Mathematics and Computer Science for Modeling
Unit 6: Differential Equations

Stephan Sehring
based on materials by Jan Tekülve and Daniel Sabinasz

Institut für Neuroinformatik, Ruhr-Universität Bochum

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Overview

1. **Differential Equations**
   - Application: Dynamical Systems
   - Solving Differential Equations
   - Dynamical Systems: Stability
   - Numeric integration

Application: Dynamical Systems

- A **dynamical system** is a system of one or more variables that change in time

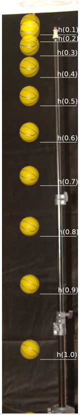
- e.g., the location of a falling ball

- Dynamical systems can often be described with a **differential equation** that describes the rate of change of the system at each point in time, e.g.,

$$h''(t) = -g$$

- **Solving** this differential equation means finding a function $h(t)$ that describes the location of the ball at each point in time.

$$h(t) = h_0 - \frac{1}{2}gt^2$$



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- A differential equation describes how a system should change in a given state.

Solving Differential Equations

- Given a differential equation of the form $f'(x) = g(f(x)) \ldots$ the original function $f(x)$ is usually not known.
- Solving a differential equation describes the process of finding an $f(x)$ that satisfies the differential equation for all x
  - Example:

$$f'(x) = 4f(x) + 5 \Rightarrow f(x) = \frac{e^{4x+c}}{4} - \frac{5}{4} \text{ and } f'(x) = e^{4x+c}$$

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  - The rate of change is a function of its antiderivative
  - In this case, the only function that stays the same when differentiated is the exponential function e^x
  - Considering the chain rule, the derivative of e^{cx} is exactly ce^{cx} therefore $f(x) = ce^{cx}$

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- The same is true for differential equations.
- But: Given some initial condition $(f(x_0), x_0)$ we find a unique solution $f(x)$.

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- General solution: $f(x) = (\int b(x)e^{A(x)}dx + c)e^{-A(x)}$ with $A(x) = \int a(x)dx$

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- $f(0) = 1$
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- $\Rightarrow f(x) = \frac{e^x + e^{-2x}}{3}$

Higher-order and non-linear differential equations

- Differential equations are not always easy to solve or might not even have any analytical solution.
- Higher-order: $a(x)f(x) + b(x)f'(x) + c(x)f''(x) + \ldots = b(x)$
- Non-linear: eg. $\tau a'(t) = -a(t) - h + k\sigma(a(t))$

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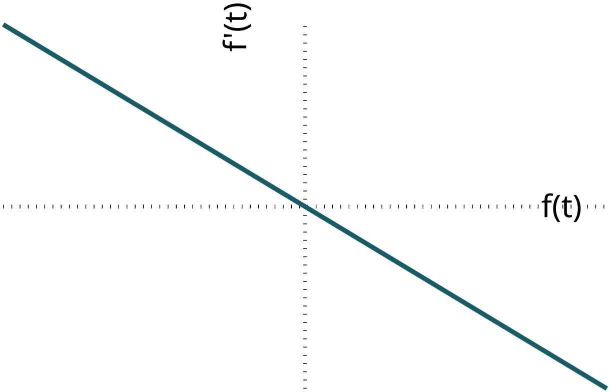
- Fixed points are of major interest to determine the stability of a dynamical system.
- A fixed point f_0 is a point where $f' = 0 \Rightarrow$ no change in time.
- The sign of f' around the fixed point gives us information about the stability of the system.

Phase Plots of Dynamical Systems

$$f'(t) = -f(t)$$



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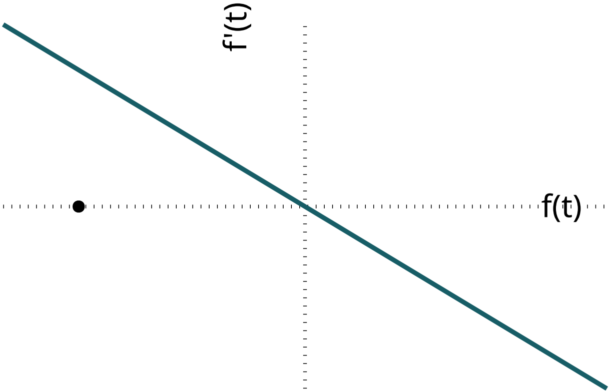

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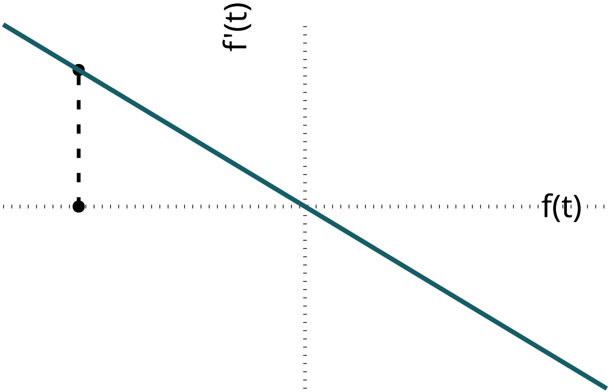

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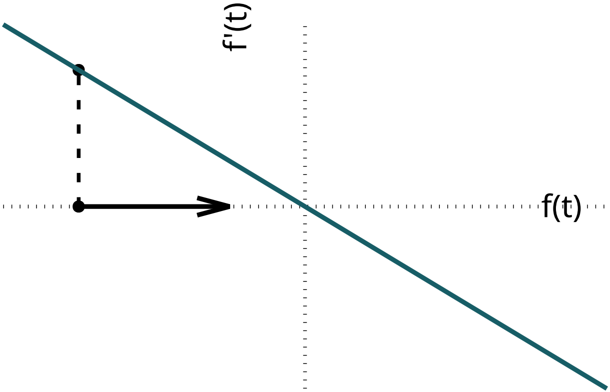

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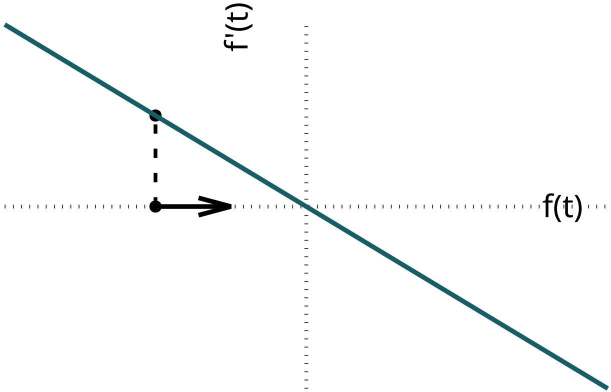

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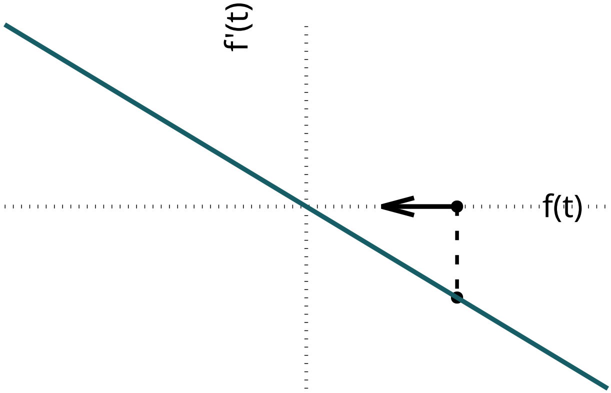

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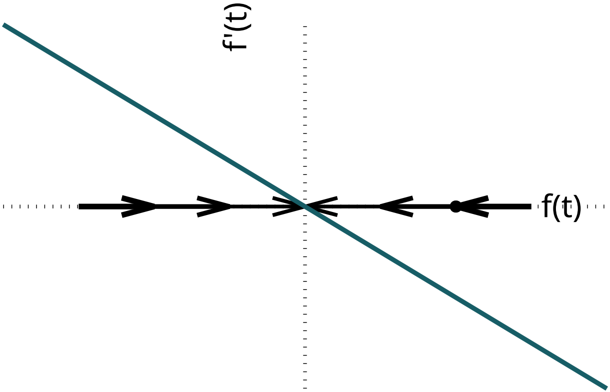
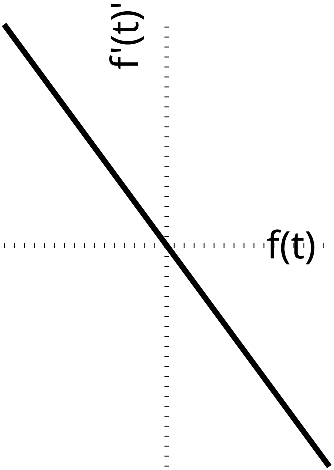


Phase Plot

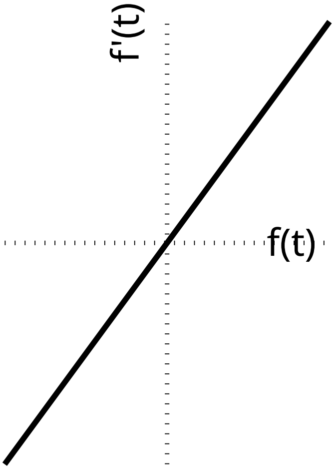
Attractors

$$f'(t) = -f(t)$$



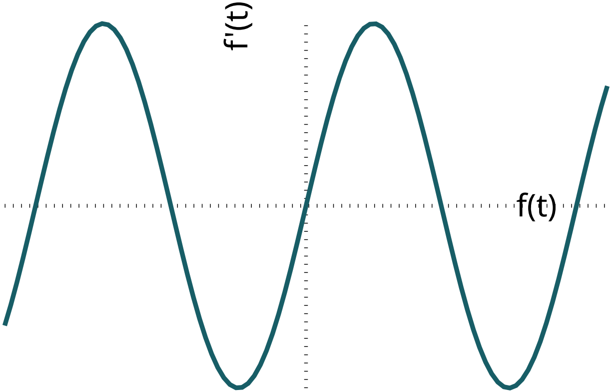
Repellors

$$f'(t) = -f(t)$$



Initial Condition Matters

$$f'(t) = sin(f(t))$$



Exercise1

- Revisit the example of the falling ball in the first slide of this lecture. We will now include a term to model the air resistance of the ball as it speeds up.
- Let $v'(t) = -\gamma v(t) + mg$ be the differential equation we want to solve, where v is the velocity of the ball, mg models the gravitational force and $-\gamma v(t)$ models the drag force of the air as the ball speeds up.
- Draw the phase plot of this linear differential equation. Is the system stable? How should $v(t)$ develop for different initial conditions?
- Can you find a solution to this differential equation? Start by assuming $v(t) = Ae^{-ct} + B$ and see if you can determine the free parameters (A, B, c). Assume the initial condition to be $v(0) = 0$. Does the solution behave as expected?

Euler Method

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- $\frac{d}{dt}f(t) = g(f(t))$ and $f(t_0) = f_0$
- $f(t_1) = f(t_0) + \frac{d}{dt}f(t_0)\Delta t = f_0 + g(f_0)\Delta t$

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- $f(t_2) = f(t_1) + \frac{d}{dt}f(t_1)\Delta t = f(t_1) + g(f(t_1))\Delta t$
- $f(t_i) = f(t_{i-1}) + g(f(t_i - 1))\Delta t$

Exercise2

- Write a python program that numerically integrates a differential equation using the Euler method.
- Numerically integrate the differential equation from exercise 1 and compare the result to the analytical solution.
- How could you compute the position of the ball from the known velocities at different points in time?

Advanced Exercise

- Coupled differential equations of multiple variables $(X(t), Y(t), Z(t))$ can be written as $\begin{pmatrix} X'(t) \\ Y'(t) \\ Z'(t) \end{pmatrix} = \begin{pmatrix} f(X,Y,Z) \\ g(X,Y,Z) \\ h(X,Y,Z) \end{pmatrix}$
- We can still use the Euler method to solve this equation for some initial condition (X_0, Y_0, Z_0).
- $\begin{pmatrix} X(t_i) \\ Y(t_i) \\ Z(t_i) \end{pmatrix} = \begin{pmatrix} X(t_{i-1}) \\ Y(t_{i-1}) \\ Z(t_{i-1}) \end{pmatrix} + \begin{pmatrix} f(X(t_{i-1}), Y(t_{i-1}), Z(t_{i-1})) \\ g(X(t_{i-1}), Y(t_{i-1}), Z(t_{i-1})) \\ h(X(t_{i-1}), Y(t_{i-1}), Z(t_{i-1})) \end{pmatrix} \Delta t$
- See if you can find the **Lorenz Attractor** by numerically solving the following system $\begin{pmatrix} f(X,Y,Z) \\ g(X,Y,Z) \\ h(X,Y,Z) \end{pmatrix} = \begin{pmatrix} a(Y-X) \\ X(b-Z)-Y \\ XY-cZ \end{pmatrix}$, for $a = 10$, $b = 28$ and c=8/3.