Mathematics and Computer Science for Modeling
Unit 2: Functions in Math

Daniel Sabinasz
based on materials by Jan Tekülve and Daniel Sabinasz

Institut für Neuroinformatik, Ruhr-Universität Bochum

September 27, 2023
Dates

1. Mon 25.09. 15-17:30
2. Tue 26.09. 09:00-11:30, 15-17:30
3. Wed 27.09. 15-17:30
4. Thu 28.09. 15-17:30
5. Fri 29.09. 15-17:30
6. Mon 02.10. 09:00-11:30, 15-17:30
7. Wed 04.10. 15-17:30
## Course Structure

<table>
<thead>
<tr>
<th>Unit</th>
<th>Title</th>
<th>Topics</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Intro to Programming in Python</td>
<td>Variables, if Statements, Loops, Functions, Lists</td>
</tr>
<tr>
<td>-</td>
<td>Full-Time Programming Session</td>
<td>Deepen Programming Skills</td>
</tr>
<tr>
<td>2</td>
<td>Functions in Math</td>
<td>Function Types and Properties, Plotting Functions</td>
</tr>
<tr>
<td>3</td>
<td>Linear Algebra</td>
<td>Vectors, Trigonometry, Matrices</td>
</tr>
<tr>
<td>4</td>
<td>Calculus</td>
<td>Derivative Definition, Calculating Derivatives</td>
</tr>
</tbody>
</table>
# Course Structure

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<thead>
<tr>
<th>Unit</th>
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<tbody>
<tr>
<td>5</td>
<td>Integration</td>
<td>Geometrical Definition, Calculating Integrals</td>
</tr>
<tr>
<td>6</td>
<td>Differential Equations</td>
<td>Properties of Differential Equations</td>
</tr>
<tr>
<td>-</td>
<td>04.10.23: Test</td>
<td></td>
</tr>
</tbody>
</table>
Lecture Slides/Material

Use the following URL to access the lecture slides:

https://www.ini.rub.de/teaching/courses/preparatory_course_mathematics_and_computer_science_for_modeling_winter_term_2023
1. Sets and Number Systems

2. Functions in Math
   - Definition
   - Function Types
   - Parametrization
   - Multiple Arguments
   - Properties
1. Sets and Number Systems

2. Functions in Math
   ➤ Definition
   ➤ Function Types
   ➤ Parametrization
   ➤ Multiple Arguments
   ➤ Properties
Sets

- For practical purposes, think of a **set** as a container of objects
- e.g., the set of natural numbers
Sets

- Notation: \( \mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, \ldots \} \)
- Something is either in the set or not in the set
- If something is in the set, we call it an **element** of the set
- e.g., 5 is an element of \( \mathbb{N} \), but -3 is not an element of \( \mathbb{N} \)
- Write 5 \( \in \mathbb{N} \) and -3 \( \notin \mathbb{N} \)
Sets

- Instead of listing all the elements, you can describe in natural language what the elements should be

- e.g., $A = \{ x \mid x \text{ is an even number} \} = \{0, 2, 4, 6, 8, \ldots \}$
## Number Systems

- **Natural Numbers**: \( \mathbb{N} = \{0, 1, 2, 3, 4, \ldots \} \)
- **Integer Numbers**: \( \mathbb{Z} = \ldots \) (not fully visible)
- **Rational Numbers**: \( \mathbb{Q} \)
- **Real Numbers**: \( \mathbb{R} \)
Number Systems

- **Natural Numbers**: \( \mathbb{N} = \{0, 1, 2, 3, 4, \ldots \} \)
- **Integer Numbers**: \( \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots \} \)
- **Rational Numbers**: \( \mathbb{Q} \)
- **Real Numbers**: \( \mathbb{R} \)
Number Systems

- **Natural Numbers:** \( \mathbb{N} = \{0, 1, 2, 3, 4, \ldots \} \)
- **Integer Numbers:** \( \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots \} \)
- **Rational Numbers:** \( \mathbb{Q} = \{ \frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0 \} \)
- **Real Numbers:** \( \mathbb{R} \)
Number Systems

- **Natural Numbers**: \( \mathbb{N} = \{0, 1, 2, 3, 4, \ldots\} \)
- **Integer Numbers**: \( \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \)
- **Rational Numbers**: \( \mathbb{Q} = \frac{a}{b} \), where \( a, b \in \mathbb{Z} \) and \( b \neq 0 \)
- **Real Numbers**: \( \mathbb{R} = \mathbb{Q} \cup \) irrational numbers
Number Systems

\[ \mathbb{R} \supseteq \mathbb{Q} \supseteq \mathbb{Z} \supseteq \mathbb{N} \]
1. Sets and Number Systems

2. Functions in Math
   ➤ Definition
   ➤ Function Types
   ➤ Parametrization
   ➤ Multiple Arguments
   ➤ Properties
Function Intuition

- Function example: \( f(x) = 2x + 3 \)

- A function, written like this, can be thought of as a formula that can be evaluated to give the value of the function

- e.g.,
  - \( f(1) = 2 \cdot 1 + 3 = 5 \)
  - \( f(2) = 2 \cdot 2 + 3 = 6 \)
Plotting Functions

Tabular Interpretation of: \( f(x) = 2x + 3 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
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<tr>
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<td>5</td>
<td></td>
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\[ y = 2x + 3 \]
Plotting Functions

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<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>5</td>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{array}{c|ccccccc}
 x & 0 & 1 & 2 & 3 & 4 & 5 \\
y & 5 & 9 &   &   &   &   \\
\end{array}
\]
Plotting Functions

Tabular Interpretation of: \( f(x) = 2x + 3 \)

<table>
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<th>x</th>
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<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>13</td>
</tr>
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</table>
Plotting Functions

Tabular Interpretation of: \( f(x) = 2x + 3 \)

<table>
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<th>( x )</th>
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Function Definition

**Function**

X and Y are two sets.

A function \( f : X \to Y \) is a mathematical object that assigns each element \( x \in X \) exactly one element \( y \in Y \).

\[
x \mapsto y = f(x)
\]

- \( x \) is called the **function argument**
- \( y \) is called the **function value**
- \( X \) is called the **domain**
- \( Y \) is called the **codomain**
- The **image** \( W \) of \( f(x) \) are all values in \( Y \) that can be assumed by the function.
Matplotlib allows to plot functions:

```python
import matplotlib.pyplot as plt

numbers = [2*x+3 for x in range(6)]

plt.plot(numbers)
plt.show()
```
Function Types

- **Linear Functions**
  \[ y = mx + b \]
Function Types

▶ Linear Functions
\[ y = mx + b \]

▶ Power Functions
\[ y = ax^n \]
Function Types

▶ Linear Functions
\[ y = mx + b \]

▶ Power Functions
\[ y = ax^n \]

▶ Polynomial Functions
\[ y = \sum_{i=0}^{n} a_i x^i \]
\[ y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n \]

describes a polynomial of degree \( n \), where \( a_n \neq 0 \)
The Summation Symbol

- $\sum_{i=0}^{n} T(i)$ denotes a sum of multiple terms
- The bottom row defines an indexing variable, here $i$, and specifies an initial value, here 0
- That variable takes on increasing values (0, 1, 2, 3, ..., $n$)
- The top row specifies the maximum value for $i$, here $n$
- $T(i)$ specifies a term for each $i$
- $\sum_{i=0}^{n} T(i)$ sums up $T(i)$ for each $i$
- Thus, $\sum_{i=0}^{n} T(i) = T(0) + T(1) + T(2) + \ldots + T(n)$
- e.g., $\sum_{i=0}^{5} i = 0 + 1 + 2 + 3 + 4 + 5$
Exponentials Functions

Exponential Functions

\[ f(x) = e^x \]
\[ g(x) = 10^x \]

Logarithmic Functions

\[ h(x) = \ln(x) \]
\[ j(x) = \log_{10}(x) \]
The Gaussian Function

\[ f(x) = e^{-x^2} \]
The Gaussian Function

\[ f(x) = e^{-x^2} \]
\[ g(x) = e^{-(x-2)^2} \]
Trigonometric Functions

\[ f(x) = \sin(x) \quad g(x) = \cos(x) \]
Chaining Functions

\[ f(x) = e^{-x^2} \quad g(x) = \cos(x) \]
Chaining Functions

\[ f(x) = e^{-x^2} \quad g(x) = \cos(x) \quad h(x) = g(f(x)) \]
Chaining Functions

\[ f(x) = e^{-x^2} \quad g(x) = \cos(x) \quad h(x) = g(f(x)) \quad j(x) = f(g(x)) \]
Function Translation

- Translation in $y$-direction: $\hat{f}(x) = f(x) + b$
- Translation in $x$-direction: $\hat{f}(x) = f(x - a)$
Function Translation

- Translation in $y$-direction: $\hat{f}(x) = f(x) + b$
- Translation in $x$-direction: $\hat{f}(x) = f(x - a)$

$f(x) = e^x$

$g(x) = e^x + 3$
Function Translation

- Translation in $y$-direction: $\hat{f}(x) = f(x) + b$
- Translation in $x$-direction: $\hat{f}(x) = f(x - a)$

$f(x) = e^x$
$g(x) = e^x + 3$
$h(x) = e^{x-2}$
Function Stretching and Compression

- Stretching/Compression in $\mathbf{y}$-direction: $\hat{f}(x) = df(x), \, d > 0$

- Stretching/Compression in $\mathbf{x}$-direction: $\hat{f}(x) = f(cx), \, c > 0$

$\hat{f}(x) = e^x$
Function Stretching and Compression

- Stretching/Compression in \textbf{y-direction}: \( \hat{f}(x) = df(x), \ d > 0 \)
- Stretching/Compression in \textbf{x-direction}: \( \hat{f}(x) = f(cx), \ c > 0 \)

\[
\begin{align*}
&f(x) = e^x \\
g(x) = \frac{1}{2}e^x
\end{align*}
\]
Function Stretching and Compression

- Stretching/Compression in $y$-direction: $\hat{f}(x) = df(x), d > 0$
- Stretching/Compression in $x$-direction: $\hat{f}(x) = f(cx), c > 0$

\[ f(x) = e^x \quad g(x) = \frac{1}{2}e^x \quad h(x) = 4e^x \]
Function Stretching and Compression

- **Stretching/Compression in \( y \)-direction:** \( \hat{f}(x) = df(x), \, d > 0 \)
- **Stretching/Compression in \( x \)-direction:** \( \hat{f}(x) = f(cx), \, c > 0 \)

\[
\begin{align*}
  f(x) &= e^x \\
  g(x) &= \frac{1}{2}e^x \\
  h(x) &= 4e^x \\
  j(x) &= e^{4x}
\end{align*}
\]
Example

\[ e^{-x^2} \]

\[ e^{-(x-3)^2} \]

\[ e^{-(2(x-3))^2} \]

\[ 1.5e^{-(2(x-3))^2} \]
Function Reflection

- Reflection across the **y-axis**: \( \hat{f}(x) = f(-x) \)
- Reflection across the **x-axis**: \( \hat{f}(x) = -f(x) \)

\[
\begin{align*}
  f(x) &= e^x \\
  g(x) &= -e^x \\
  h(x) &= e^{-x}
\end{align*}
\]
Exercise 1

1. Give an example for a natural number, a negative integer, a rational number and an irrational number

2. Which of the following is true? (a) Every real number is rational. (b) Every integer is rational. (c) Every natural number is a real number.

3. Let \( f : \mathbb{N} \to \mathbb{R}, x \to 2x + 3 \). Identify the function argument, the function value, the domain, the codomain and the image.

4. Create a function \( \hat{f}(x) \) by translating \( f(x) = e^x \) by \(-2\) in y-direction and by \(3\) in x-direction.

5. Create a function \( \hat{f}(x) \) by stretching \( f(x) = e^x \) along the y-axis and compressing it along the x-axis.

6. Create a function \( \hat{f}(x) \) by compressing \( f(x) = e^x \) along the y-axis and stretching it along the x-axis.
Exercise 2

1. Write a python function that calculates $f(x) = 4x + 3$ and plot it.

2. Define a second function $g(x, a_0, a_1, a_2, a_3)$ that calculates a polynomial of degree 3 with variable coefficients $a_0$ to $a_3$ and plot $g(x, 3, 0, 2, 1)$

3. Calculate $f(x)$ or $g(x, 3, 0, 2, 1)$ for $x$ values from 0 to 20. Store the result in a list.

4. (optional) Define a function ‘polynomial(a, x)‘ that receives a list of coefficients ‘a‘ ($a_0$, $a_1$, $a_2$, ..., $a_n$) with a flexible number of items and computes $\sum_{i=0}^{n} a_i x^i$. 
Multiple Arguments

\[ f(x, y) = x + y \]
Multiple Arguments

\[ f(x, y) = \sin(x) + y \]
Multiple Arguments

$$f(x, y) = e^{-(x^2 + y^2)}$$
Multiple Arguments

\[ f(x, y) = e^{-(x-2)^2 + (y+1)^2} \]
Injective, Surjective and Bijective Functions

- An image $f$ is **injective**, if two different elements $x_1 \neq x_2$ are always projected to two different elements $y_1 \neq y_2$.
- An image $f$ is **surjective**, if for each element $y \in Y$ one $x \in X$ exists, such that $y = f(x)$.
- An image $f$ is **bijective**, if it is injective and surjective.

Image source:
https://commons.wikimedia.org/wiki/File:Injective,_Surjective,_Bijective.svg
Injective, Surjective and Bijective Functions

- An image $f$ is **injective**, if two different elements $x_1 \neq x_2$ are always projected to two different elements $y_1 \neq y_2$

- An image $f$ is **surjective**, if for each element $y \in Y$ one $x \in X$ exists, such that $y = f(x)$

Injective, but not surjective

$$f(x) = e^x$$

Surjective, but not injective

$$f(x) = x \sin(x)$$
**Bijective Function Example**

\[ f(x) = 4x \]

\[ f(x) = x^3 \]
Inverse Function

Definition

Given a bijective function \( f : X \rightarrow Y, f^{-1} : Y \rightarrow X \) denotes the inverse function of \( f \).

It holds that \( f^{-1}(f(x)) = x \) for all \( x \in X \).
Inverse Function

Definition

Given a bijective function $f : X \rightarrow Y, f^{-1} : Y \rightarrow X$ denotes the inverse function of $f$.

It holds that $f^{-1}(f(x)) = x$ for all $x \in X$.

Image source: https://www.geogebra.org/m/Efs8QRRF
Monotonicity

Definition

- A function $f : \mathbb{R} \to \mathbb{R}$ is called **monotonically increasing**, if for all $x_1, x_2$ order is preserved by applying $f$:

  $$x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$$

- A function $f : \mathbb{R} \to \mathbb{R}$ is called **monotonically decreasing**, if for all $x_1, x_2$ order is reversed by applying $f$:

  $$x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$$
Monotonicity Examples

monotonically increasing

monotonically decreasing
Functions Exercise 3

1. Write a python function that calculates \( f(x, y) = 4x^2 + 2(y - 2)^2 \) and plot it.

2. Determine the inverse \( f^{-1}(x) \) of \( f(x) = 2x + 3 \)

3. For each of the following functions, determine if they are monotonically increasing, monotonically decreasing or neither: \( f(x) = x^2, f(x) = -x^5, f(x) = x^7 \)