

Mathematics and Computer Science for Modeling

Unit 2: Functions in Math

Stephan Sehring

based on materials by Jan Tekülve and Daniel Sabinasz

Institut für Neuroinformatik, Ruhr-Universität Bochum

24.09.2024

1. Sets and Number Systems

2. Functions in Math

- Definition
- Function Types
- Parametrization
- Multiple Arguments
- Properties

1. Sets and Number Systems

2. Functions in Math

- Definition
- Function Types
- Parametrization
- Multiple Arguments
- Properties

Sets

- ▶ For practical purposes, think of a **set** as a container of objects
- ▶ e.g., the set of natural numbers



Sets

- ▶ Notation: $\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, \dots\}$
- ▶ Something is either in the set or not in the set
- ▶ If something is in the set, we call it an **element** of the set
- ▶ e.g., 5 is an element of $\mathbb{N}$, but -3 is not an element of $\mathbb{N}$
- ▶ Write $5 \in \mathbb{N}$ and $-3 \notin \mathbb{N}$

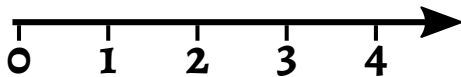
Sets

- ▶ Instead of listing all the elements, you can describe in natural language what the elements should be
- ▶ e.g., $A = \{x \mid x \text{ is an even number}\} = \{0, 2, 4, 6, 8, \dots\}$

Number Systems

Number Systems

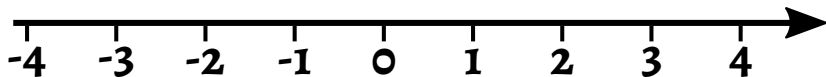
- ▶ **Natural Numbers:** $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$
- ▶ **Integer Numbers:** $\mathbb{Z} =$
- ▶ **Rational Numbers:** $\mathbb{Q}$
- ▶ **Real Numbers:** $\mathbb{R}$



Number Systems

Number Systems

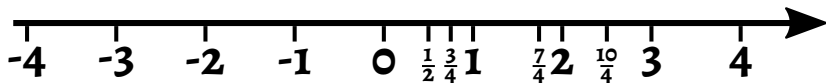
- ▶ **Natural Numbers:** $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$
- ▶ **Integer Numbers:** $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- ▶ **Rational Numbers:** $\mathbb{Q}$
- ▶ **Real Numbers:** $\mathbb{R}$



Number Systems

Number Systems

- ▶ **Natural Numbers:** $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$
- ▶ **Integer Numbers:** $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- ▶ **Rational Numbers:** $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}$
- ▶ **Real Numbers:** $\mathbb{R}$

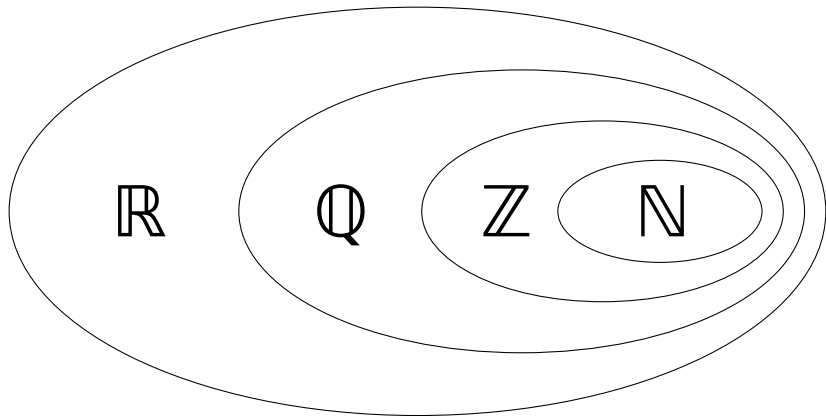


Number Systems

Number Systems

- ▶ **Natural Numbers:** $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$
- ▶ **Integer Numbers:** $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- ▶ **Rational Numbers:** $\mathbb{Q} = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$
- ▶ **Real Numbers:** $\mathbb{R} = \mathbb{Q} \cup \text{irrational numbers}$

Number Systems



1. Sets and Number Systems

2. Functions in Math

- Definition
- Function Types
- Parametrization
- Multiple Arguments
- Properties

Function Intuition

- ▶ Function example: $f(x) = 2x + 3$
- ▶ A function, written like this, can be thought of as a formula that can be evaluated to give the value of the function
- ▶ e.g.,
 - ▶ $f(1) = 2 \cdot 1 + 3 = 5$
 - ▶ $f(2) = 2 \cdot 2 + 3 = 6$

Plotting Functions

Tabular Interpretation of: $f(x) = 2x + 3$

x		0	1	2	3	4	5
y							

Plotting Functions

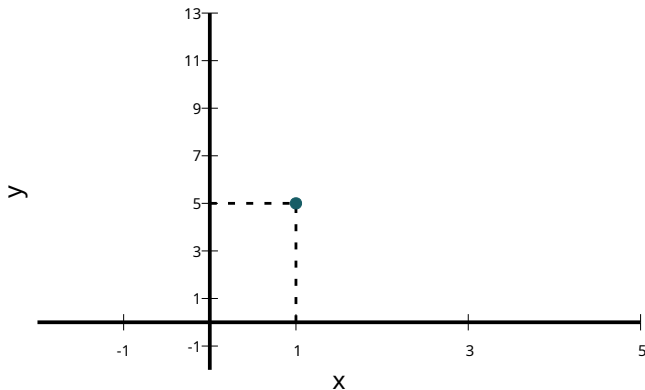
Tabular Interpretation of: $f(x) = 2x + 3$

x	0	1	2	3	4	5
y		5				

Plotting Functions

Tabular Interpretation of: $f(x) = 2x + 3$

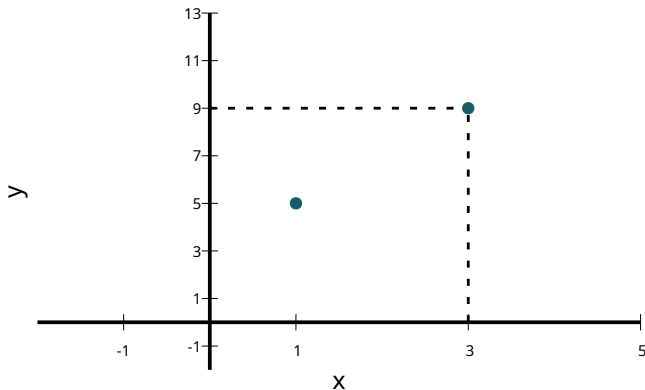
x	0	1	2	3	4	5
y		5				



Plotting Functions

Tabular Interpretation of: $f(x) = 2x + 3$

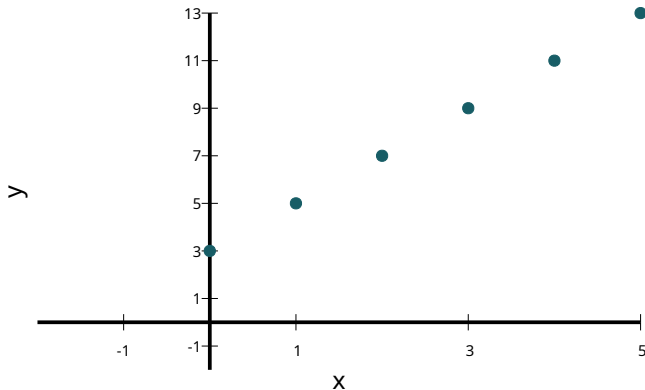
x	0	1	2	3	4	5
y		5		9		



Plotting Functions

Tabular Interpretation of: $f(x) = 2x + 3$

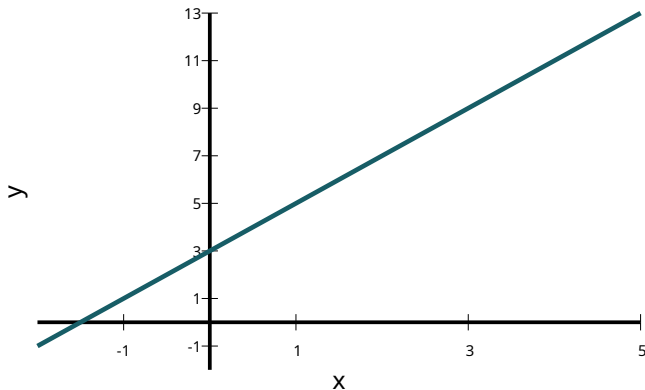
x	0	1	2	3	4	5
y	3	5	7	9	11	13



Plotting Functions

Tabular Interpretation of: $f(x) = 2x + 3$

x	0	1	2	3	4	5
y	3	5	7	9	11	13



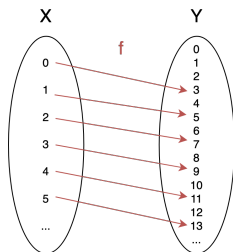
Function Definition

Function

X and Y are two sets.

A **function** $f : X \rightarrow Y$ is a mathematical object that assigns each element $x \in X$ exactly one element $y \in Y$.

$$x \rightarrow y = f(x)$$



- ▶ x is called the **function argument**
- ▶ y is called the **function value**
- ▶ X is called the **domain**
- ▶ Y is called the **codomain**
- ▶ The **image** W of $f(x)$ are all values in Y that can be assumed by the function.

Matplotlib



- ▶ Matplotlib allows to plot functions:

```
import matplotlib.pyplot as plt
```

```
numbers = [2*x+3 for x in range(6)]
```

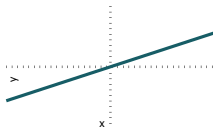
```
plt.plot(numbers)
```

```
plt.show()
```

Function Types

► Linear Functions

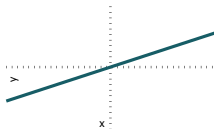
$$y = mx + b$$



Function Types

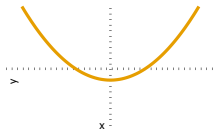
► Linear Functions

$$y = mx + b$$



► Power Functions

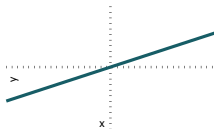
$$y = ax^n$$



Function Types

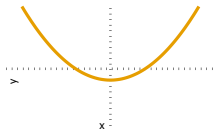
► Linear Functions

$$y = mx + b$$



► Power Functions

$$y = ax^n$$

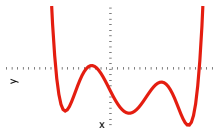


► Polynomial Functions

$$y = \sum_{i=0}^n a_i x^i$$

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots a_nx^n$$

describes a polynomial of degree n , where $a_n \neq 0$



The Summation Symbol

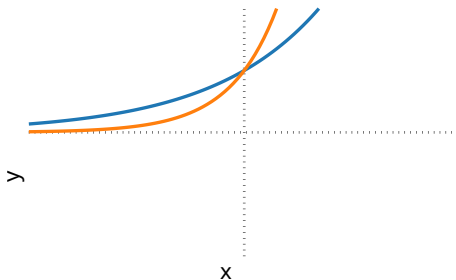
- ▶ $\sum_{i=0}^n T(i)$ denotes a sum of multiple terms
- ▶ The bottom row defines an indexing variable, here i , and specifies an initial value, here 0
- ▶ That variable takes on increasing values (0, 1, 2, 3, ..., n)
- ▶ The top row specifies the maximum value for i , here n
- ▶ $T(i)$ specifies a term for each i
- ▶ $\sum_{i=0}^n T(i)$ sums up $T(i)$ for each i
- ▶ Thus, $\sum_{i=0}^n T(i) = T(0) + T(1) + T(2) + \dots + T(n)$
- ▶ e.g., $\sum_{i=0}^5 i = 0 + 1 + 2 + 3 + 4 + 5$

Exponentials Functions

Exponential Functions

$$f(x) = e^x$$

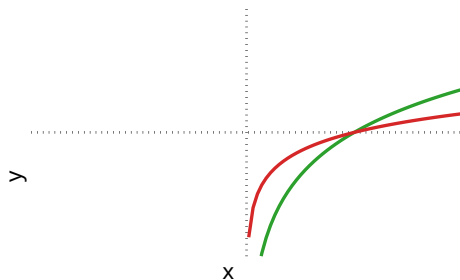
$$g(x) = 10^x$$



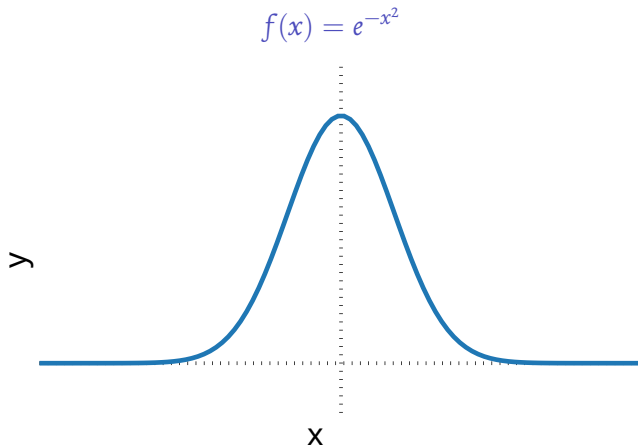
Logarithmic Functions

$$h(x) = \ln(x)$$

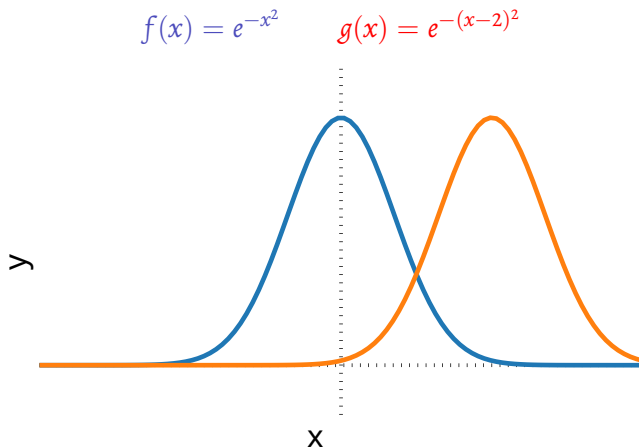
$$j(x) = \log_{10}(x)$$



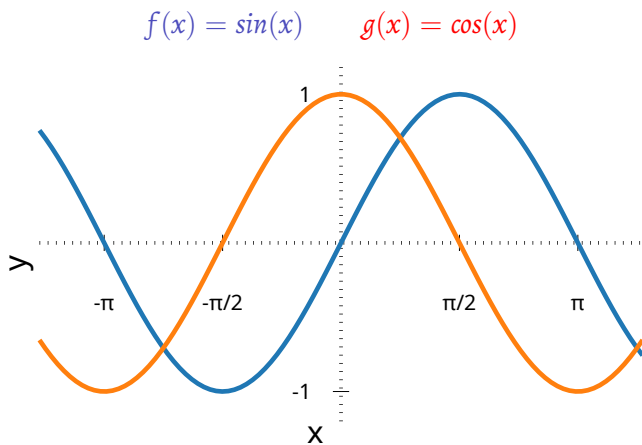
The Gaussian Function



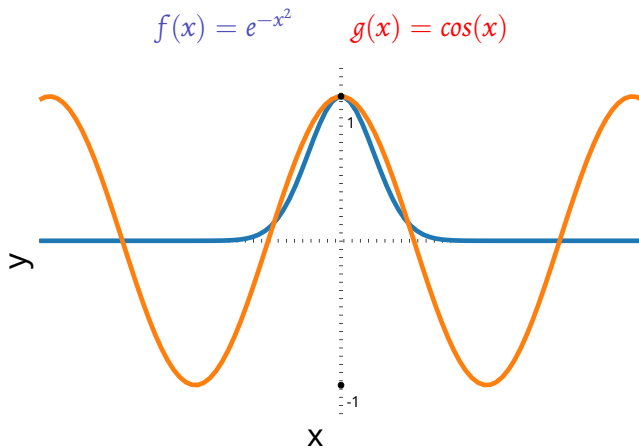
The Gaussian Function



Trigonometric Functions



Chaining Functions

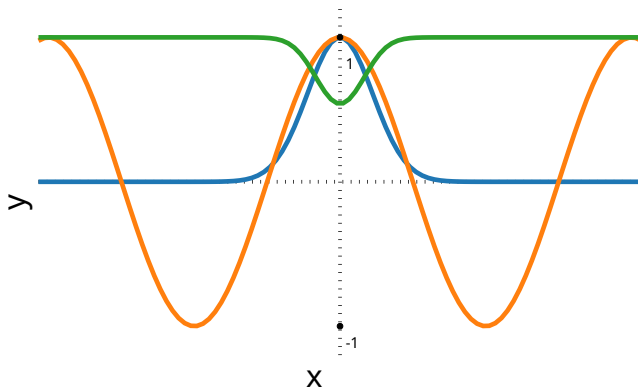


Chaining Functions

$$f(x) = e^{-x^2}$$

$$g(x) = \cos(x)$$

$$h(x) = g(f(x))$$



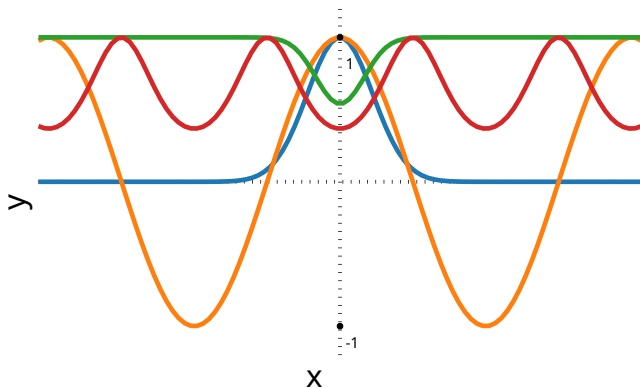
Chaining Functions

$$f(x) = e^{-x^2}$$

$$g(x) = \cos(x)$$

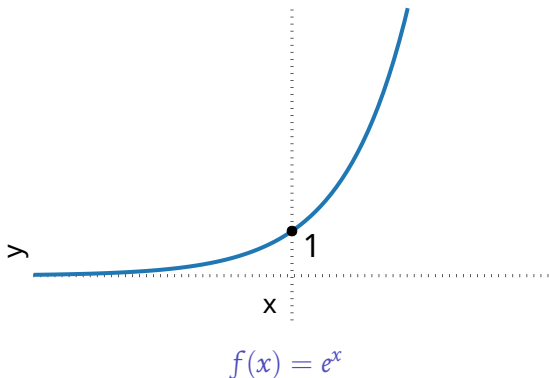
$$h(x) = g(f(x))$$

$$j(x) = f(g(x))$$



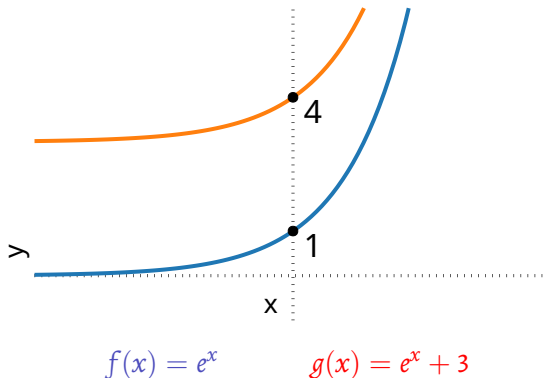
Function Translation

- ▶ Translation in y -direction: $\hat{f}(x) = f(x) + b$
- ▶ Translation in x -direction: $\hat{f}(x) = f(x - a)$



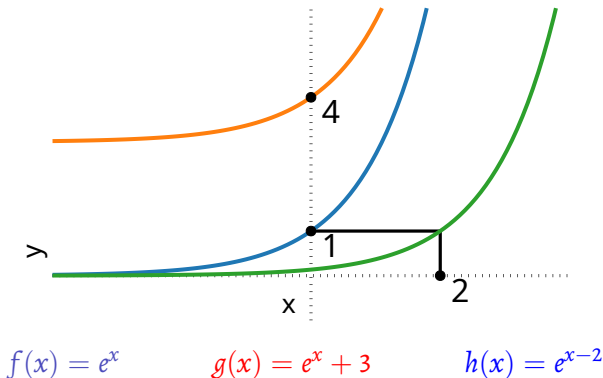
Function Translation

- ▶ Translation in y -direction: $\hat{f}(x) = f(x) + b$
- ▶ Translation in x -direction: $\hat{f}(x) = f(x - a)$



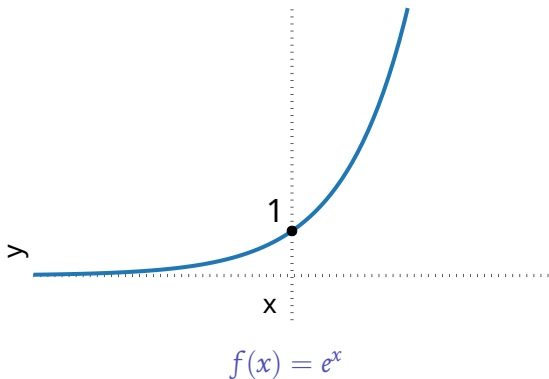
Function Translation

- ▶ Translation in y -direction: $\hat{f}(x) = f(x) + b$
- ▶ Translation in x -direction: $\hat{f}(x) = f(x - a)$



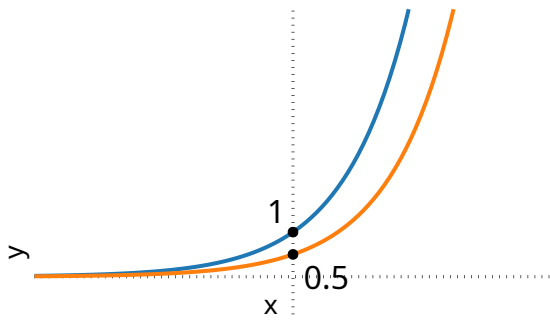
Function Stretching and Compression

- ▶ Stretching/Compression in **y-direction**: $\hat{f}(x) = d f(x), d > 0$
- ▶ Stretching/Compression in **x-direction**: $\hat{f}(x) = f(cx), c > 0$



Function Stretching and Compression

- ▶ Stretching/Compression in **y-direction**: $\hat{f}(x) = df(x), d > 0$
- ▶ Stretching/Compression in **x-direction**: $\hat{f}(x) = f(cx), c > 0$

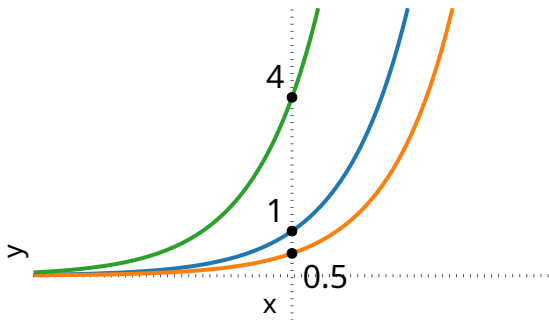


$$f(x) = e^x$$

$$g(x) = \frac{1}{2}e^x$$

Function Stretching and Compression

- ▶ Stretching/Compression in **y-direction**: $\hat{f}(x) = df(x), d > 0$
- ▶ Stretching/Compression in **x-direction**: $\hat{f}(x) = f(cx), c > 0$



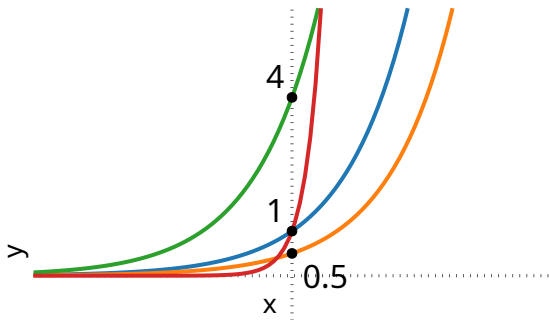
$$f(x) = e^x$$

$$g(x) = \frac{1}{2}e^x$$

$$h(x) = 4e^x$$

Function Stretching and Compression

- ▶ Stretching/Compression in **y-direction**: $\hat{f}(x) = df(x), d > 0$
- ▶ Stretching/Compression in **x-direction**: $\hat{f}(x) = f(cx), c > 0$



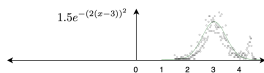
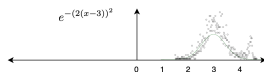
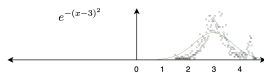
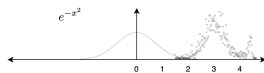
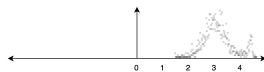
$$f(x) = e^x$$

$$g(x) = \frac{1}{2}e^x$$

$$h(x) = 4e^x$$

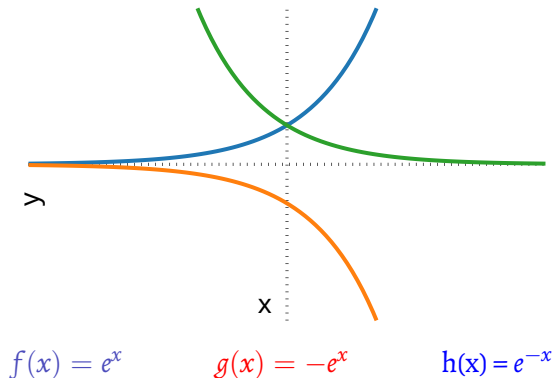
$$j(x) = e^{4x}$$

Example



Function Reflection

- ▶ Reflection across the **y-axis**: $\hat{f}(x) = f(-x)$
- ▶ Reflection across the **x-axis**: $\hat{f}(x) = -f(x)$



Exercise 1

1. Give an example for a natural number, a negative integer, a rational number and an irrational number
2. Which of the following is true? (a) Every real number is rational. (b) Every integer is rational. (c) Every natural number is a real number.
3. Let $f : \mathbb{N} \rightarrow \mathbb{R}, x \rightarrow 2x + 3$. Identify the function argument, the function value, the domain, the codomain and the image.
4. Create a function $\hat{f}(x)$ by translating $f(x) = e^x$ by -2 in y-direction and by 3 in x-direction.
5. Create a function $\hat{f}(x)$ by stretching $f(x) = e^x$ along the y-axis and compressing it along the x-axis.
6. Create a function $\hat{f}(x)$ by compressing $f(x) = e^x$ along the y-axis and stretching it along the x-axis.

Exercise 2

1. Write a python function that calculates $f(x) = 4x + 3$ and plot it.
2. Define a second function $g(x, a_0, a_1, a_2, a_3)$ that calculates a polynomial of degree 3 with variable coefficients a_0 to a_3 and plot $g(x, 3, 0, 2, 1)$
3. Calculate $f(x)$ or $g(x, 3, 0, 2, 1)$ for x values from 0 to 20. Store the result in a list.
4. (optional) Define a function 'polynomial(a, x)' that receives a list of coefficients 'a' ($a_0, a_1, a_2, \dots, a_n$) with a flexible number of items and computes $\sum_{i=0}^n a_i x^i$.

Multiple Arguments

- ▶ So far: functions of a single argument, e.g., $f(x) = 2x + 3$
- ▶ Generalization: functions of multiple arguments, e.g., $f(w, h) = w \cdot h$

Multiple Arguments

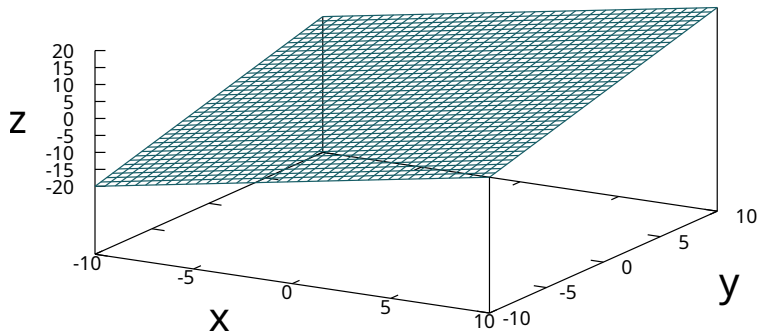
- ▶ So far: functions of a single argument, e.g., $f(x) = 2x + 3$
- ▶ Generalization: functions of multiple arguments, e.g., $f(w, h) = w \cdot h$
- ▶ Then $f(5, 2) = 5 \cdot 2 = 10$

Multiple Arguments

- ▶ So far: functions of a single argument, e.g., $f(x) = 2x + 3$
- ▶ Generalization: functions of multiple arguments, e.g., $f(w, h) = w \cdot h$
- ▶ Then $f(5, 2) = 5 \cdot 2 = 10$
- ▶ Formally: $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, f(w, h) = w \cdot h$

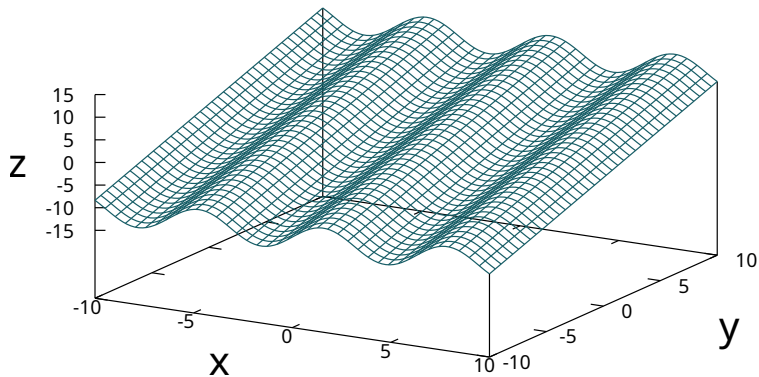
Multiple Arguments

$$f(x, y) = x + y$$



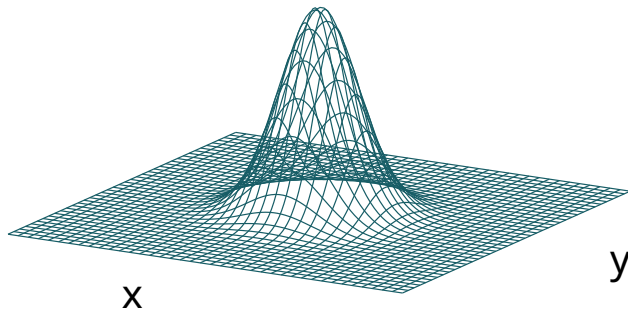
Multiple Arguments

$$f(x,y) = \sin(x) + y$$



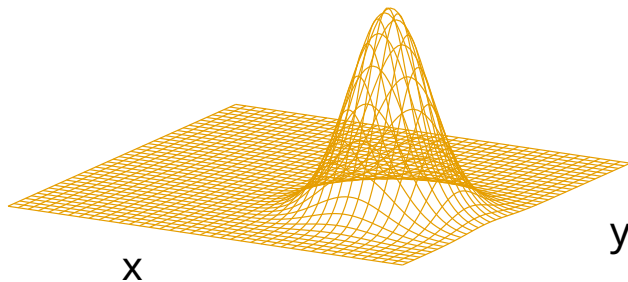
Multiple Arguments

$$f(x, y) = e^{-(x^2 + y^2)}$$



Multiple Arguments

$$f(x, y) = e^{-((x-2)^2 + (y+1)^2)}$$



Injective, Surjective and Bijective Functions

- ▶ An image f is **injective**, if two different elements $x_1 \neq x_2$ are always projected to two different elements $y_1 \neq y_2$
- ▶ An image f is **surjective**, if for each element $y \in Y$ an element $x \in X$ exists, such that $y = f(x)$
- ▶ An image f is **bijective**, if it is injective and surjective

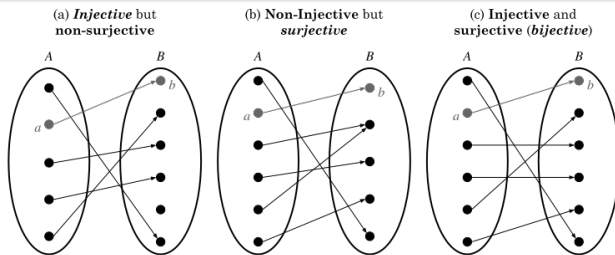


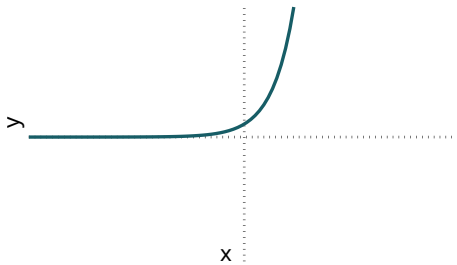
Image source:

https://commons.wikimedia.org/wiki/File:Injective,_Surjective,_Bijective.svg

Injective, Surjective and Bijective Functions

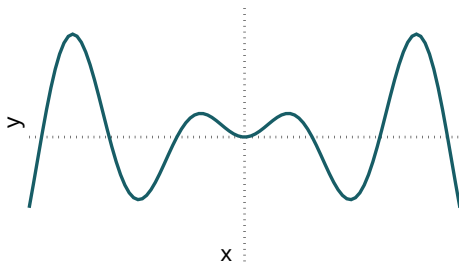
Injective, but not surjective

$$f(x) = e^x$$



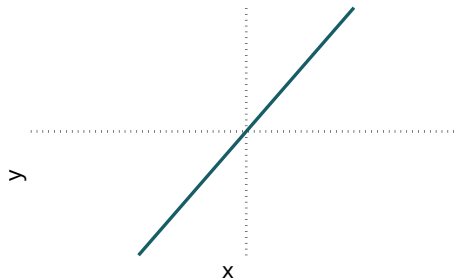
Surjective, but not Injective

$$f(x) = x \sin(x)$$

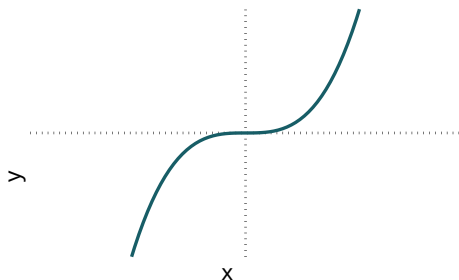


Bijjective Function Example

$$f(x) = 4x$$



$$f(x) = x^3$$

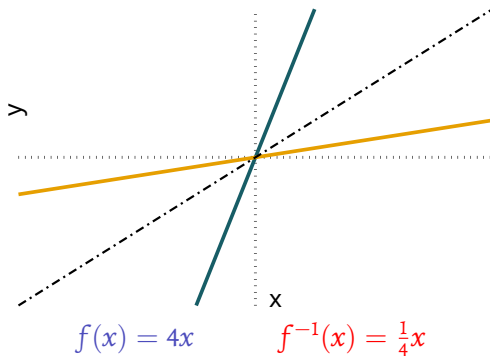


Inverse Function

Definition

Given a bijective **function** $f : X \rightarrow Y$, $f^{-1} : Y \rightarrow X$ denotes the **inverse function** of f .

It holds that $f^{-1}(f(x)) = x$ for all $x \in X$.

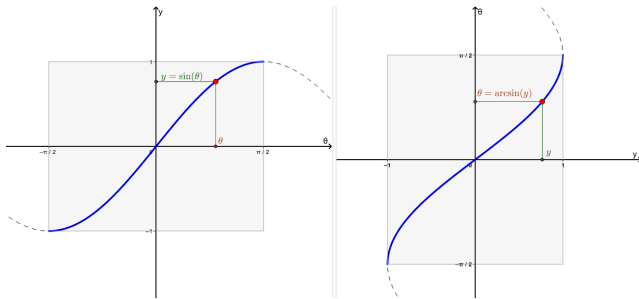


Inverse Function

Definition

Given a bijective **function** $f : X \rightarrow Y$, $f^{-1} : Y \rightarrow X$ denotes the **inverse function** of f .

It holds that $f^{-1}(f(x)) = x$ for all $x \in X$.



<https://www.geogebra.org/m/Efs8QRRF>

Image source:

Monotonicity

Definition

- ▶ A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called **monotonically increasing**, if for all x_1, x_2 order is preserved by applying f :

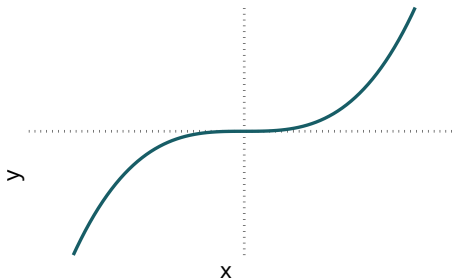
$$x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$$

- ▶ A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called **monotonically decreasing**, if for all x_1, x_2 order is reversed by applying f :

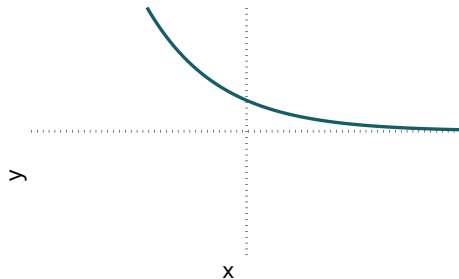
$$x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$$

Monotonicity Examples

monotonically increasing



monotonically decreasing



Functions Exercise 3

1. Determine the inverse $f^{-1}(x)$ of $f(x) = 2x + 3$
2. For each of the following functions, determine if they are monotonically increasing, monotonically decreasing or neither: $f(x) = x^2$, $f(x) = -x^5$, $f(x) = x^7$