Mathematics and Computer Science for Modeling
Unit 4: Calculus

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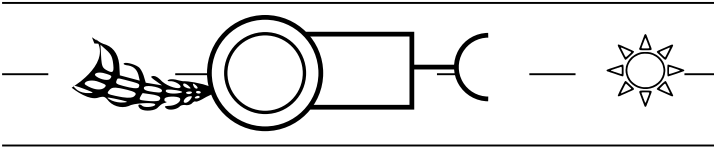
Overview

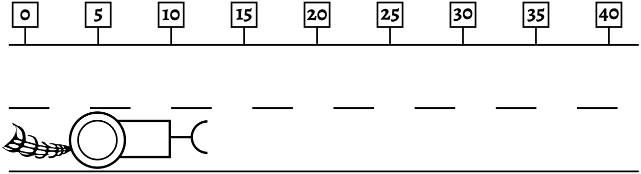
1. **Motivation**
   - Numerical Differentiation
2. **Differentiation**
   - Graphical Interpretation
   - Formal Description
   - Rules for Differentiation
3. **Exercises**

Motivation

Estimating Velocity by Differentiation



The Vehicle's Position



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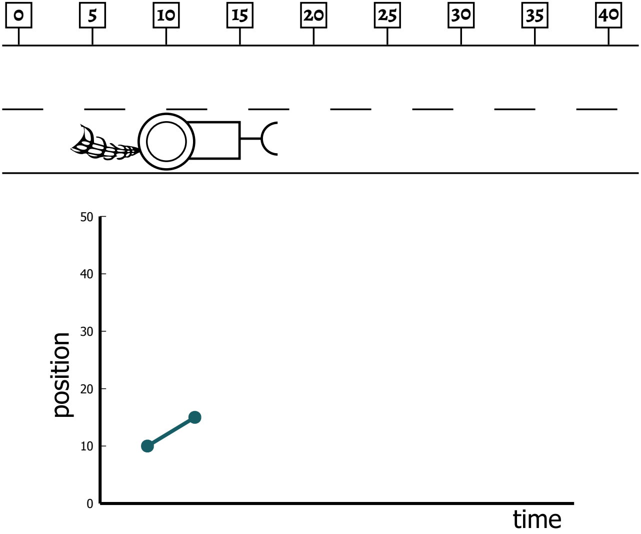

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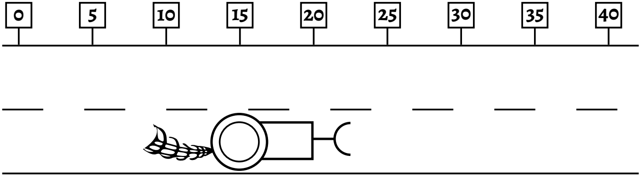

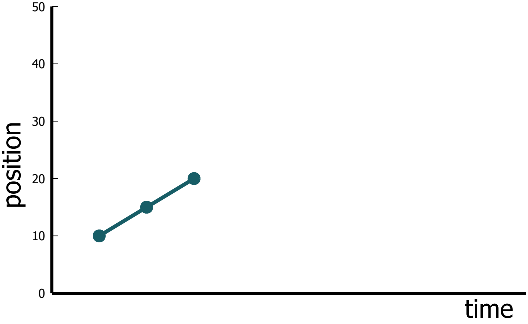

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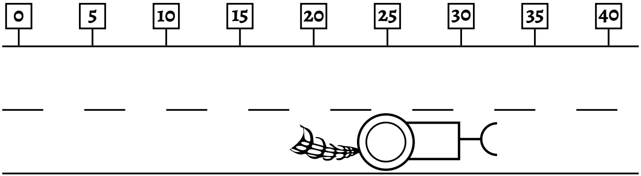

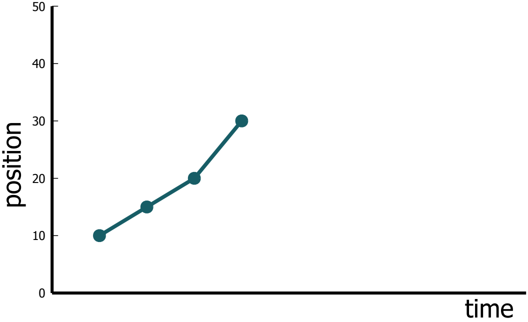

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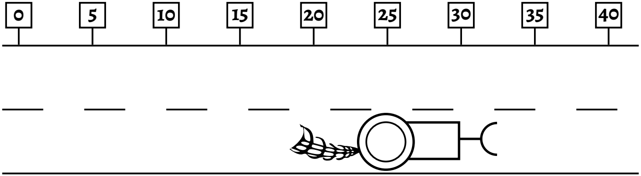

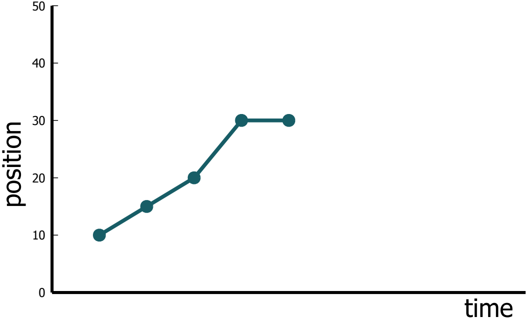

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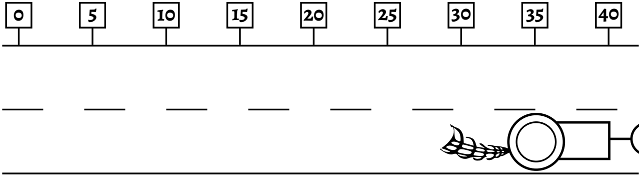

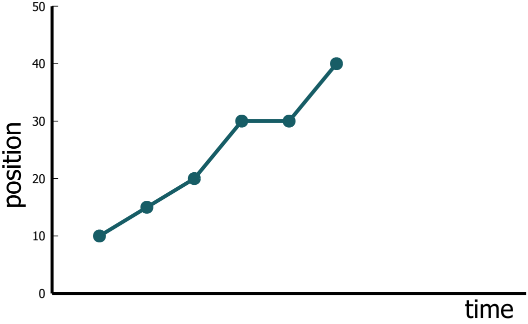

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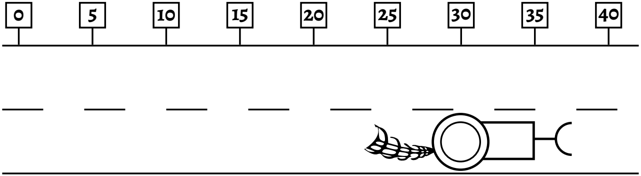

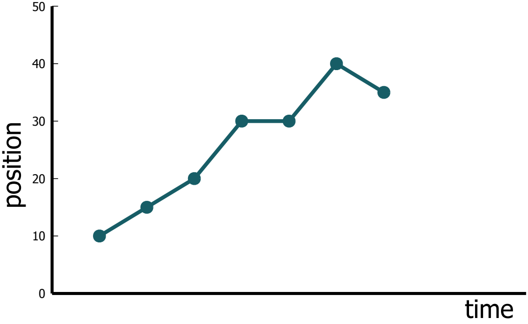

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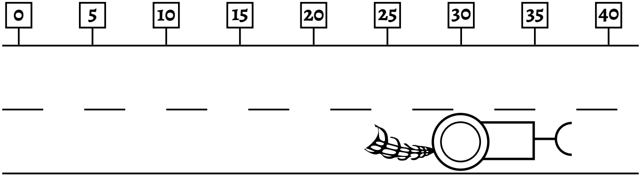

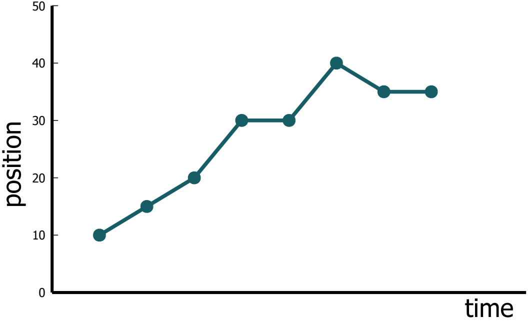

Numerical Differentiation

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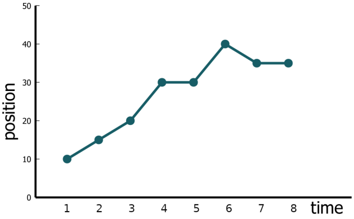
(Simple) Numerical Differentiation

It is possible to calculate the average slope of $f(x)$ between x_i and x_{i+1}.

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}.$$

Exercise 1

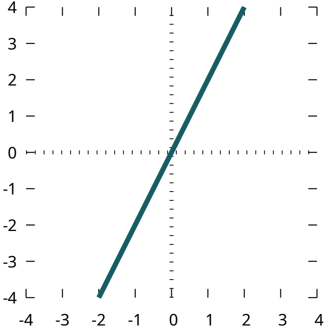


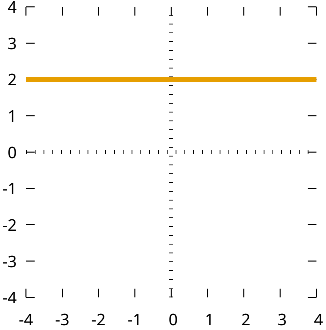
1. Calculate the change of position between time 3 and 4. Next, calculate the rate of change of the position (= velocity) between time 3 and 4.
2. Do the same for the time between 3 and 5.
3. Now assume that the position is given as $f(t) = t^2$. Plot that function from time 0 to 3. Calculate the velocity between time 0 and 2. Draw a line through the points $(0, f(0))$ and $(2, f(2))$. How does the slope of the line relate to the velocity? Why? Next, do the same for the velocity between time 1 and 2, then between time 1.5 and 2.
4. Think about what it would mean to calculate the velocity *at* time 2.

A function and its derivative

- The derivative of a function $f(x)$, denoted $f'(x)$, measures the degree to which $f(x)$ changes when x changes

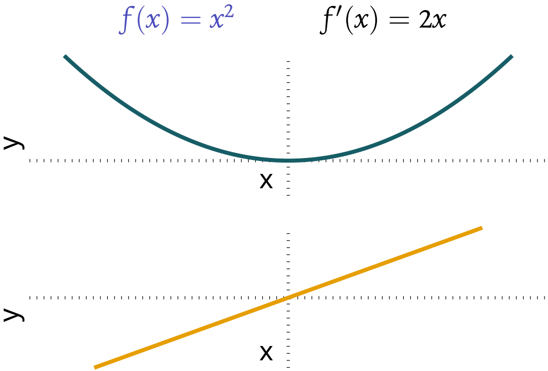
$$f(x) = x \qquad f'(x) = 1$$





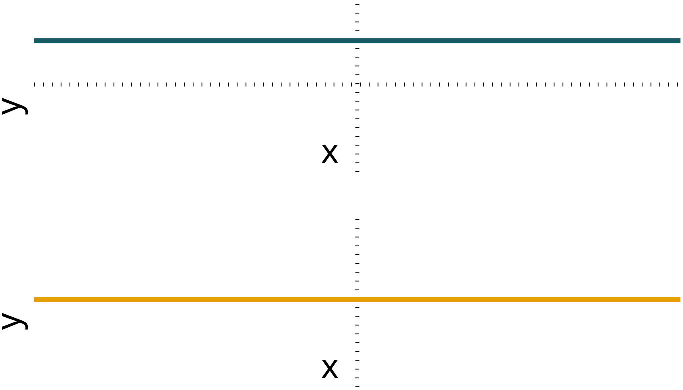
A function and its derivative

- The derivative of a function $f(x)$, denoted $f'(x)$, measures the degree to which $f(x)$ changes when x changes
- $f'(x)$ is the slope of the tangent at x



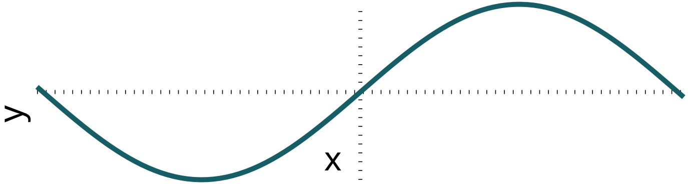
A function and its derivative

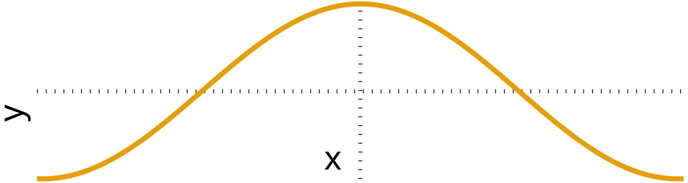
$f(x) = 0.5$ $\qquad$ $f'(x) = 0$



A function and its derivative

$f(x) = sin(x)$ $\quad$ $f'(x) = cos(x)$





Derivative as a Tangent

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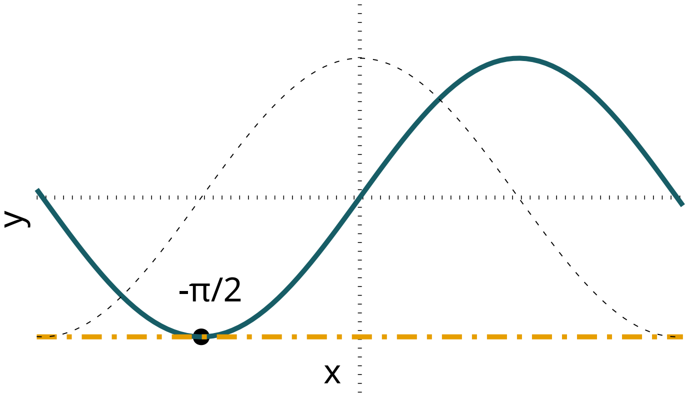

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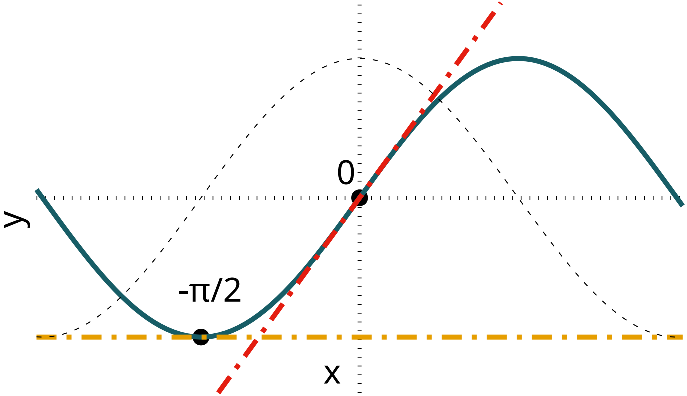

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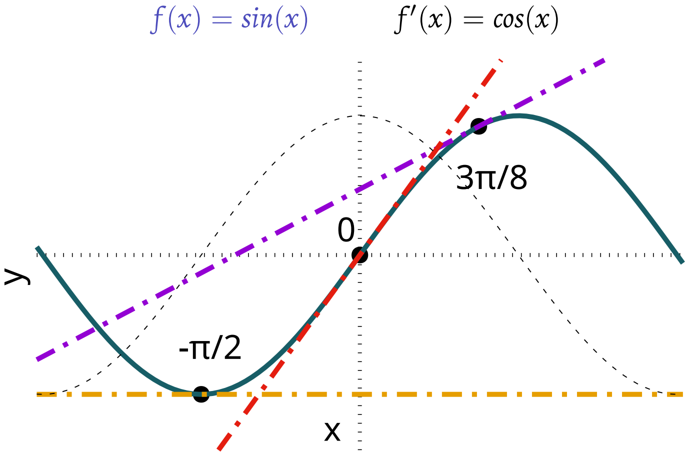

Formal Definition

Differentiable Function

- The **derivative of** f **at position** x_0, short $f'(x)$, is defined as

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

- This denotes the value of $\frac{f(x)-f(x_0)}{x-x_0}$ as x gets closer and closer to x_0.

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- Alternate notations:

$$f'(x) = \frac{df}{dx}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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- Simplifying

$$\lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\cancel{(x - x_0)}(x + x_0)}{\cancel{x - x_0}} = \lim_{x \to x_0} (x + x_0)$$

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- Applying the limit:

$$\lim_{x \to x_0} (x + x_0) = 2x$$

Differentiation is a linear operator

Rules

- **Constant Factor**

$$\frac{d}{dx}(af) = a\frac{d}{dx}(f)$$

- **Sums**

$$\frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)$$

Example:

$$\frac{d}{dx}(4x^2) = 4\frac{d}{dx}(x^2) = 4(2x) = 8x$$

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$$\frac{d}{dx}(4x^2 + x^2) = 4\frac{d}{dx}(x^2) + \frac{d}{dx}(x^2) = 4(2x) + 2x = 10x$$

Differentiation for Products and Quotients

Rules

- **Multiplication**

$$\frac{d}{dx}(fg) = \frac{d}{dx}(f)g + f\frac{d}{dx}(g)$$

- **Exponentiation**

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

- **Division**

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{d}{dx}(f)g - f\frac{d}{dx}(g)}{g^2}$$

Examples

- **Multiplication**

$$\frac{d}{dx}(x^2 sin(x)) = \frac{d}{dx}(x^2)sin(x) + x^2\frac{d}{dx}(sin(x)) = 2xsin(x) + x^2cos(x)$$

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- **Division**

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{\frac{d}{dx}(1)x - 1\frac{d}{dx}(x)}{x^2} = \frac{0-1}{x^2} = \frac{-1}{x^2}$$

Exponentiation Rule derives from Multiplication Rule

- Example $f'(x^3)$

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$$= 2xx^2 + x^2 2x = 2x^3 + 2x^3 = 4x^3$$

Special cases

- The derivative of

$$f(x) = e^x \text{ is } f'(x) = e^x$$

- The derivative of

$$f(x) = ln(x) \text{ is } f'(x) = \frac{1}{x}$$

- The derivative of

$$f(x) = sin(x) \text{ is } f'(x) = cos(x)$$

Composite functions

Chain Rule

- Function h is a composition of functions g and f

$$h(x) = (g \circ f)(x) = g(f(x))$$

- If f and g are differentiable, h is also differentiable

$$\frac{d}{dx}(h(x)) = \frac{d}{dy}(g(y))\frac{d}{dx}(f(x)), \text{ with } y = f(x)$$

- Verbal rule: **Inner derivative times outer derivative**

Chain Rule Examples

- $h(x) = 5(7x + 2)^4 = g(f(x))$

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  $g(x) = 5x^4 \wedge f(x) = 7x + 2$

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$$
\begin{aligned}
g(x) &= 5x^4 \wedge f(x) = 7x + 2 \\
g'(x) &= 20x^3 \wedge f'(x) = 7
\end{aligned}
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h'(x) &= 20(7x+2)^3 7 = 140(7x+2)^3
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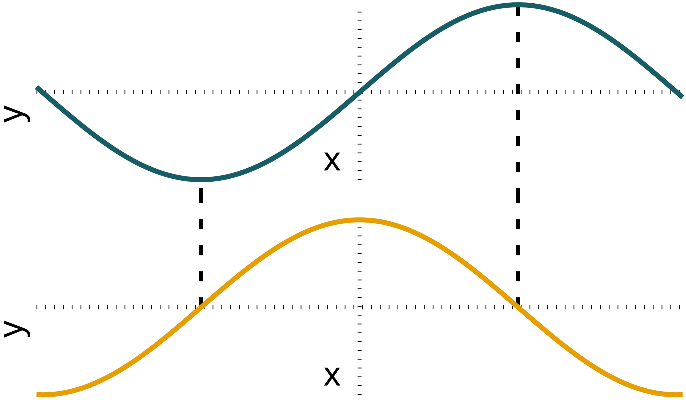
$$g(x) = e^x \wedge f(x) = 5x$$
$$g'(x) = e^x \wedge f'(x) = 5$$
$$h'(x) = e^{5x} 5 = 5e^{5x}$$

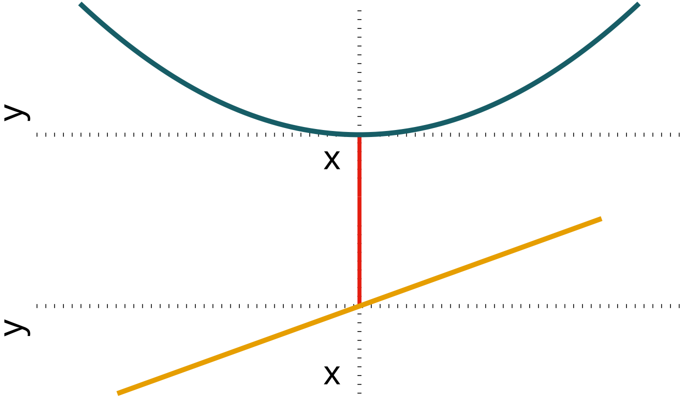
Finding Local Extrema

$f(x) = sin(x)$ $f'(x) = cos(x)$



Finding Local Extrema

$f(x) = x^2$ $\quad$ $f'(x) = 2x$



Calculation of Local Extrema

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$$\iff x = \frac{-6}{8} = \frac{-3}{4}$$

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- $f(x) = sin(x)$

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- $\boldsymbol{f(x) = sin(x)}$

  $$f'(x) = cos(x)$$

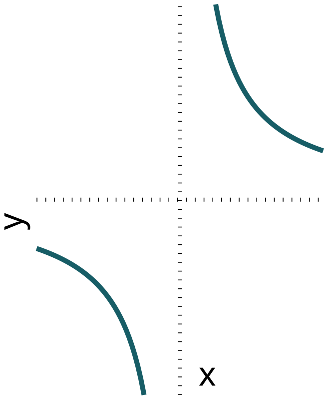
  $$f'(x) = cos(x) \overset{!}{=} 0$$

  $$\iff x = cos^{-1}(0)$$

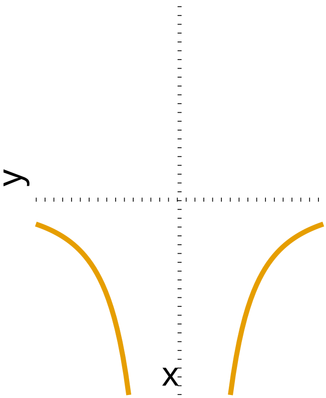
  $$\iff x = 90^\circ = \frac{\pi}{2}, 270^\circ = \frac{3\pi}{2}, \ldots$$

Differentiability is not given

$f(x) = \frac{1}{x}$



$f'(x) = \frac{-1}{x^2}$



Formal Definition

Differentiable Function

- The **derivative of f at position** x_0, short $f'(x)$, is defined as

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

- This denotes the value of $\frac{f(x)-f(x_0)}{x-x_0}$ as x gets closer and closer to x_0.
- f is called **differentiable** if and only if this limit exists.
- Alternate notations:

$$f'(x) = \frac{df}{dx}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation is a linear operator

Rules

- **Constant Factor**

$$\frac{d}{dx}(af) = a\frac{d}{dx}(f)$$

- **Sums**

$$\frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)$$

Differentiation for Products and Quotients

Rules

- **Multiplication**

$$\frac{d}{dx}(fg) = \frac{d}{dx}(f)g + f\frac{d}{dx}(g)$$

- **Exponentiation**

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

- **Division**

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{d}{dx}(f)g - f\frac{d}{dx}(g)}{g^2}$$

Composite functions

Chain Rule

- Function h is a composition of functions g and f

$$h(x) = (g \circ f)(x) = g(f(x))$$

- If f and g are differentiable, h is also differentiable

$$\frac{d}{dx}(h(x)) = \frac{d}{dy}(g(y))\frac{d}{dx}(f(x)), \text{with } y = f(x)$$

- Verbal rule: **Inner derivative times outer derivative**

Special cases

- The derivative of

$$f(x) = e^x \text{ is } f'(x) = e^x$$

- The derivative of

$$f(x) = ln(x) \text{ is } f'(x) = \frac{1}{x}$$

- The derivative of

$$f(x) = sin(x) \text{ is } f'(x) = cos(x)$$

Exercise 2

1. Calculate the derivative of the following functions (on a piece of paper)
   1.1 $f(x) = 7x^4$
   1.2 $g(x) = 2x^4 + 3x^3 + x^2 + 10x + 5$
   1.3 $h(x) = 4e^{3x}$
   1.4 $i(x) = (12x^2 + 5)3x^3$
   1.5 $j(x) = \frac{3x}{\cos(x)}$

   - First think about the rule you need to use
   - Identify the parts of the rule in the equation
   - If possible differentiate individual parts first
   - Apply the rule

2. Find the extreme value of the function $k(x) = 6x^2 + 3x + 2$