Mathematics and Computer Science for Modeling
Unit 5: Integration

Stephan Sehring
based on materials by Jan Tekülve and Daniel Sabinasz

Institut für Neuroinformatik, Ruhr-Universität Bochum

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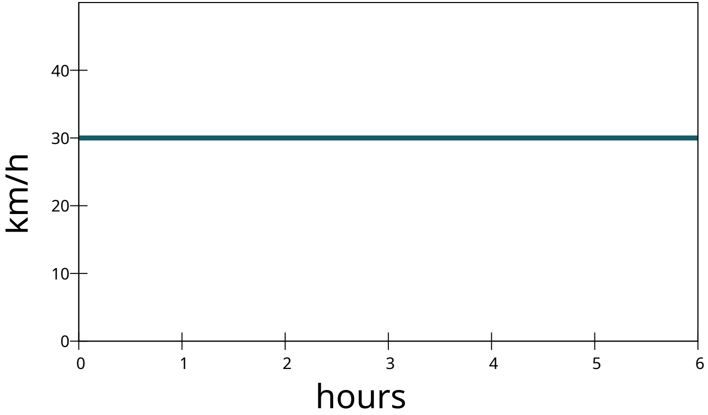
Overview

1. **Motivation**
2. **Mathematics**
   - Approximating the Area under a Curve
   - Calculating the Area under a curve
   - Improper Integrals
3. **Exercise**

From Velocity to Position

You drove 30 km/h for 6 hours. How far did you drive?



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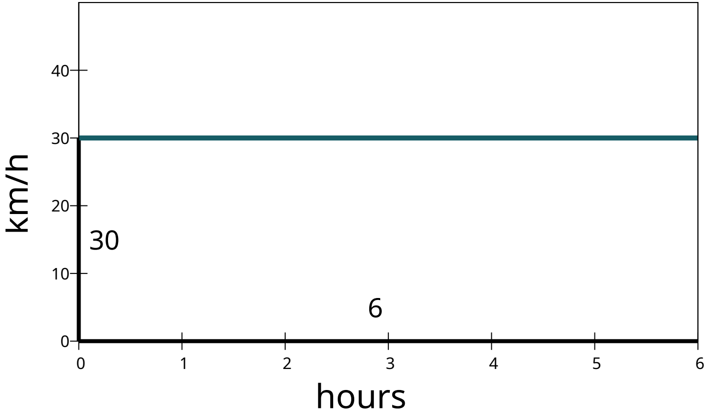

$$30\frac{\text{km}}{\text{h}} * 6\text{h} = 180\text{km}$$

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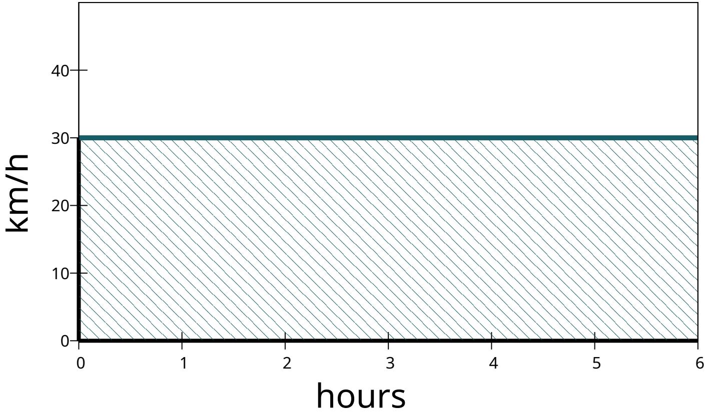

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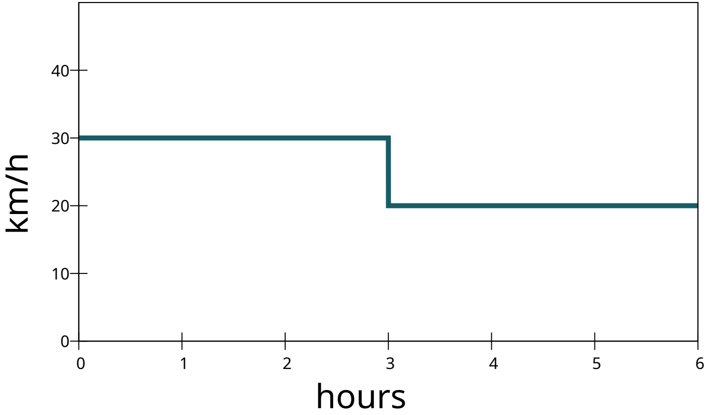
From Velocity to Position

Let's say you slowed down for the last 3 hours. How far did you get?



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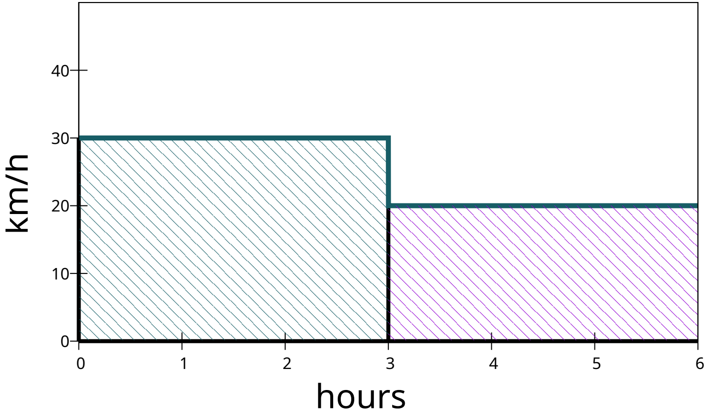
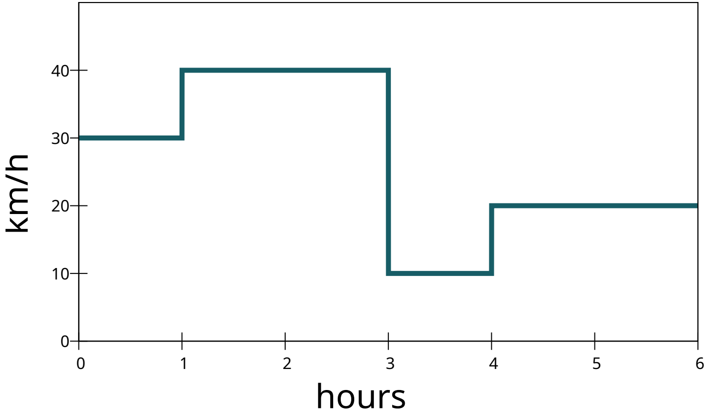


$$30\frac{\text{km}}{\text{h}} * 3\text{h} + 20\frac{\text{km}}{\text{h}} * 3\text{h} = 150\text{km}$$

From Velocity to Position

What if you mixed it up to not get bored?



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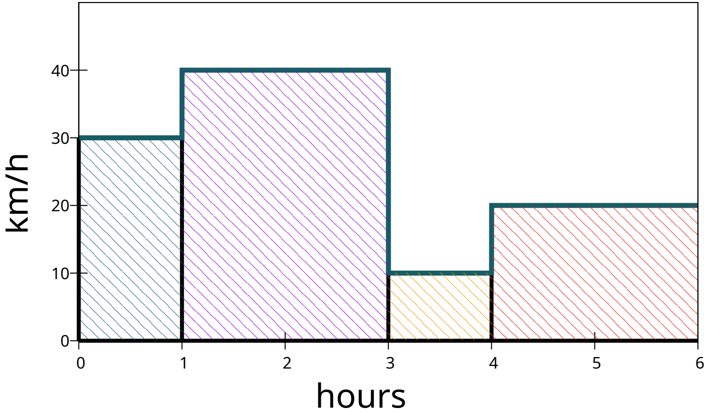
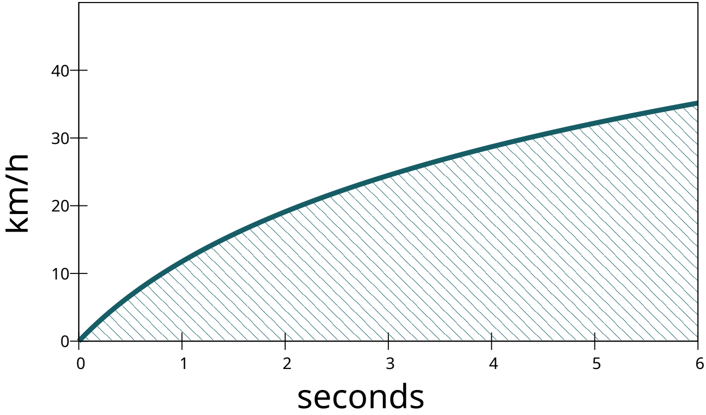


$$30\frac{\text{km}}{\text{h}} * 1\text{h} + 40\frac{\text{km}}{\text{h}} * 2\text{h} + 10\frac{\text{km}}{\text{h}} * 1\text{h} + 20\frac{\text{km}}{\text{h}} * 2\text{h} = 160\text{km}$$

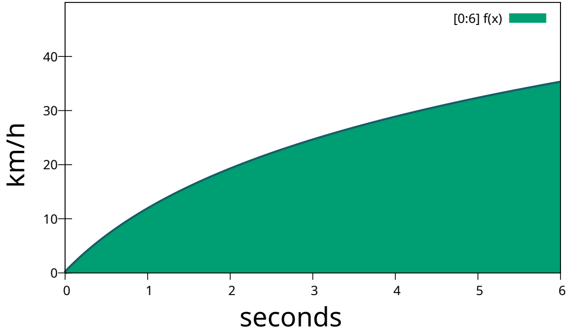
From Velocity to Position

But how about something realistic?



Approximation

- Not all areas can be calculated with rectangles



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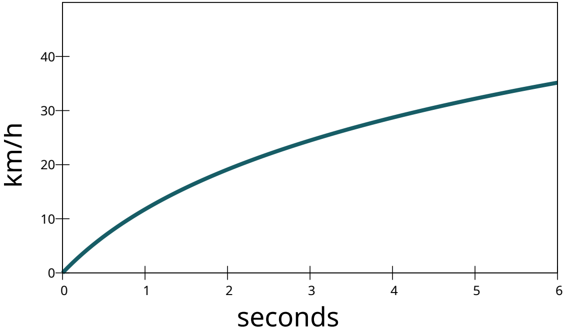

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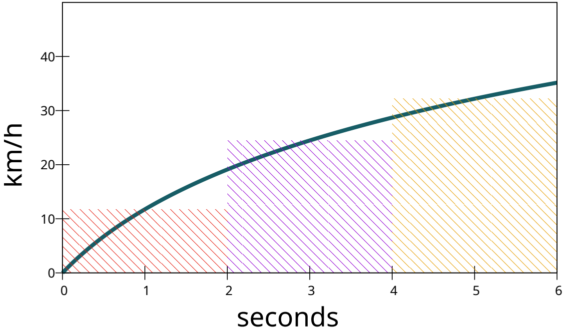

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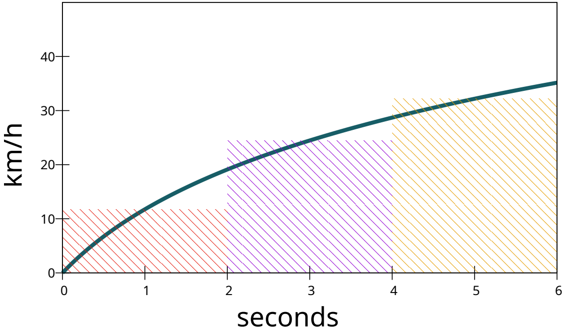

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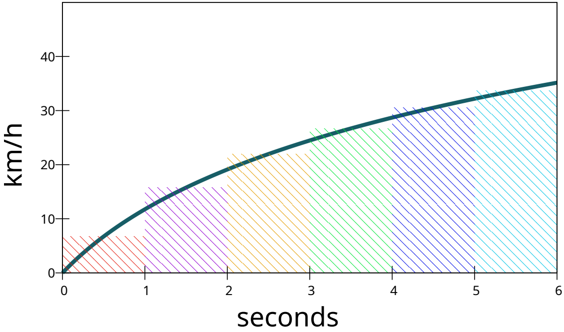


Midpoint Riemann Sum

Calculating Midpoints

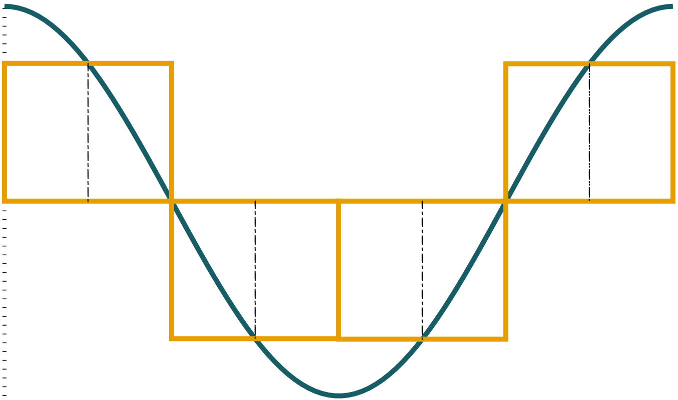
The **Midpoint Riemann Sum** is a way of approximating an integral with finite sums.
The are under the curve in a given interval $[x_i, x_{i+1}]$ can be approximated as the area of a rectangle with width $\Delta x = x_{i+1} - x_i$ and height $f(\frac{x_i + x_{i+1}}{2})$:

$$f(\frac{x_i + x_{i+1}}{2})\Delta x$$

The sum over all intervals yields an estimation of the area under the curve

$$I_M = \sum_{i}^{n} f(\frac{x_i + x_{i+1}}{2})\Delta x$$

Midpoint Sums



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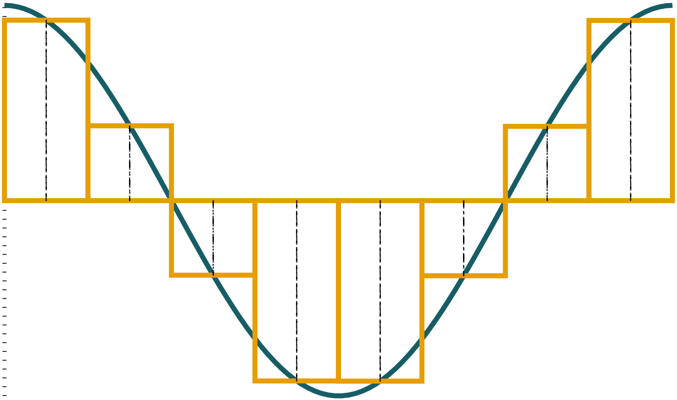

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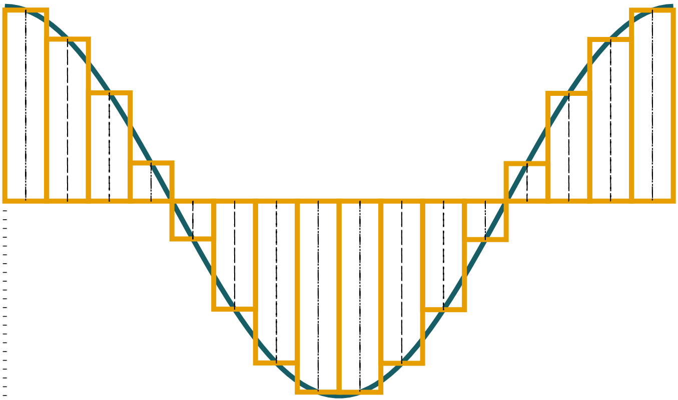

Exercise1

1. Download the exercise_template.py file from the course webpage. Run the script. What does it do?
2. Write a function 'midpoint_sum(f, interval, dx)' that calculates the Midpoint Riemann Sum by taking as input any callable function 'f', an integration 'interval' defined as a list [lower_bound, upper_bound]and a step-size dx.
3. Test the function by calculating and plotting the Riemann Sum of the function $f(x) = x$ in the interval [0,10] for different dx.
4. What should be the area A under $f(x) = x$ for this interval? Compare your result to the Riemann Sum by plotting the absolute difference between the real and approximated area ($abs(A - sum)$) for different dx.
5. (optional) What happens for some interval [-a,a]?

From Sums to Integrals

Midpoint Sum: $f\left(\frac{x_i+x_{i+1}}{2}\right)\Delta x$

The larger the number n of intervals, the smaller Δx and the better our approximation.

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Definite Integral

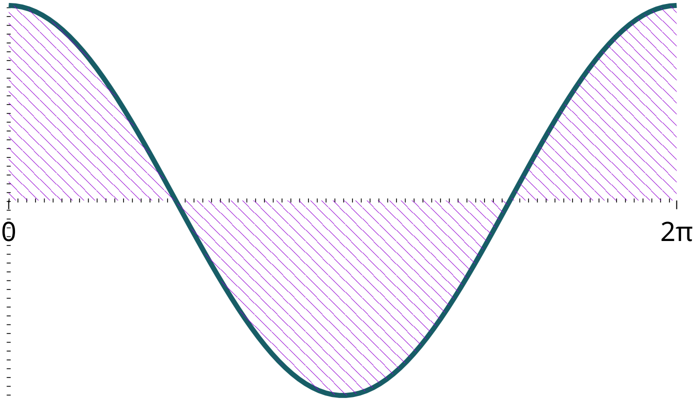
The **definite integral** of a function $f(x)$ between the **lower boundary** a and the **upper boundary** b

$$\int_a^b f(x)dx$$

is defined as the size of the area between f and the x-axis inside the boundaries. Areas above the x-axis are considered positively and areas below negatively.

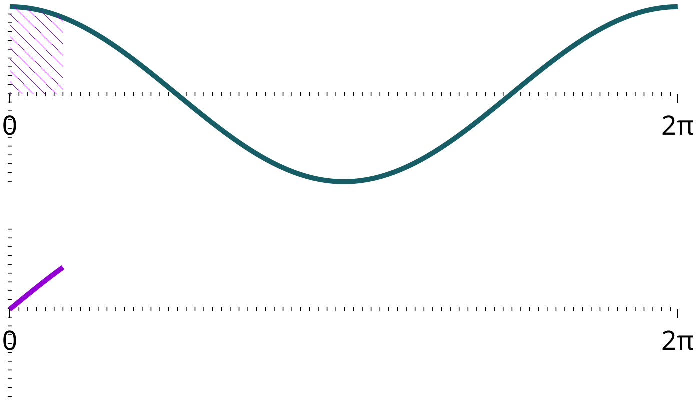
Definite Integral

$f(x) = cos(x)$ $\int_0^{2\pi} cos(x)dx$



Indefinite Integral

$f(x) = cos(x)$ $\int_0^x cos(x')dx'$



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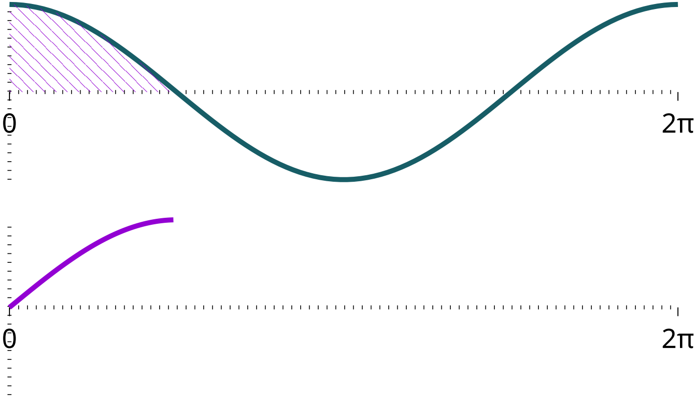

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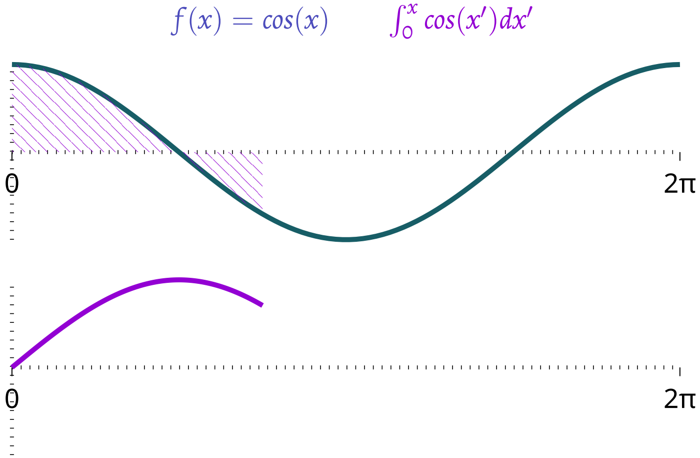

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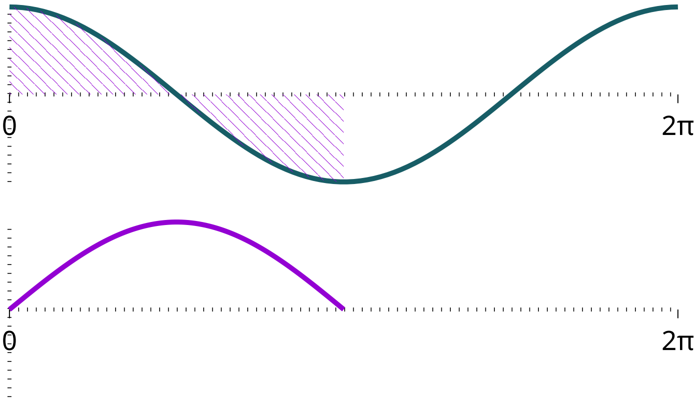

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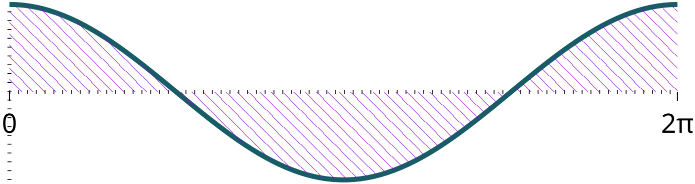
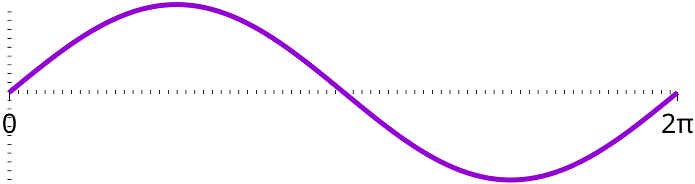
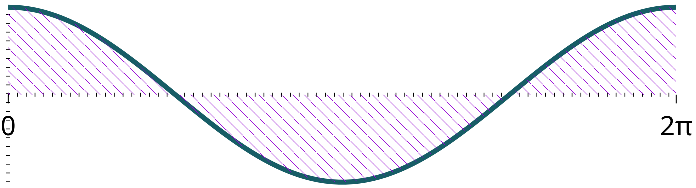
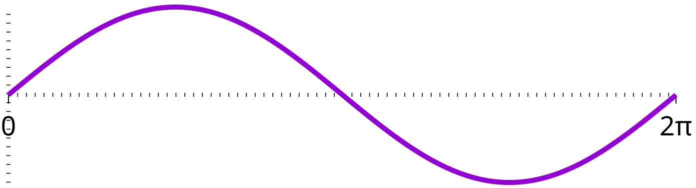




Indefinite Integral

$f(x) = cos(x)$ $\int_0^x cos(x')dx' = sin(x)$





Indefinite Integral

$f(x) = cos(x) \qquad \int_0^x cos(x')dx' = \int cos(x')dx'$

The Antiderivative

Definition

If f is a function with domain $[a, b] \to \mathbb{R}$ and there is a function F, which is differentiable in the interval $[a, b]$ with the property that

$$F'(x) = f(x),$$

then F is considered an **antiderivative** of f

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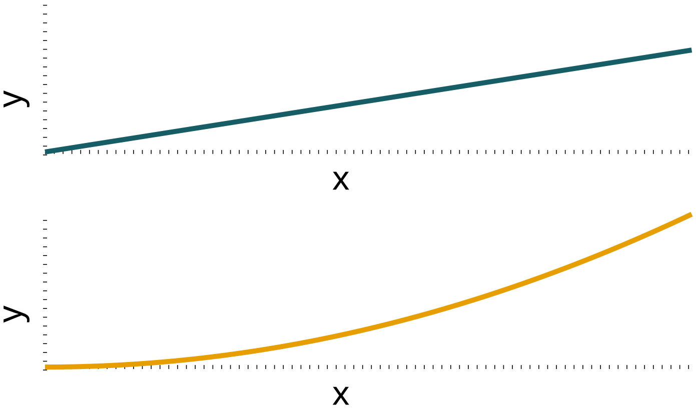
Properties of an antiderivative

- Differentiation removes constants, therefore $F(x) + c$ for any constant c is also an antiderivative
- Unlike with differentiation there are no fixed rules to compute an antiderivative from a given f

A function and its antiderivative

$f(x) = x$ $\qquad$ $F(x) = \frac{1}{2}x^2$



The Fundamental Theorem of Calculus

First Fundamental Theorem of Calculus

One of the antiderivatives of a function can be obtained as the indefinite integral:

$$\int f(x')dx' = F(x)$$

- Intuition: The rate of change of the area under $f(x)$ is $f(x)$

The Fundamental Theorem of Calculus

Second Fundamental Theorem of Calculus

If f is integrable and continuous in $[a, b]$, then the following holds for each antiderivative F of f

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

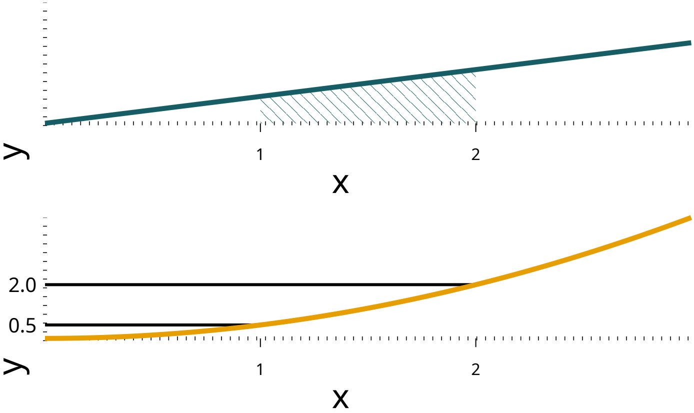
Example:

- Area under $f(x) = x$ between values 1 and 2

$$\int_1^2 xdx = \left[\frac{1}{2}x^2\right]_1^2 = \frac{1}{2}2^2 - \frac{1}{2}1^2 = 1.5$$

Definite Integral Example

$f(x) = x$ $\qquad F(x) = \frac{1}{2}x^2$ $\qquad \int_1^2 f(x)dx = F(2) - F(1)$



The Integral is a Linear Operator

Integration Rules

- **Summation**

$$\int_a^b f(x) + g(x) = \int_a^b f(x) + \int_a^b g(x)$$

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$$\int_a^b cf(x) = c\int_a^b f(x)$$

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- **Boundary Transformations**

$$\int_a^b f(x) + \int_b^c f(x) = \int_a^c f(x) \qquad \int_a^b f(x) = -\int_b^a f(x)$$

Non-Linear rules

Integration Rules

- **Integration by Parts**

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx)dx$$

- **Substitution Rule**

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Other rules

Integration Rules

- **Power Rule**

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

- **Exponential**

$$\int e^x dx = e^x + C$$

- **log**

$$\int ln(x) dx = xln(x) - x + C$$

Improper Integrals

Infinite Intervals

It is possible to calculate the area in infinitely large intervals. Intervals with an infinite boundary are called **Improper Integrals**

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

Example:

- Convergent improper integral

$$\int_1^{\infty} x^{-2}dx = \lim_{b\to\infty} \int_1^b x^{-2}dx = \lim_{b\to\infty} \left[-x^{-1}\right]_1^b = \lim_{b\to\infty} \left(-b^{-1} + 1\right) = 1$$

Exercise2

Answer the following tasks using a piece of paper and a pocket calculator.

1. Given the Antiderivative $F(x) = 12x^2 + 5x$ of the function $f(x)$, calculate the area between $f(x)$ and the x-axis in the interval of $[-3, 5]$.

2. Calculate $\int_0^\pi \cos(x)dx$. Before applying the formula, look at a plot of $\cos(x)$. What kind of result would you expect?

3. Use the midpoint_sum() function from the last exercise to numerically approximate the anti-derivative of an arbitrary function f. Plot the result. Try this out for different functions and compare to the analytic results.

4. (optional) Think of the function $f(x) = sin(\frac{1}{x})$. What does this function look like? Is the integral over $[-a, a]$ well defined? Do you expect a numeric Riemann Sum evaluation to yield good results?

Exercise Solutions

Exercise Solutions

1. The antiderivative is already given, therefore you only need to plug-in the beginning and the end of the interval.

$$\begin{aligned}[F(x)]_a^b =& F(b) - F(a) = F(5) - F(3)\\ =& 12 * 5^2 + 5 * 5 - \left(12 * (-3)^2 + 5 * (-3)\right) = 325 - 93 = 232\end{aligned}$$

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2. Looking at the plot of $\cos(x)$ you can see that exactly the same area is enclosed above the x-axis as below the x-axis, therefore the total area has to be zero.
   To verify this analytically, you need to figure out the antiderivative of $\cos(x)$ first. From the lecture you know that $F(x) = sin(x)$.

$$[F(x)]_a^b = F(b) - F(a) = F(\pi) - F(0) = \sin(\pi) - \sin(0) = 0 - 0 = 0$$