Dynamical Systems Tutorial

PART 1 OF 2 – THE BASICS

What is a Dynamical System?

A way to take a state and predict its future

More formally:

- A state space S of possible configurations (e.g. location, some vector, …)
- A range of times t where it works
- A rule that takes a state x from S at time t and tells us how s changes
- We call this rule the dynamics (or vector field)

For us, the rule is a differential equation

A time course of a dynamical variable is called a trajectory

What is a Dynamical System?

A differential equation gives us the rate of change of a variable in time a function of that variable

For instance: We have a position, we get a velocity

Simplest example: $$\frac{dx}{dt} = -\frac{x}{\tau}$$

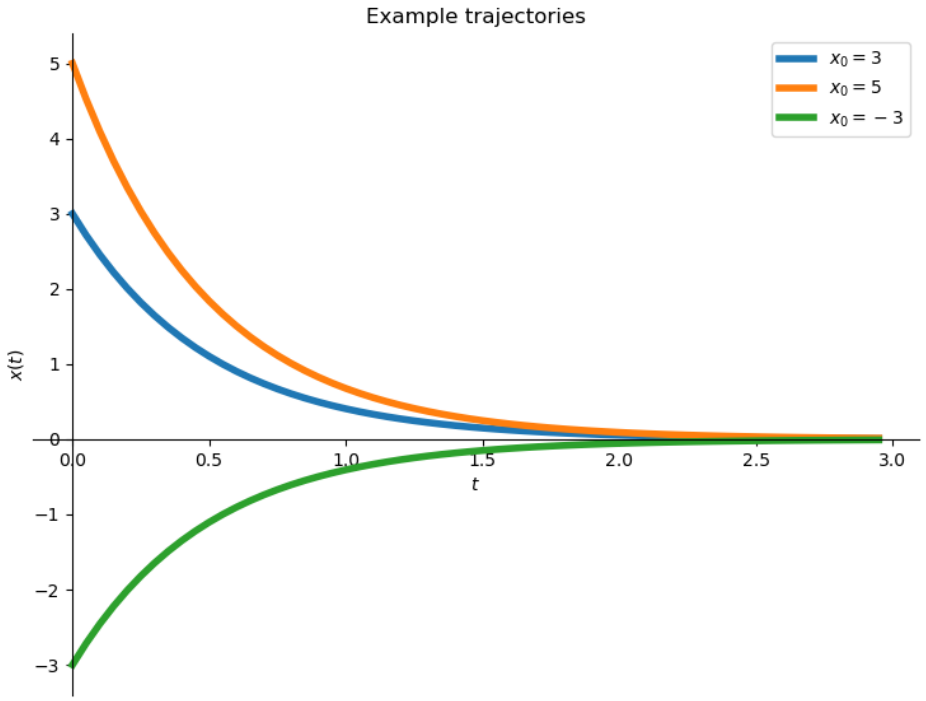
At the heart of the neuron model we use

We often use the abbreviation of the time derivative $$\dot{x}(t) = \frac{dx}{dt}$$

Separation of Variables

Procedure to solve simple differential equations

Not technically quite clean, but works anyway



$$\frac{dx}{dt} = -\frac{x}{\tau}$$

$$\frac{dx}{x} = -\frac{1}{\tau}dt$$

$$\int \frac{dx}{x} = \int -\frac{1}{\tau}dt$$

$$[\log(x)]_{x_0}^{x} = \left[-\frac{t}{\tau}\right]_0^t$$

$$\log(x) - \log(x_0) = -\frac{t}{\tau}$$

$$\frac{x}{x_0} = \exp\left(-\frac{t}{\tau}\right)$$

$$x(t) = x_0 \exp\left(-\frac{t}{\tau}\right)$$

Numerical Methods

Used to simulate systems

Always have an error

Simplest method: Euler step

Sample time discretely: $t_i \text{ with } i \in \{1, 2, ..., N\}$

Then $t_i = i\Delta t$

Approximate change Δx_i during step Δt via derivative

$$\frac{\Delta x(t_i)}{\Delta t} \approx \left.\frac{dx}{dt}\right|_{t=t_i} = f(x(t_i), t_i)$$

$$x(t_{i+1}) = x(t_i) + \Delta x(t_i) \approx x(t_i) + \Delta t \; f(x(t_i), t_i)$$

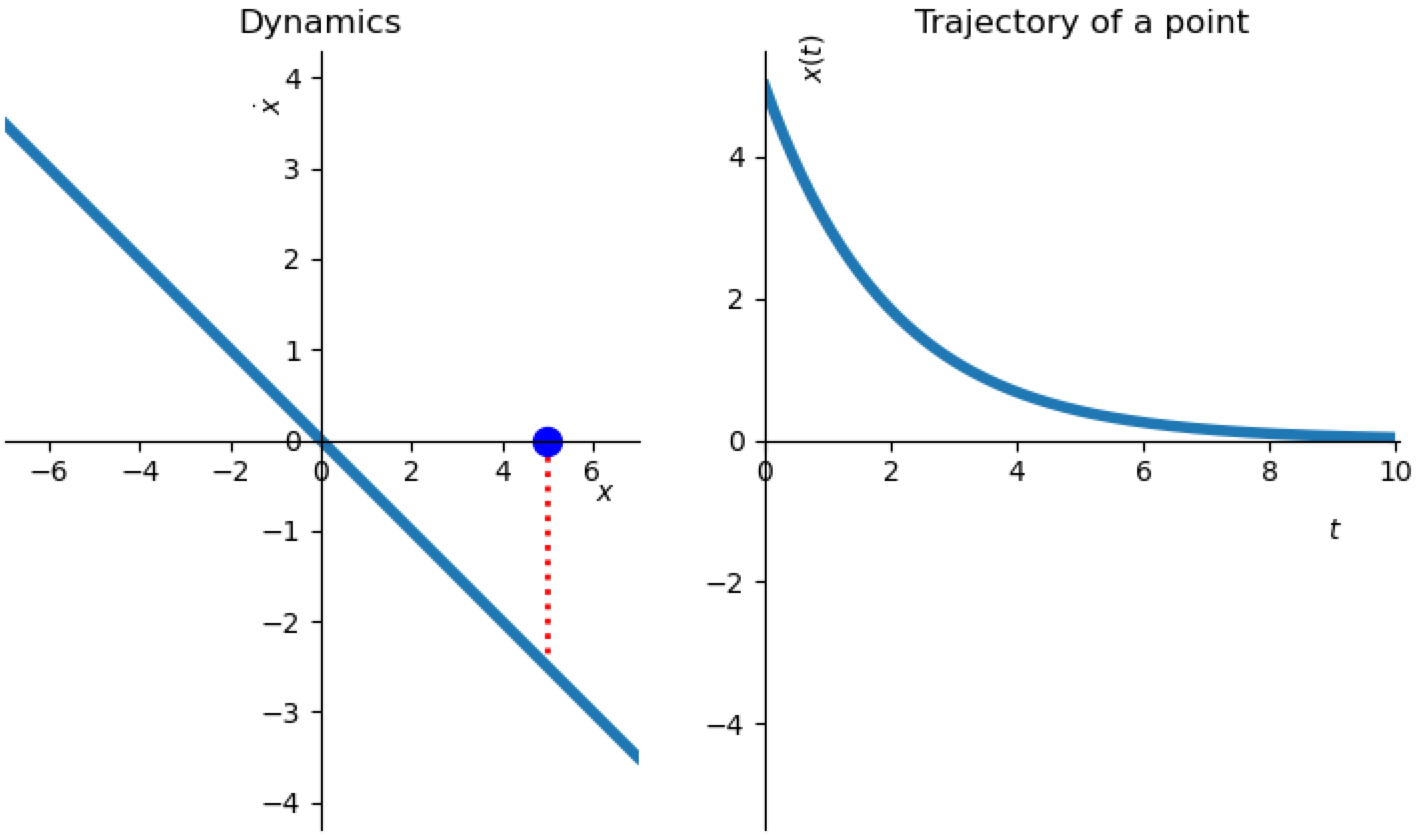
Different form of Dynamical Systems

We saw an example of a one-dimensional differential equation

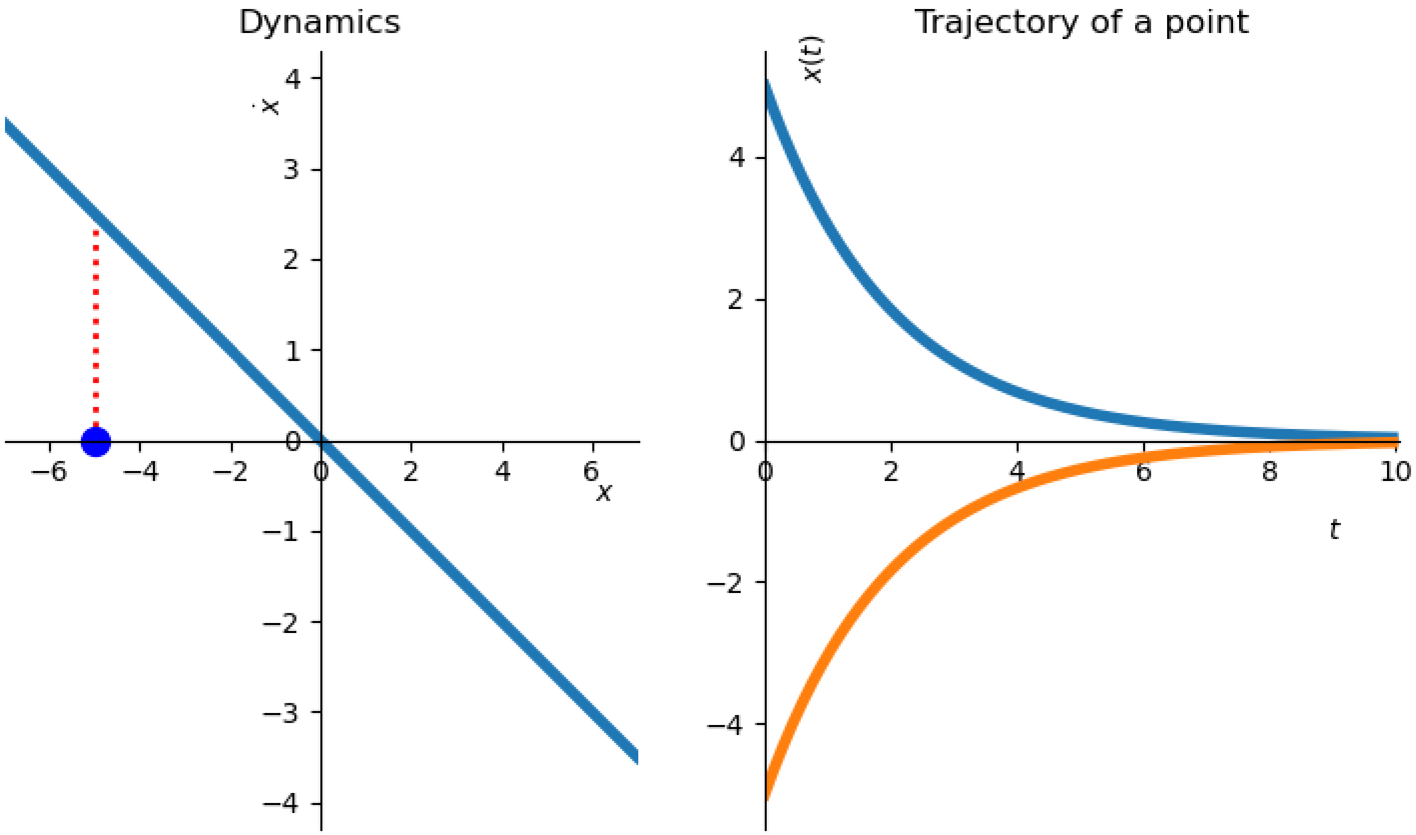
There are also

- Vector valued (N-dimensional) equations
- integro-differential equations
- partial differential equations
- functional differential equations
- delay differential equations
- …

$$\frac{dx}{dt} = -\frac{x}{\tau}$$



$$\frac{dx}{dt} = -\frac{x}{\tau}$$



Fixed Points

When the rate of change is zero we have, of course, no change

Our system is in balance!

But is that balance stable?

$$\dot{x} = 0$$

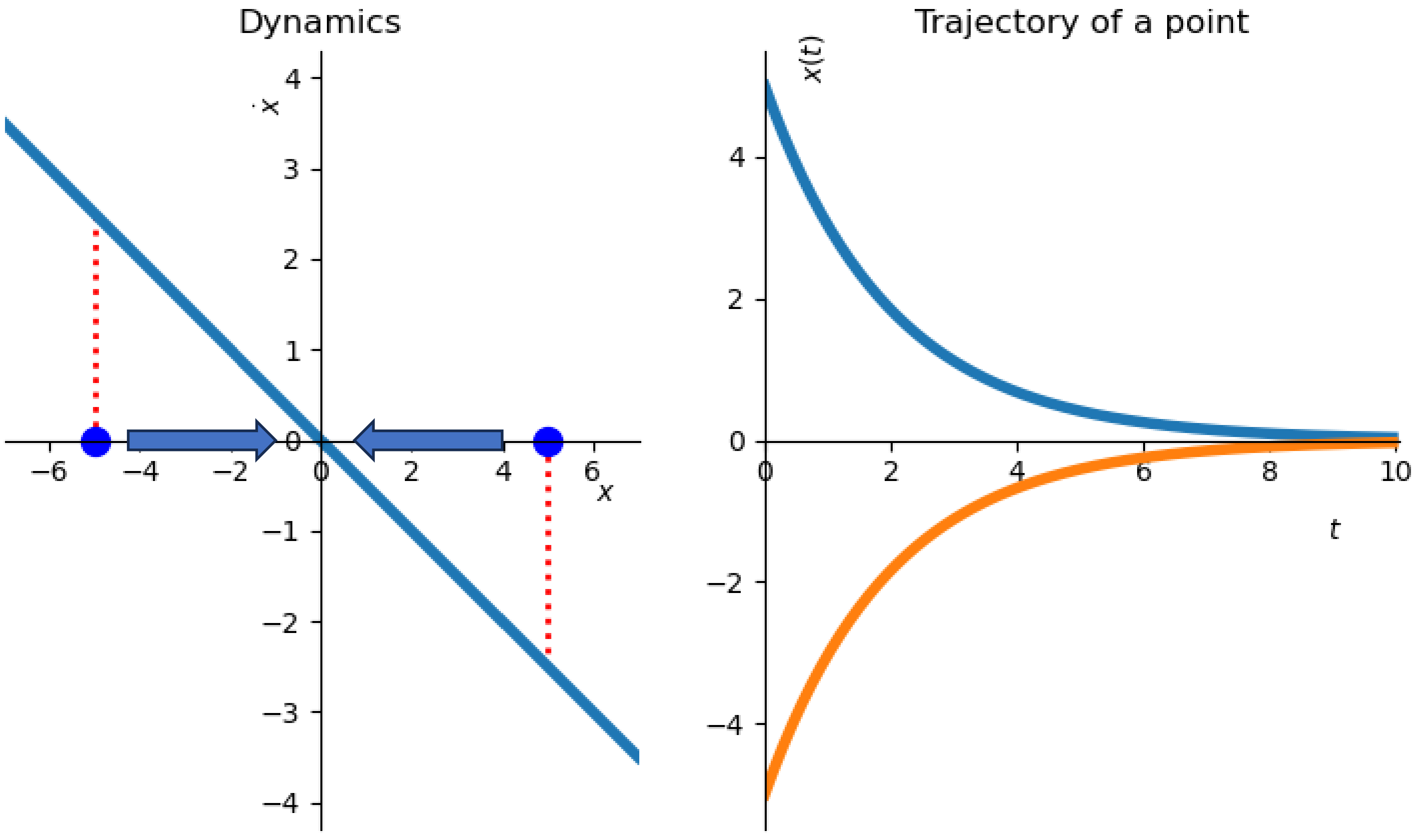
Stability

How does the system react if you disturb it slightly?

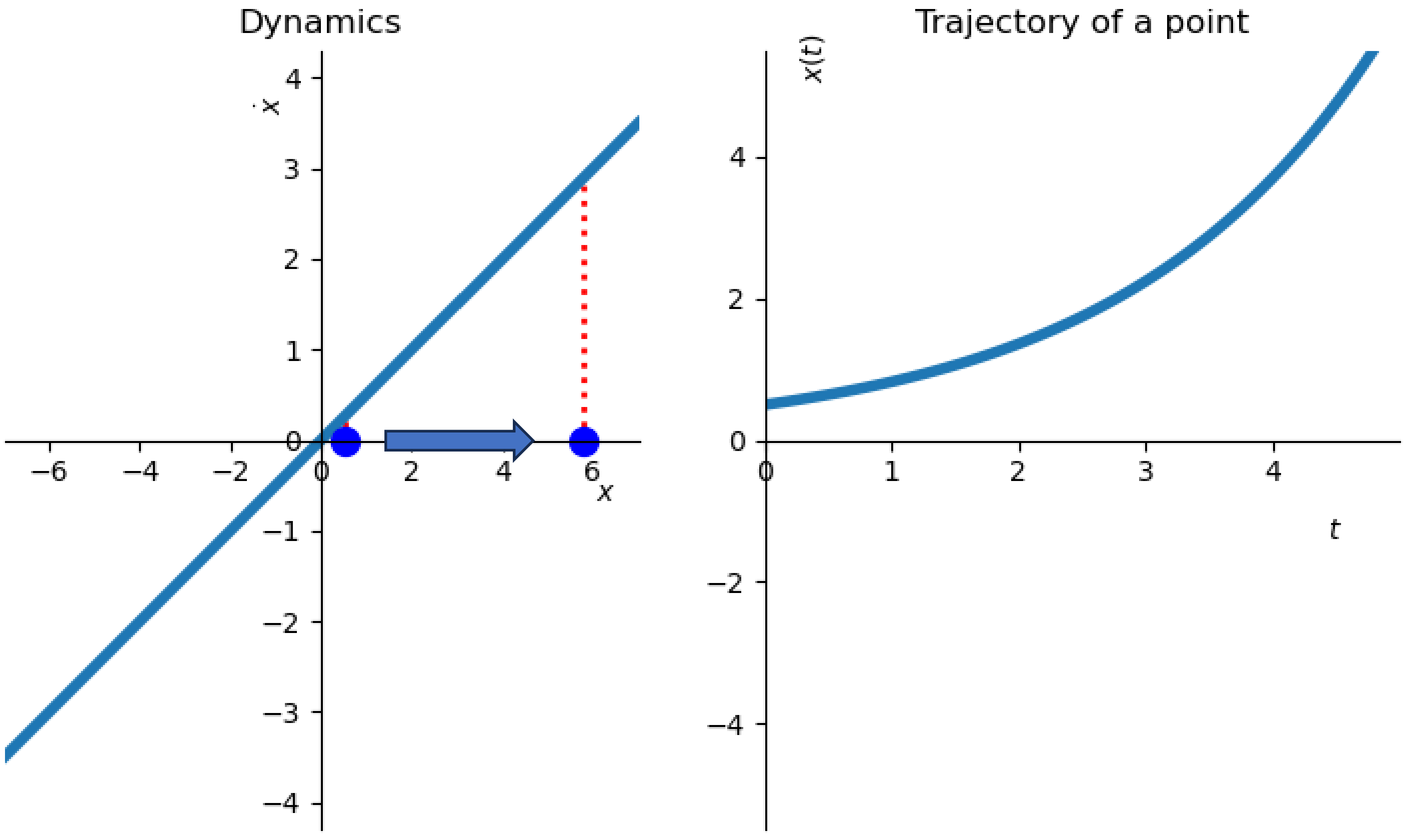
- Moves away -> unstable
- Stays in the vicinity -> stable
- Goes back to fixed point -> asymptotically stable

Unstable | Stable | Asymptotically stable

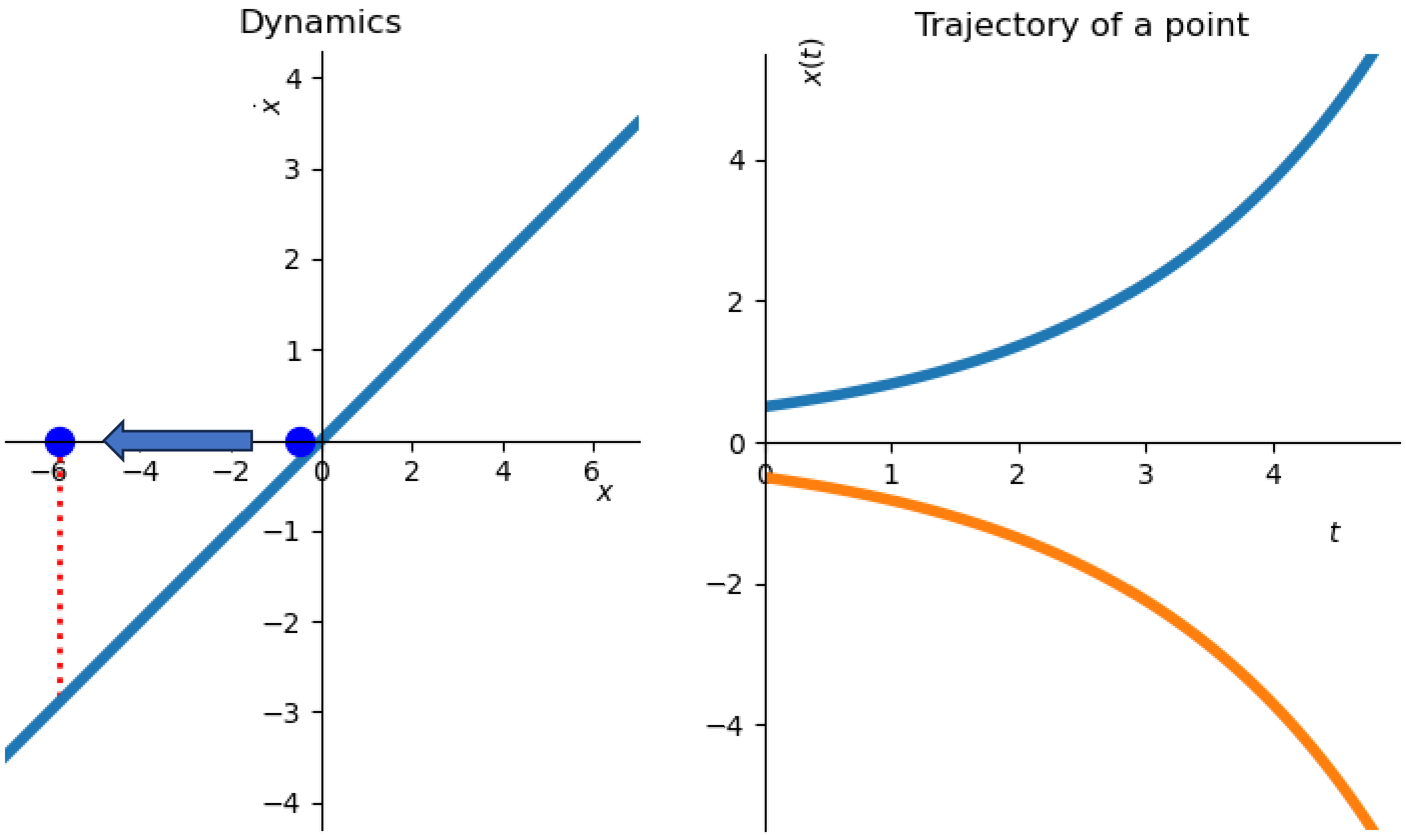
$$\frac{dx}{dt} = -\frac{x}{\tau}$$



$$\frac{dx}{dt} = \frac{x}{\tau}$$



$$\frac{dx}{dt} = \frac{x}{\tau}$$



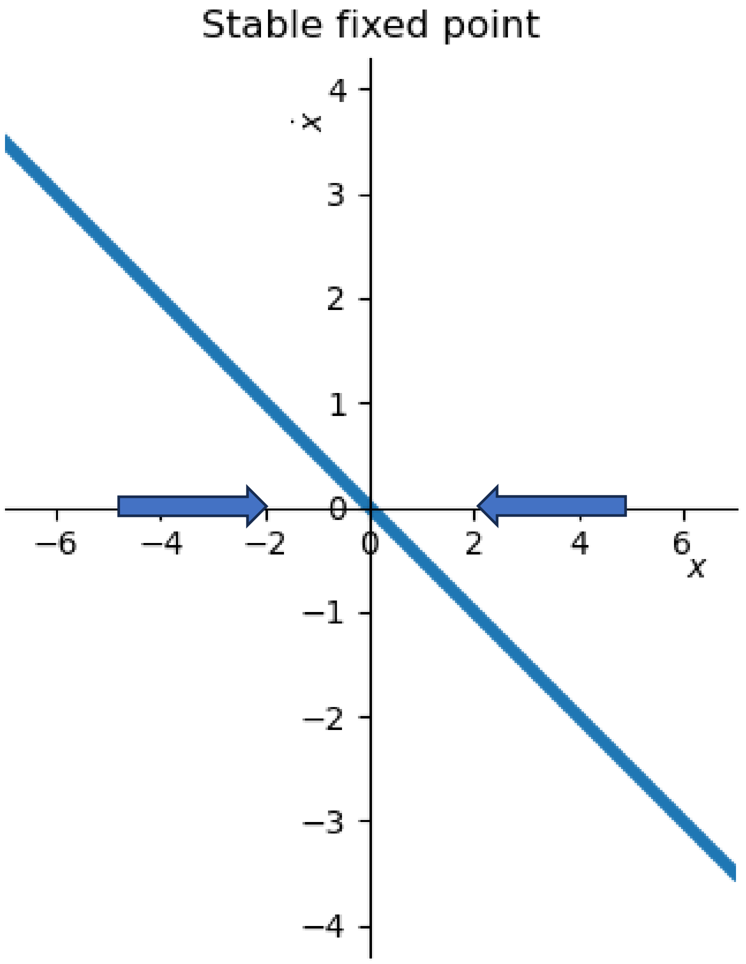
Stability

For linear system: Look at slope at fixed point

- Negative slope -> stable
- Positive slope -> unstable

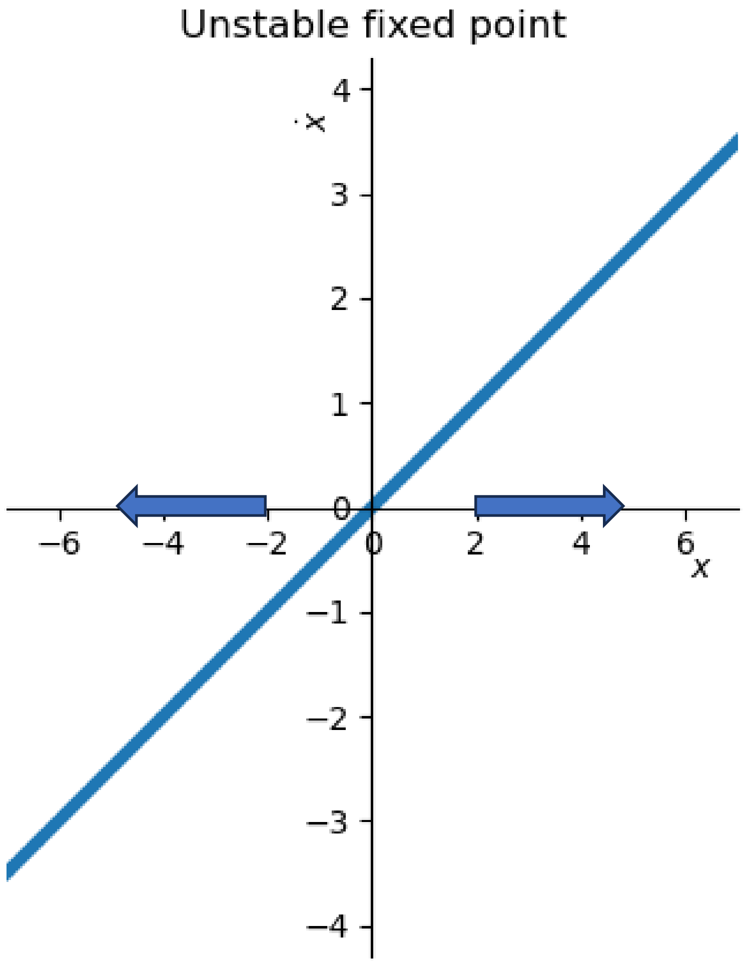
In practice, there is always noise pushing us away from reppelors

Attractor



$$\frac{dx}{dt} = -\frac{x}{\tau}$$

Repellor



$$\frac{dx}{dt} = \frac{x}{\tau}$$

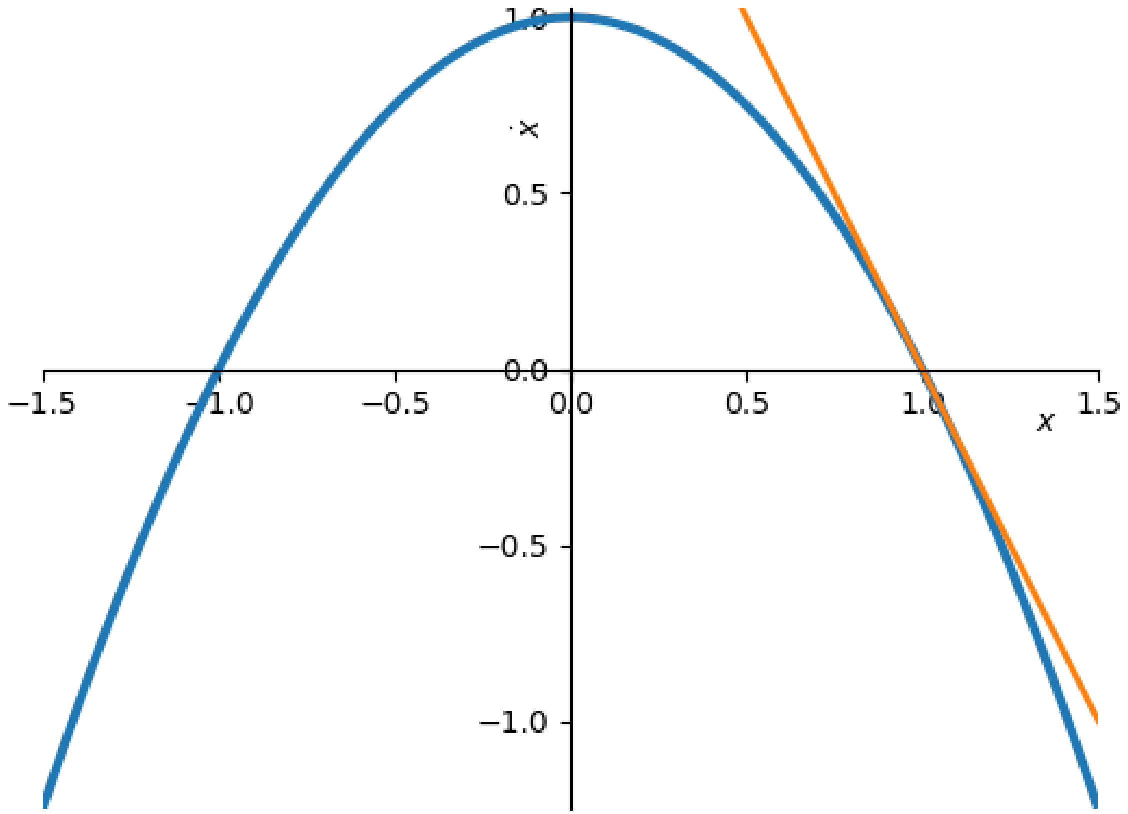
Linear Approximation

We mostly only understand linear systems well

What to do with non-linear problems?

- Make it linear!

We can still use the sign of the derivative

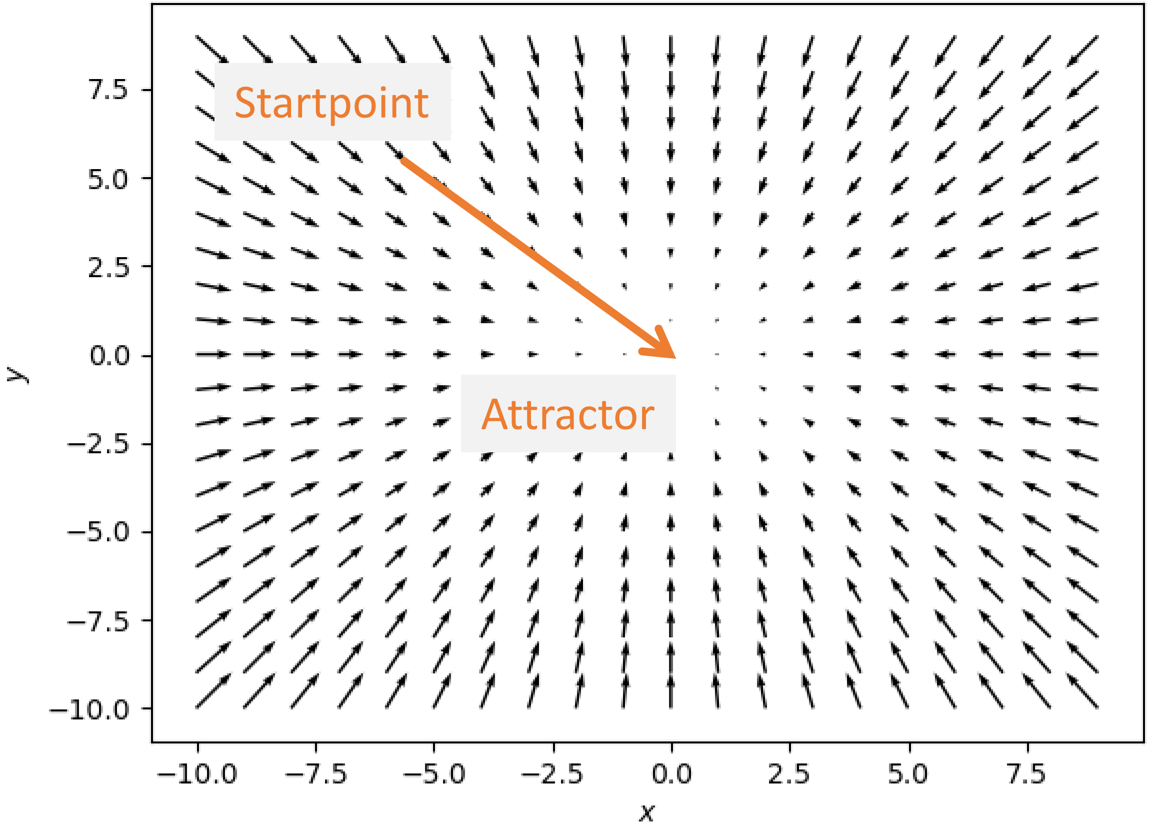


Linear Multidimensional Systems

$$\dot{\vec{x}} = M\vec{x}$$

Stability depends on eigenvalues of M

- if the real-parts of all Eigenvalues are negative: stable
- if the real-part of any Eigenvalue is positive: unstable
- if the real-part of any Eigenvalue is zero: marginally stable in that direction (stability depends on other eigenvalues)



Dynamical Systems Tutorial

PART 2 OF 2 - BIFURCATIONS

What is a bifurcation

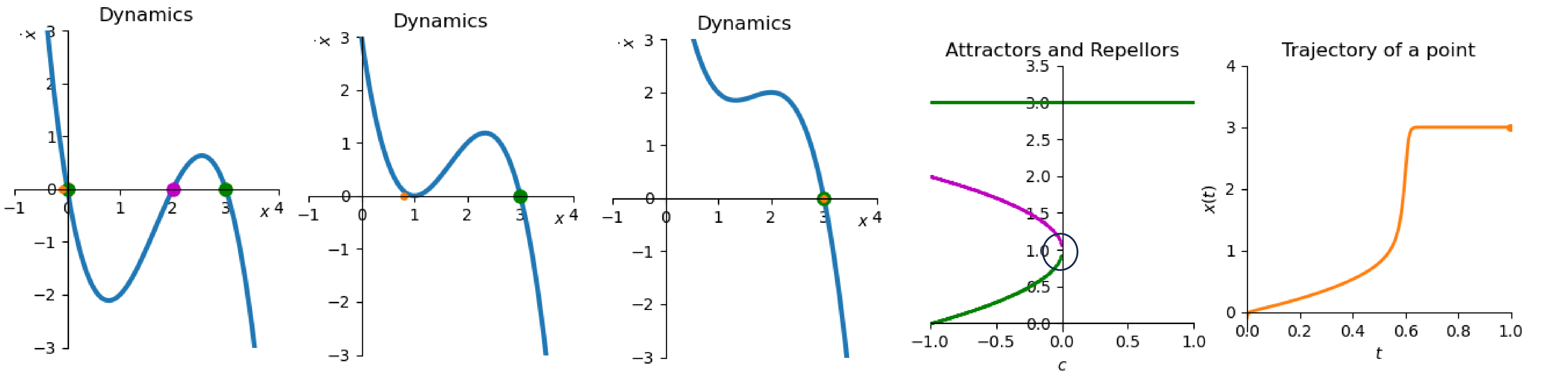
We have a dynamical system with a parameter c

As c changes smoothly, the behavior of the system as an abrupt change

Technically: Infinitesimal parameter change make for topologically inequivalent systems

Tangent Bifurkation

$$\dot{x} = (-(x-1)^2 - c)(x-3)$$



t=0
c=-1

t=0.5
c=0

t=1
c=1

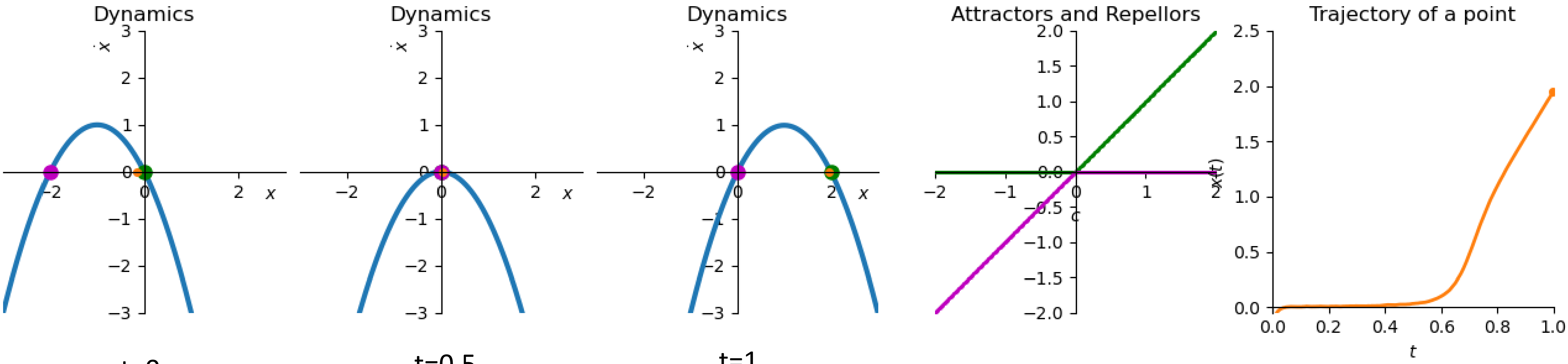
Hopf Theorem

When a single (or pair of complex conjugate) eigenvalue crosses the imaginary axis, one of four bifurcations occur

- tangent bifurcation
- transcritical bifurcation
- pitchfork bifurcation
- Hopf bifurcation

Transcritical Bifurkation

$$\dot{x} = \alpha x - x^2$$



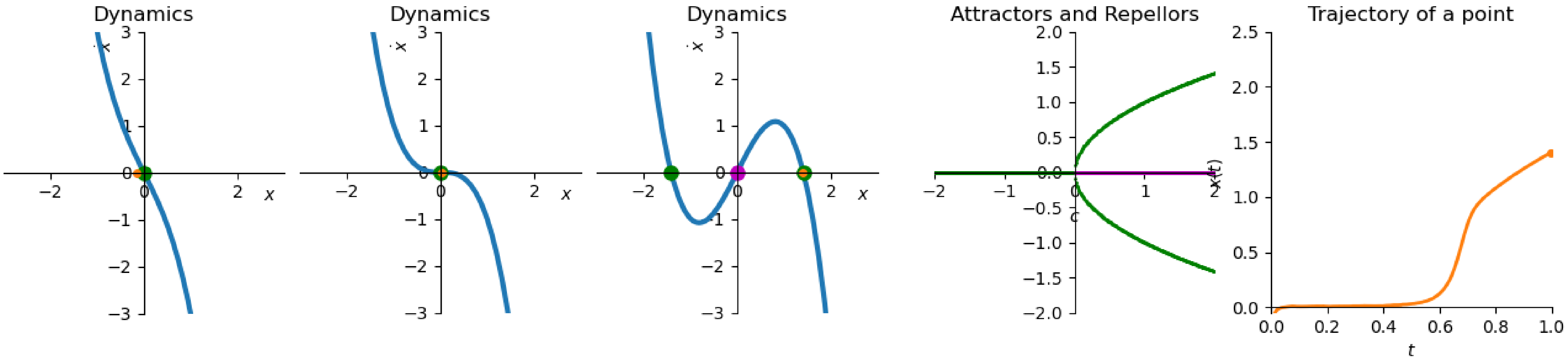
t=0
c=-1

t=0.5
c=0

t=1
c=1

Pitchfork Bifurkation

$$\dot{x} = cx - x^3$$



t=0
c=-1

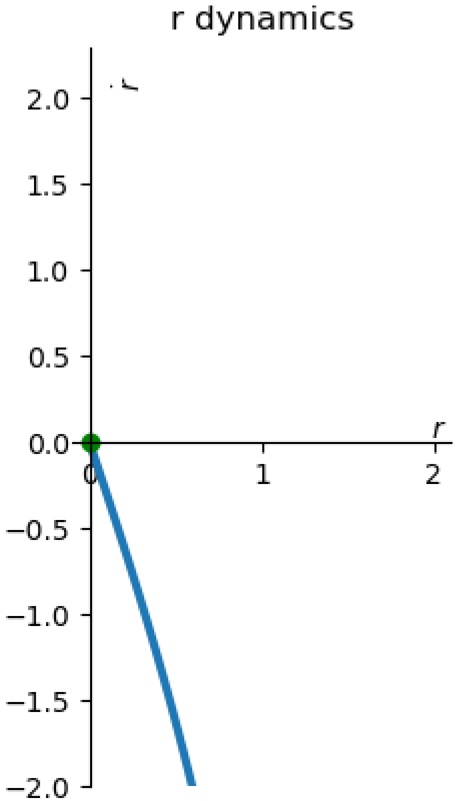
t=0.5
c=0

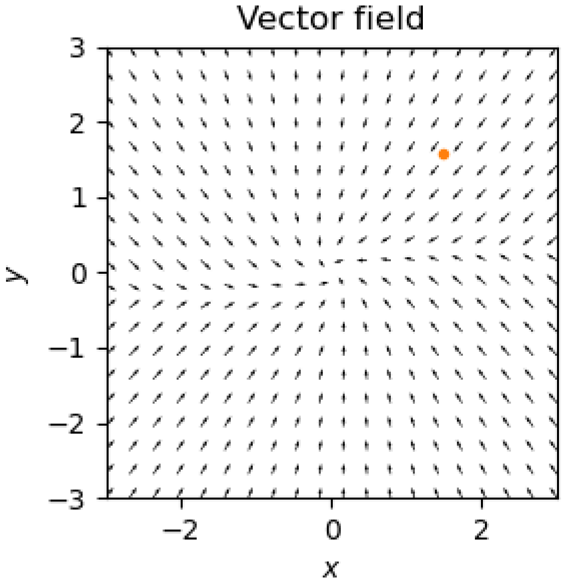
t=1
c=1

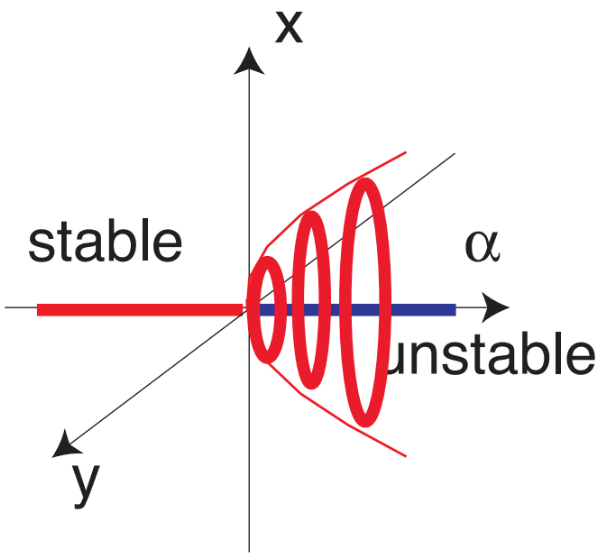
Hopf Bifurkation

$$\dot{r} = \alpha r - r^3$$
$$\dot{\phi} = \omega$$

t=0
c=-1



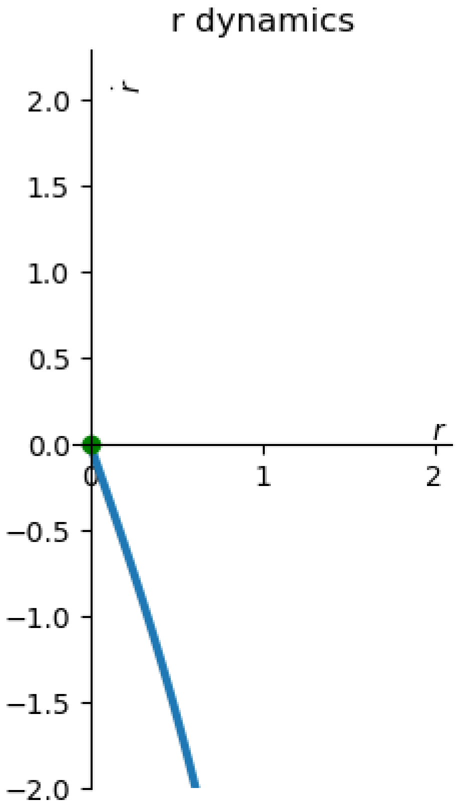


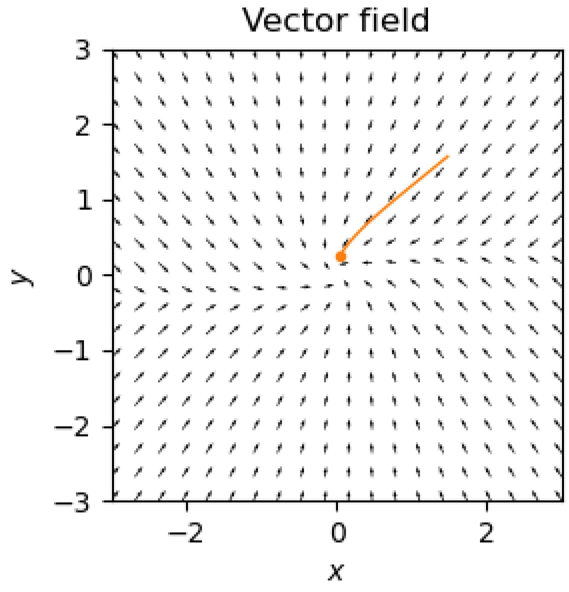


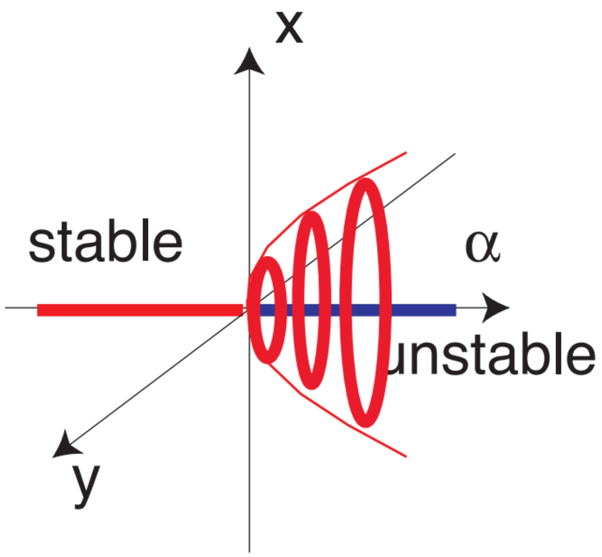
Hopf Bifurkation

$$\dot{r} = \alpha r - r^3$$
$$\dot{\phi} = \omega$$

t=0.02
c=-2.8



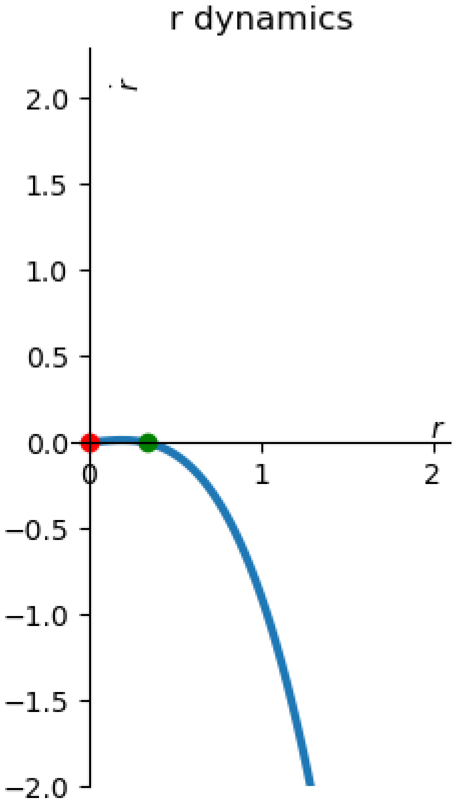


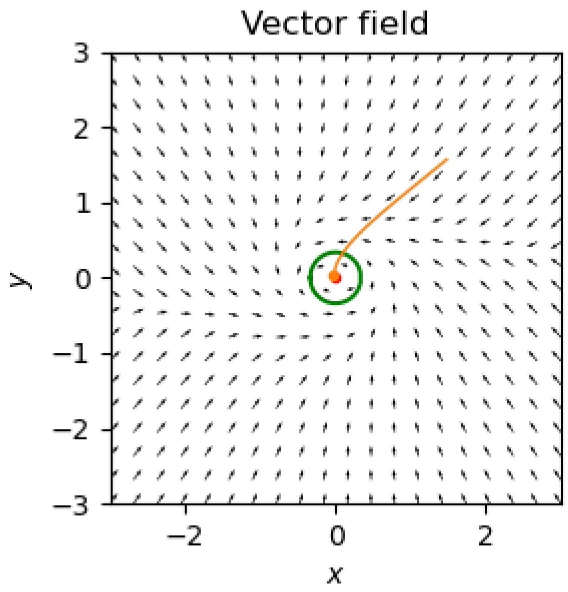


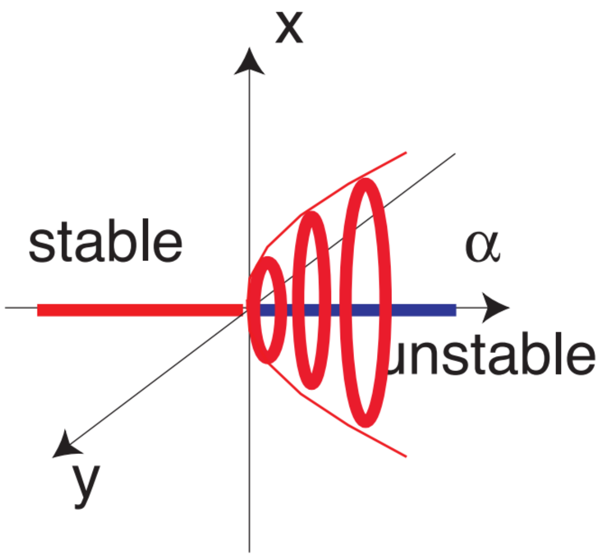
Hopf Bifurkation

$$\dot{r} = \alpha r - r^3$$
$$\dot{\phi} = \omega$$

t=0.52
c=0.123



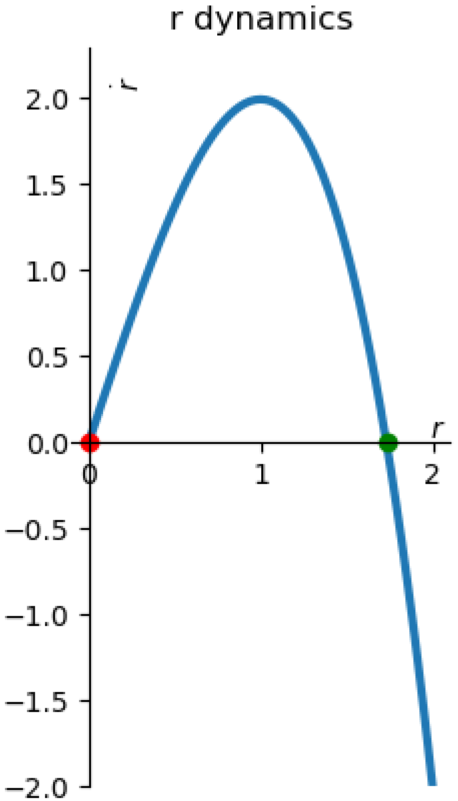


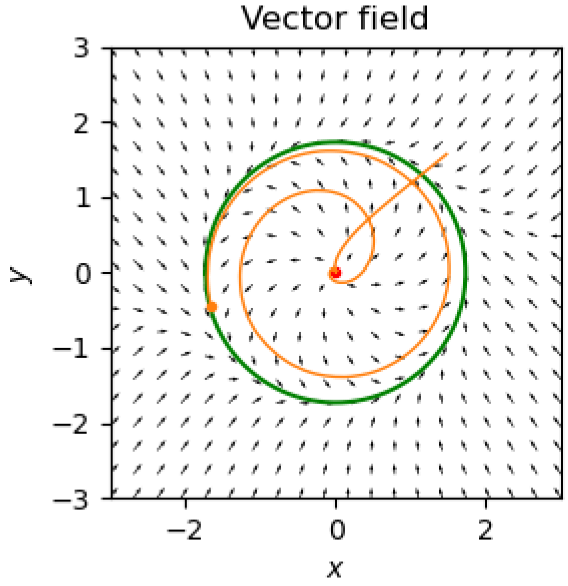


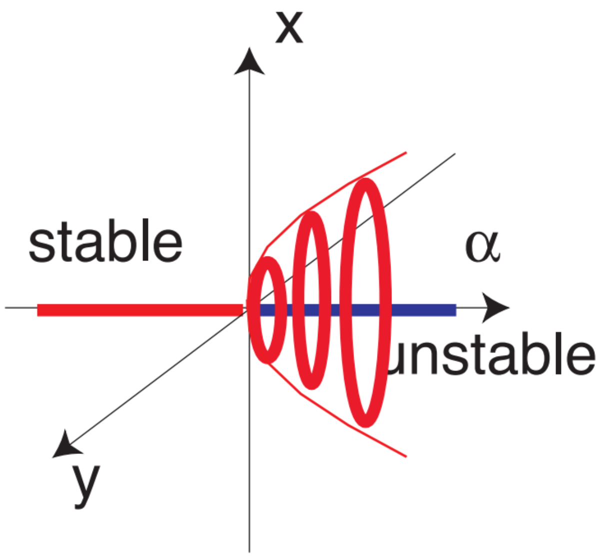
Hopf Bifurkation

$$\dot{r} = \alpha r - r^3$$
$$\dot{\phi} = \omega$$

t=1
c=3







Forward dynamics

given known equation, determined fixed points / limit cycles and their stability

more generally: determine invariant solutions (stable, unstable and center manifolds)

Basically, what we covered here

Inverse Dynamics

Given a desired behavior, construct a system that fits

Given for instance:

- Stable states
- Attractors
- Time courses
- …