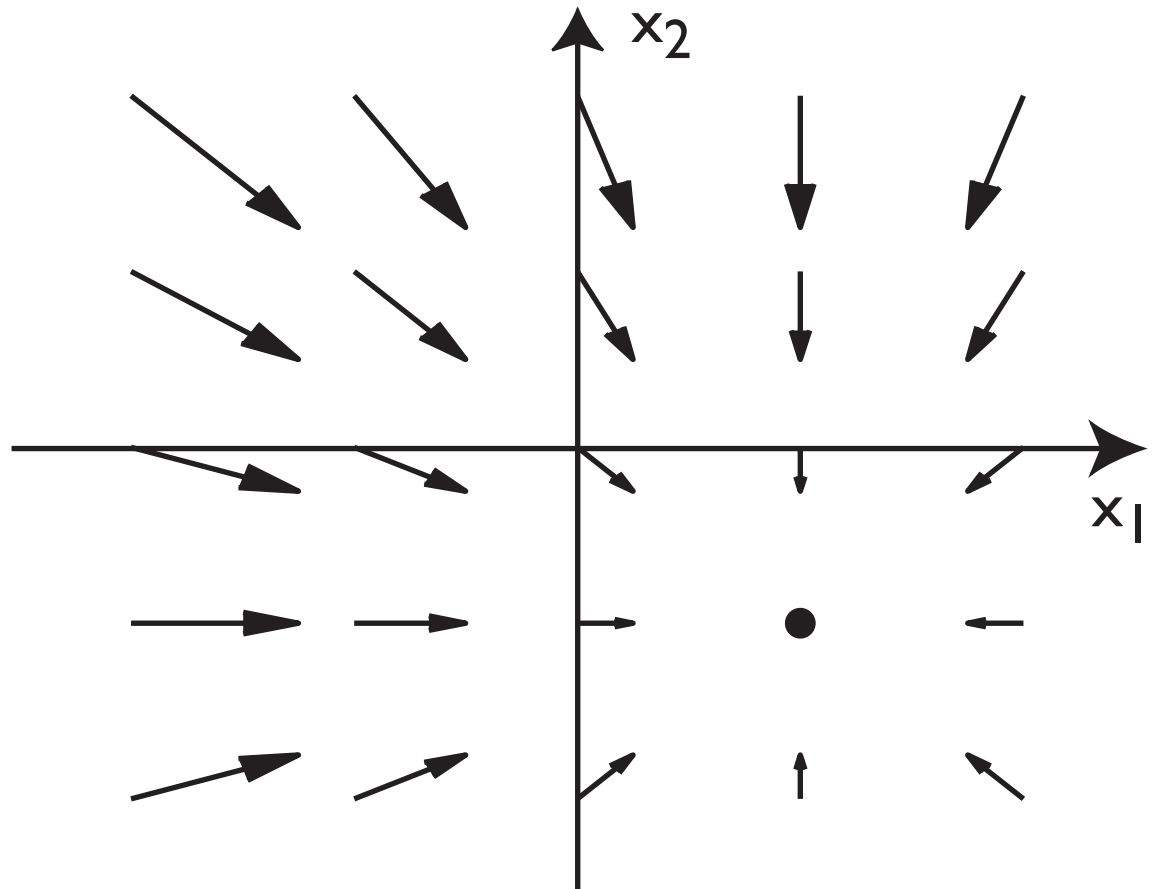
$$\dot{x} = \alpha x - x^3$$



Hopf: need higher dimensions

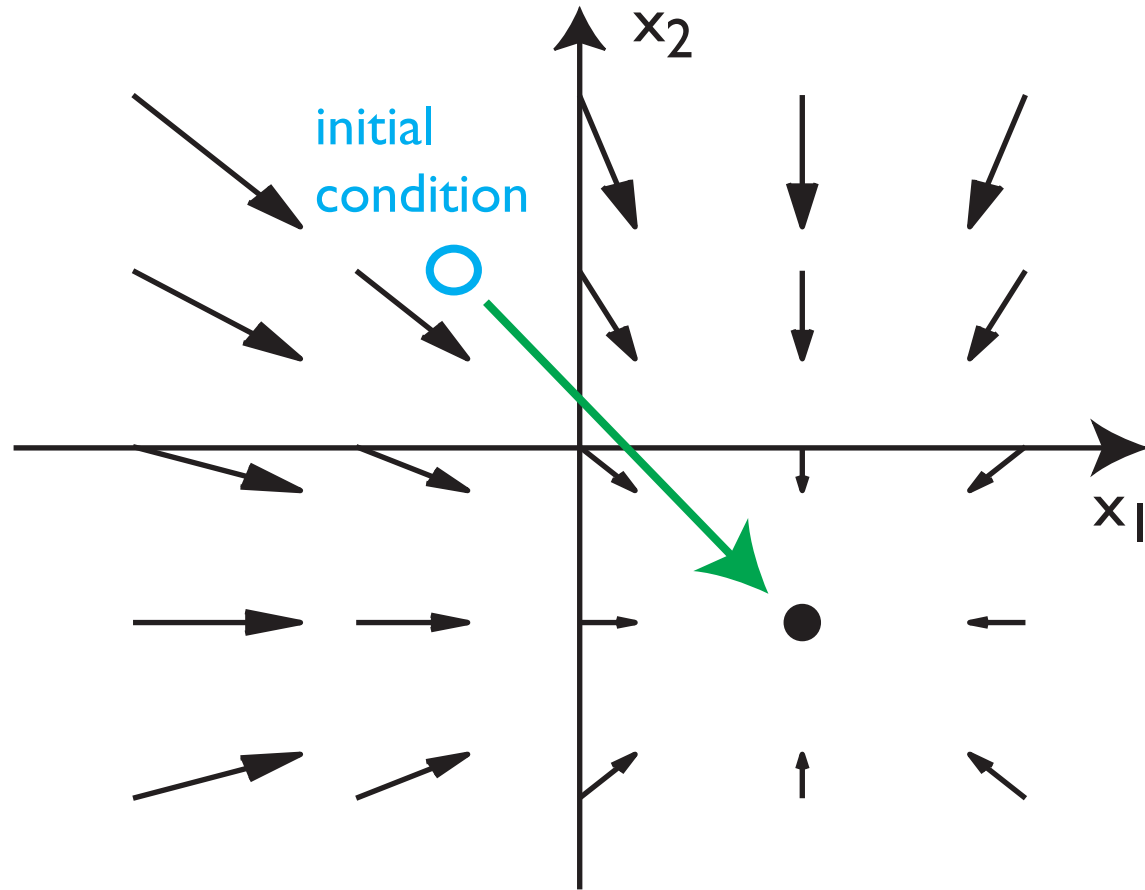
2D dynamical system: vector-field

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$



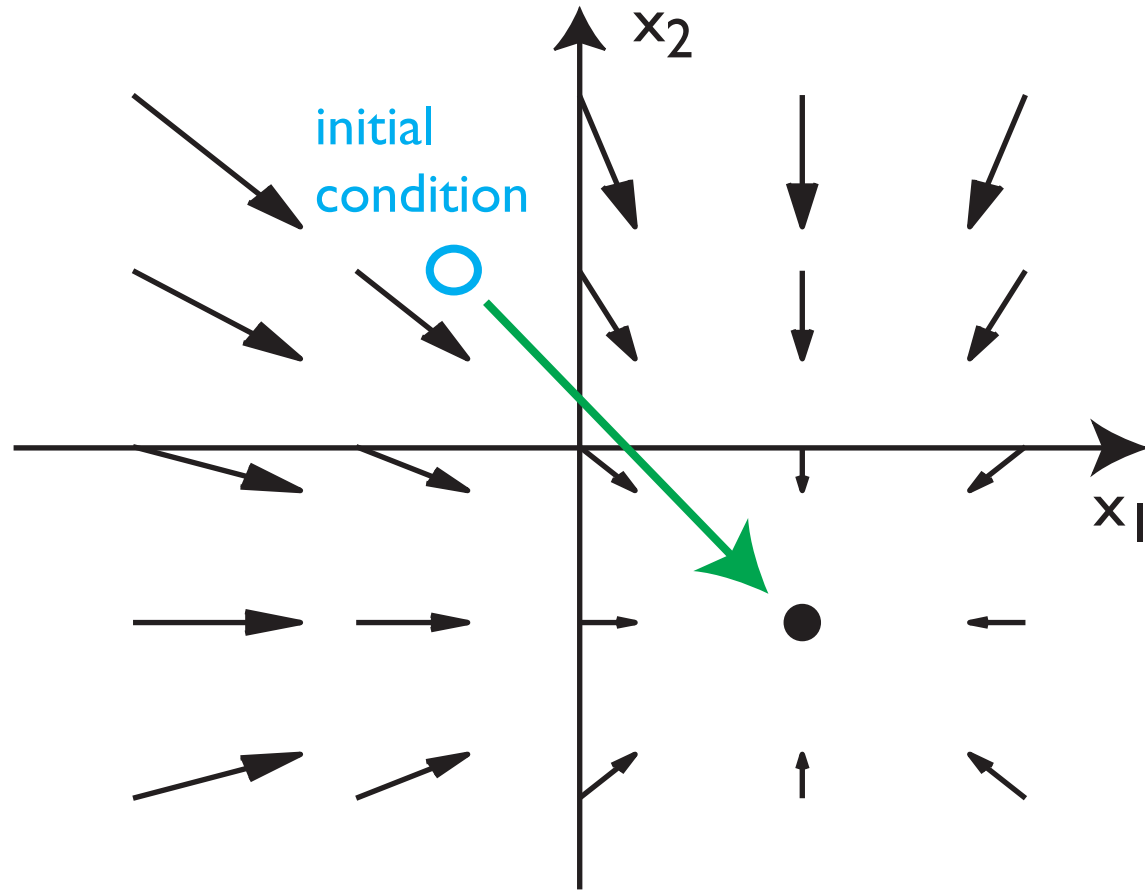
vector-field

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$



fixed point, stability, attractor

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

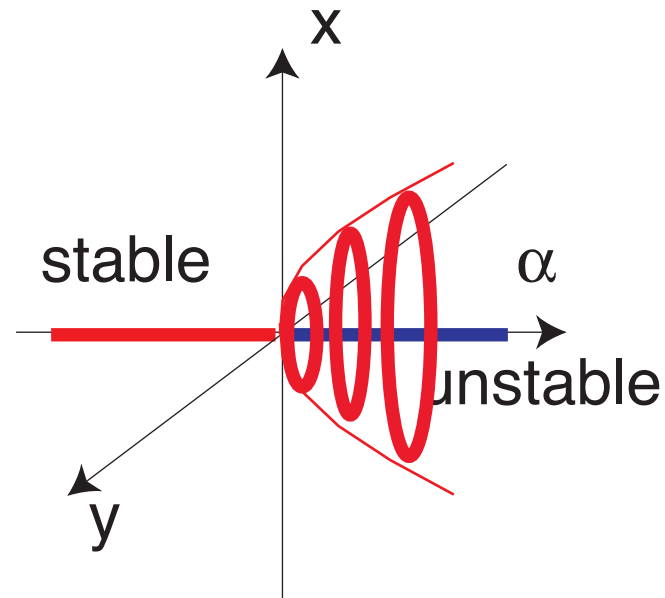
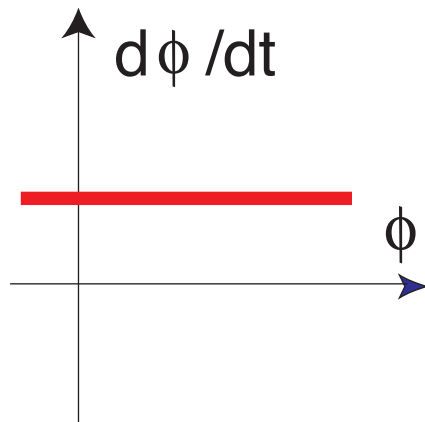
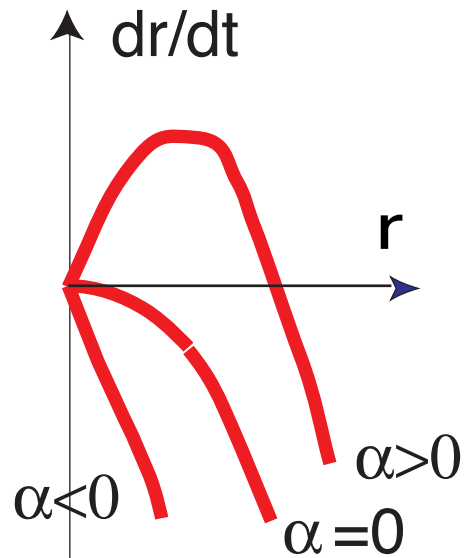


Hopf bifurcation

$$\dot{r} = \alpha r - r^3$$

$$\dot{\phi} = \omega$$

■ normal form



forward dynamics

- given known equation, determined fixed points / limit cycles and their stability
- more generally: determine invariant solutions (stable, unstable and center manifolds)

inverse dynamics

- given solution, find the equation...
- this is the problem faced in design of behavioral dynamics...

inverse dynamics: design

- in the design of behavioral dynamics... you may be given:
- attractor solutions/stable states
- and how they change as a function of parameters/conditions
- => identify the class of dynamical systems using the 4 elementary bifurcations
- and use normal form to provide an exemplary representative of the equivalence class of dynamics