Dynamical Systems Tutorial

PART 1 OF 2 – THE BASICS
What is a Dynamical System?

A way to take a state and predict its future

More formally:
- A state space \( S \) of possible configurations (e.g. location, some vector, ...)
- A range of times \( t \) where it works
- A rule that takes a state \( x \) from \( S \) at time \( t \) and tells us how \( s \) changes
- We call this rule the dynamics (or vector field)

For us, the rule is a differential equation

A time course of a dynamical variable is called a trajectory
What is a Dynamical System?

A differential equation gives us the rate of change of a variable in time a function of that variable.

For instance: We have a position, we get a velocity.

Simplest example: \( \frac{dx}{dt} = -\frac{x}{\tau} \)

At the heart of the neuron model we use

We often use the abbreviation of the time derivative \( \dot{x}(t) = \frac{dx}{dt} \)
Separation of Variables

Procedure to solve simple differential equations

Not technically quite clean, but works anyway

\[ \frac{dx}{dt} = -\frac{x}{\tau} \]

\[ \frac{dx}{x} = -\frac{1}{\tau} dt \]

\[ \int \frac{dx}{x} = \int -\frac{1}{\tau} dt \]

\[ \log(x) \bigg|_{x_0}^x = \left[-\frac{t}{\tau}\right]_0^t \]

\[ \log(x) - \log(x_0) = -\frac{t}{\tau} \]

\[ \frac{x}{x_0} = \exp \left(-\frac{t}{\tau}\right) \]

\[ x(t) = x_0 \exp \left(-\frac{t}{\tau}\right) \]
Numerical Methods

Used to simulate systems

Always have an error

Simplest method: Euler step

Sample time discretely:  \( t_i \) with \( i \in \{1, 2, \ldots, N\} \)

Then \( t_i = i\Delta t \)

Approximate change \( \Delta x_i \) during step \( \Delta t \) via derivative

\[
\frac{\Delta x(t_i)}{\Delta t} \approx \frac{dx}{dt}\bigg|_{t=t_i} = f(x(t_i), t_i)
\]

\[
x(t_{i+1}) = x(t_i) + \Delta x(t_i) \approx x(t_i) + \Delta t \ f(x(t_i), t_i)
\]
Different form of Dynamical Systems

We saw an example of a one-dimensional differential equation

There are also
- Vector valued (N-dimensional) equations
- integro-differential equations
- partial differential equations
- functional differential equations
- delay differential equations
- ...
\[ \frac{dx}{dt} = -\frac{x}{\tau} \]
\[ \frac{dx}{dt} = -\frac{x}{\tau} \]
Fixed Points

When the rate of change is zero we have, of course, no change

Our system is in balance!

But is that balance stable?

\[ \dot{x} = 0 \]
Stability

How does the system react if you disturb it slightly?

- Moves away → unstable
- Stays in the vicinity → stable
- Goes back to fixed point → asymptotically stable
\[
\frac{dx}{dt} = -\frac{x}{\tau}
\]
\[ \frac{dx}{dt} = \frac{x}{\tau} \]
\[ \frac{dx}{dt} = \frac{x}{\tau} \]
Stability

For linear system: Look at slope at fixed point
- Negative slope -> stable
- Positive slope -> unstable

In practice, there is always noise pushing us away from repellors

\[
\frac{dx}{dt} = -\frac{x}{\tau} \quad \text{Attractor}
\]

\[
\frac{dx}{dt} = \frac{x}{\tau} \quad \text{Repellor}
\]
We mostly only understand linear systems well

What to do with non-linear problems?

➢ Make it linear!

We can still use the sign of the derivative
Linear Multidimensional Systems

\[ \dot{\vec{x}} = M \vec{x} \]

Stability depends on eigenvalues of $M$

- if the real-parts of all Eigenvalues are negative: stable
- if the real-part of any Eigenvalue is positive: unstable
- if the real-part of any Eigenvalue is zero: marginally stable in that direction (stability depends on other eigenvalues)
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PART 2 OF 2 - BIFURCATIONS
What is a bifurcation

We have a dynamical system with a parameter c

As c changes smoothly, the behavior of the system as an abrupt change

Technically: Infinitesimal parameter change make for topologically inequivalent systems
Tangent Bifurkation

\[ \dot{x} = (-(x - 1)^2 - c)(x - 3) \]
Hopf Theorem

When a single (or pair of complex conjugate) eigenvalue crosses the imaginary axis, one of four bifurcations occur
- tangent bifurcation
- transcritical bifurcation
- pitchfork bifurcation
- Hopf bifurcation
Transcritical Bifurcation

\[ \dot{x} = \alpha x - x^2 \]
Pitchfork Bifurcation

\[ \dot{x} = cx - x^3 \]

- **t=0**
  - $c=-1$

- **t=0.5**
  - $c=0$

- **t=1**
  - $c=1$
Hopf Bifurkation

\[ \dot{r} = \alpha r - r^3 \]
\[ \dot{\phi} = \omega \]
Hopf Bifurcation

\[ \dot{r} = \alpha r - r^3 \]
\[ \dot{\phi} = \omega \]
Hopf Bifurcation

\[ \dot{r} = \alpha r - r^3 \]
\[ \phi = \omega \]
Hopf Bifurcation

\[ \dot{r} = \alpha r - r^3 \]
\[ \phi = \omega \]
Forward dynamics

given known equation, determined fixed points / limit cycles and their stability
more generally: determine invariant solutions (stable, unstable and center manifolds)

Basically, what we covered here
Inverse Dynamics

Given a desired behavior, construct a system that fits

Given for instance:
- Stable states
- Attractors
- Time courses
- ...