Dynamical systems tutorial

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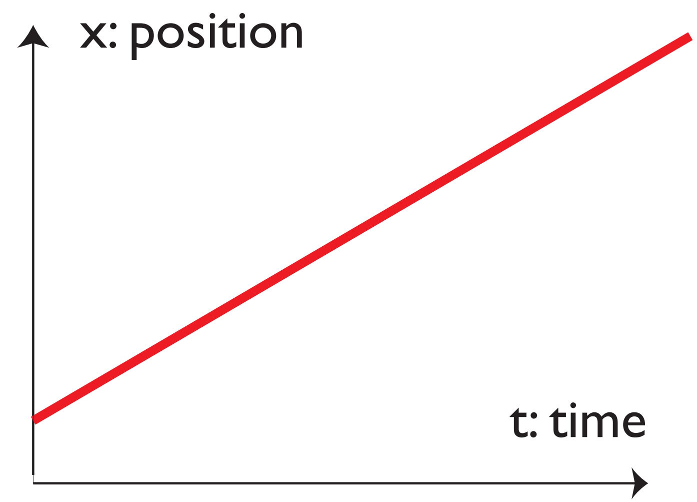
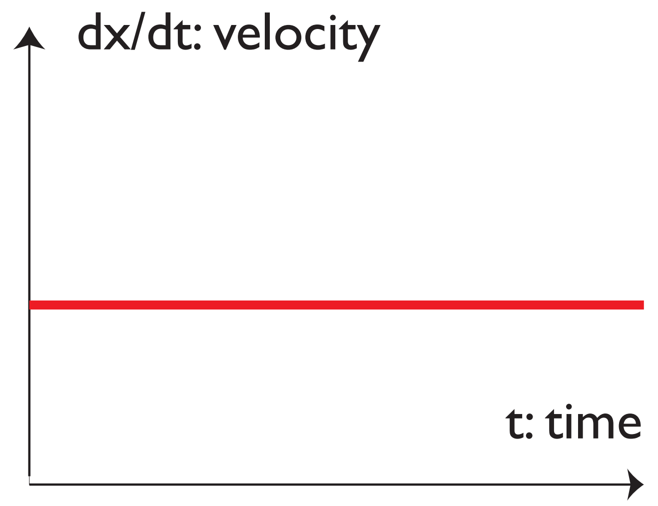
“dynamics”

- the word “dynamics”
  - time-varying measures
  - range of a quantity
  - forces causing/accounting for movement => dynamical systems
- dynamical systems are the universal language of science
  - physics, engineering, chemistry, theoretical biology, economics, quantitative sociology, ...

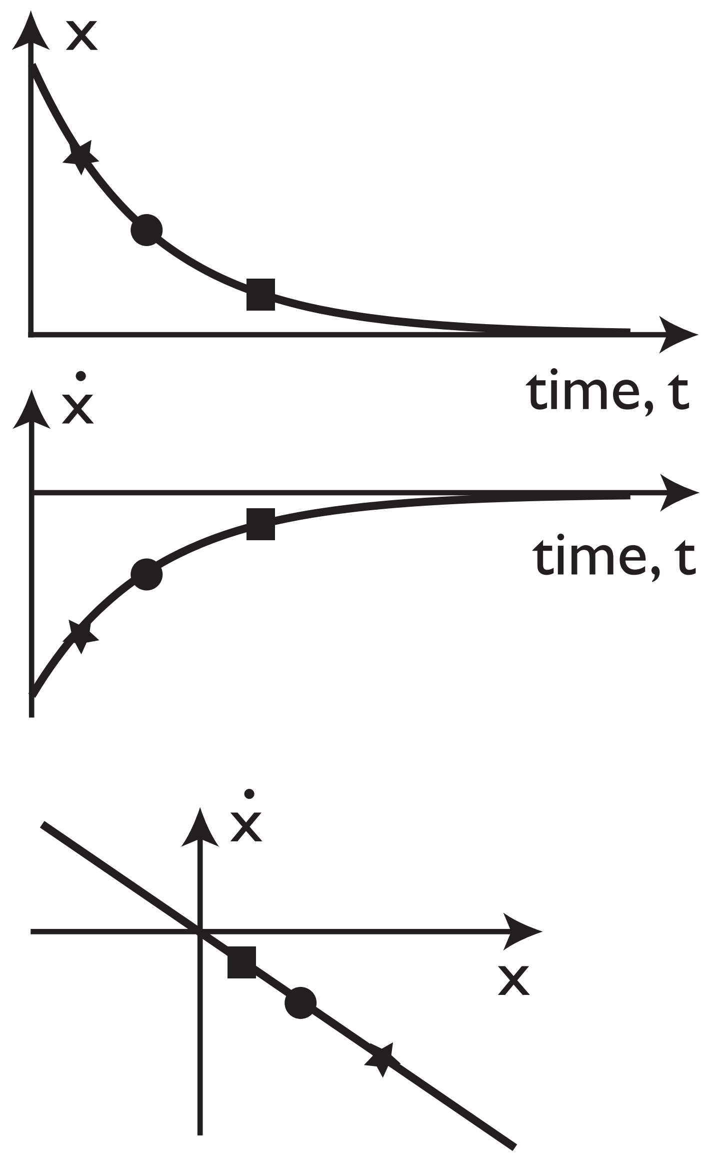
time-variation and rate of change

- variable x(t);
- rate of change dx/dt



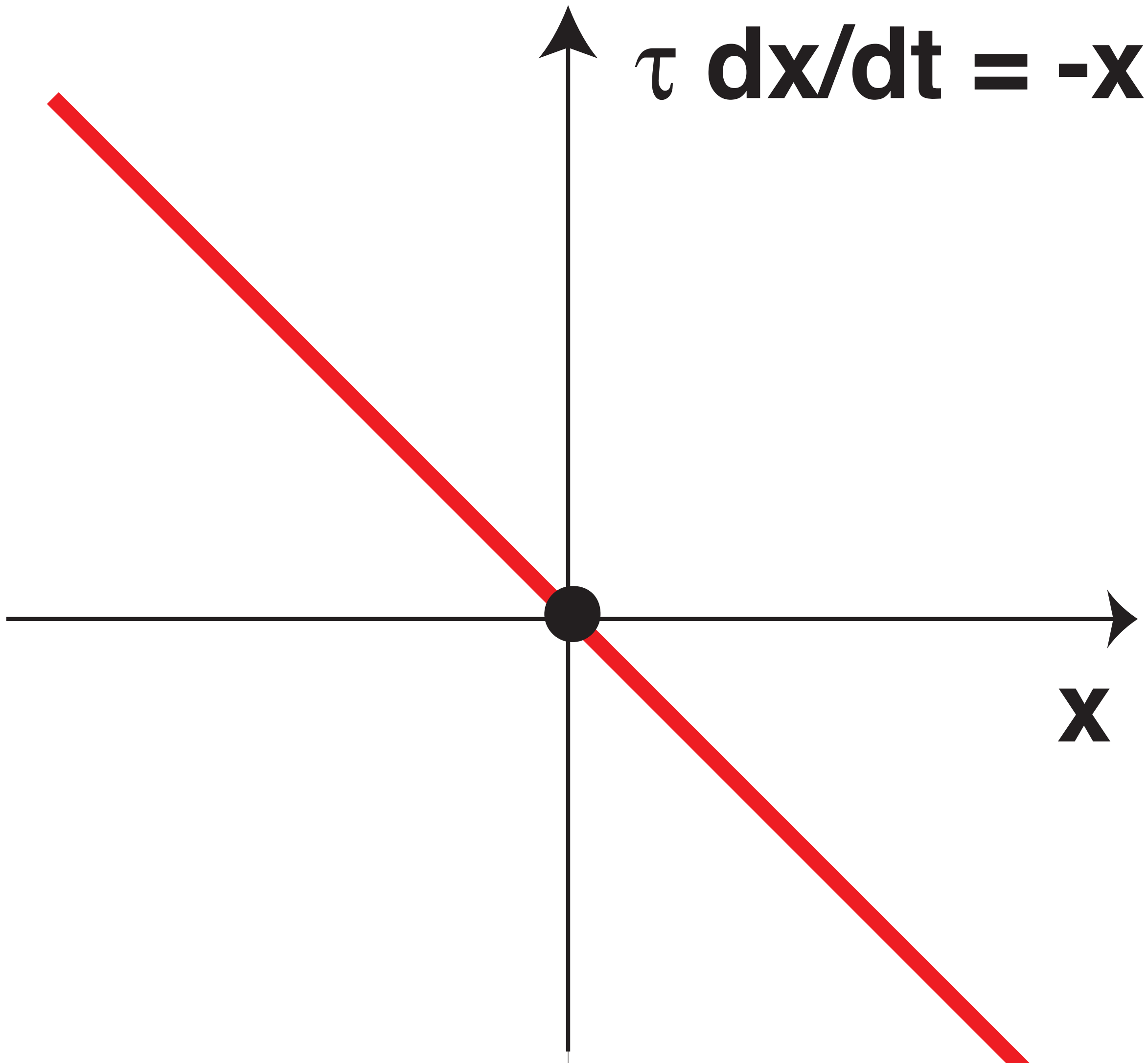


functional relationship between a variable and its rate of change



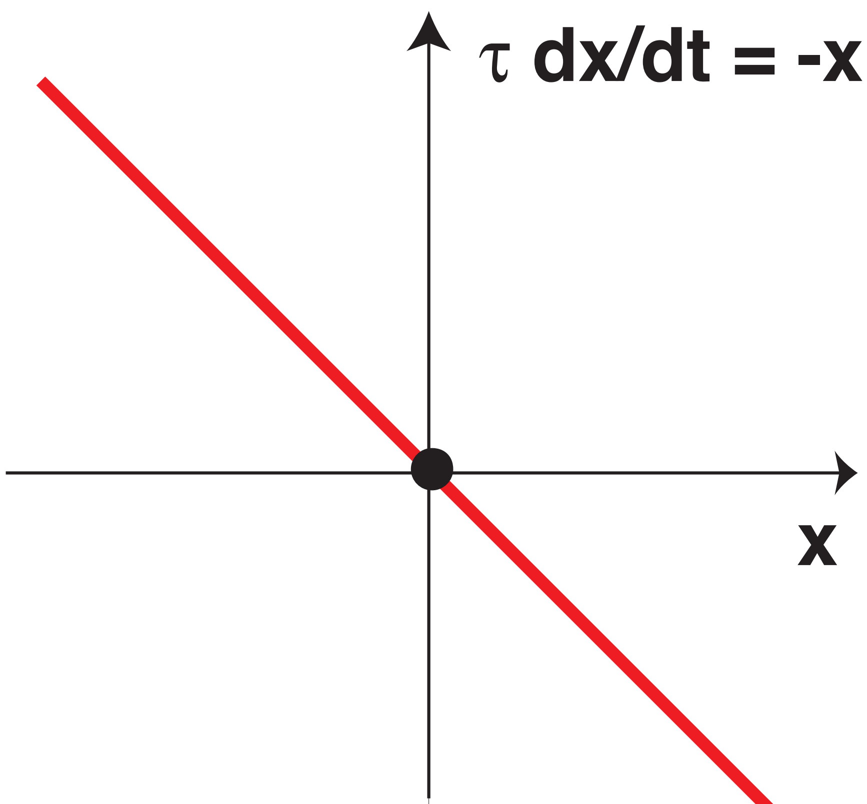
=> dynamical system

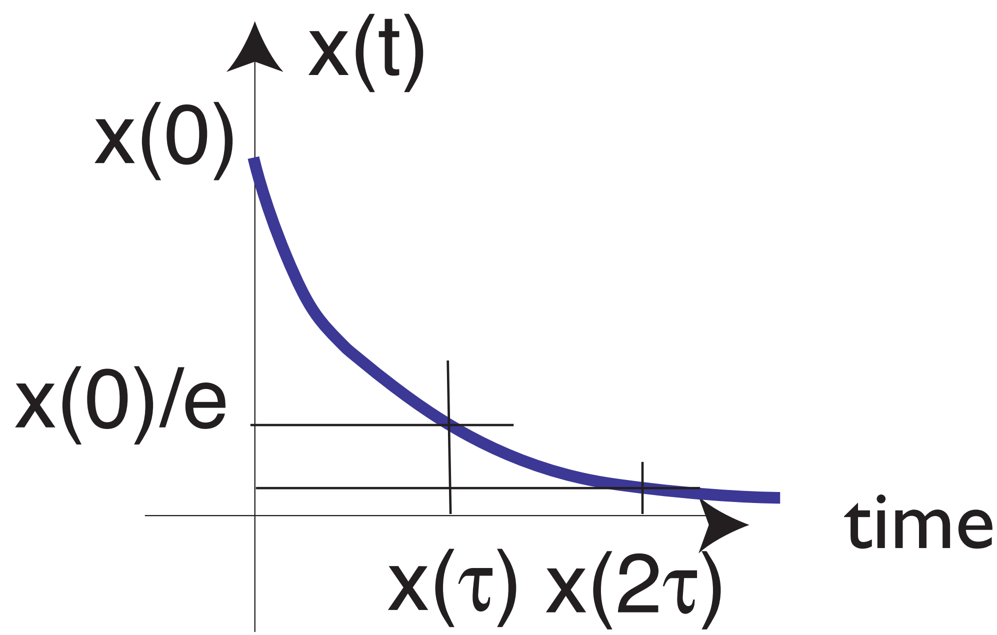
(linear) dynamical system



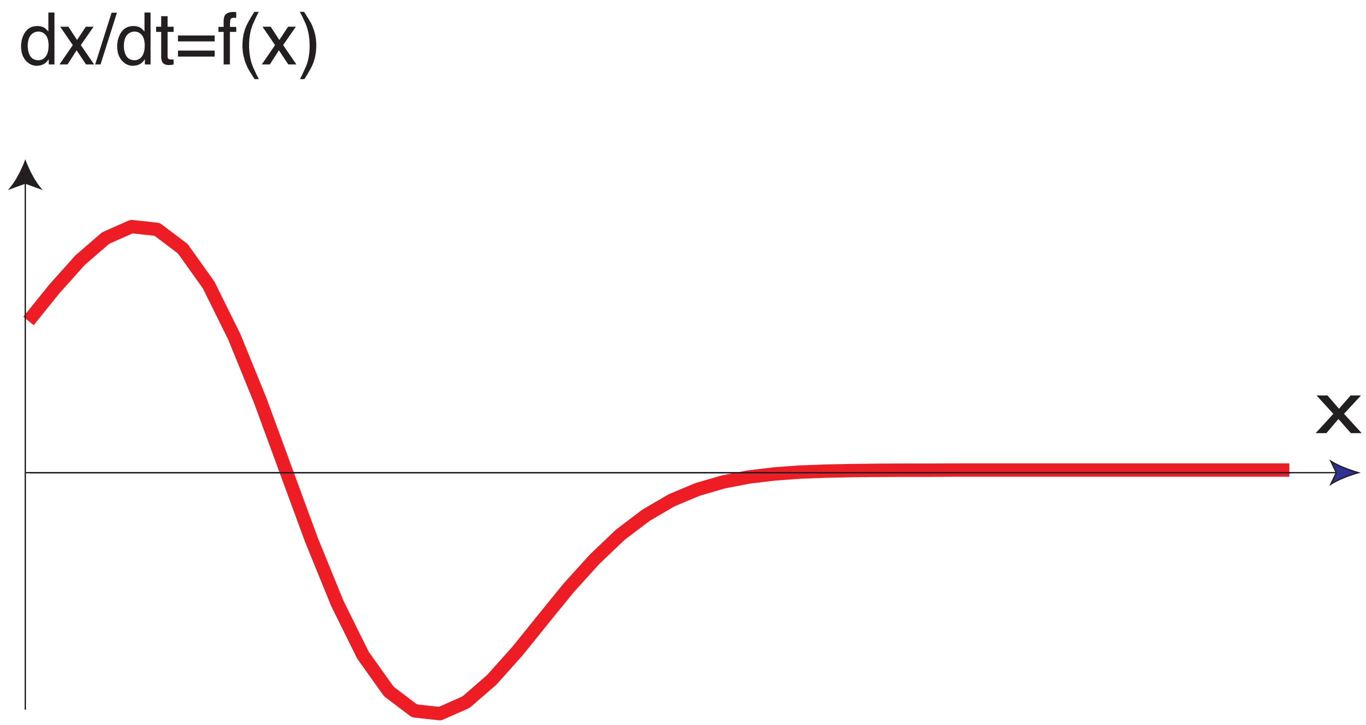
exponential relaxation to attractors

- $\tau\dot{x} = -x$ => $x(t) = x(0)\exp[-t/\tau]$ (check!)
- => has a well-defined time scale





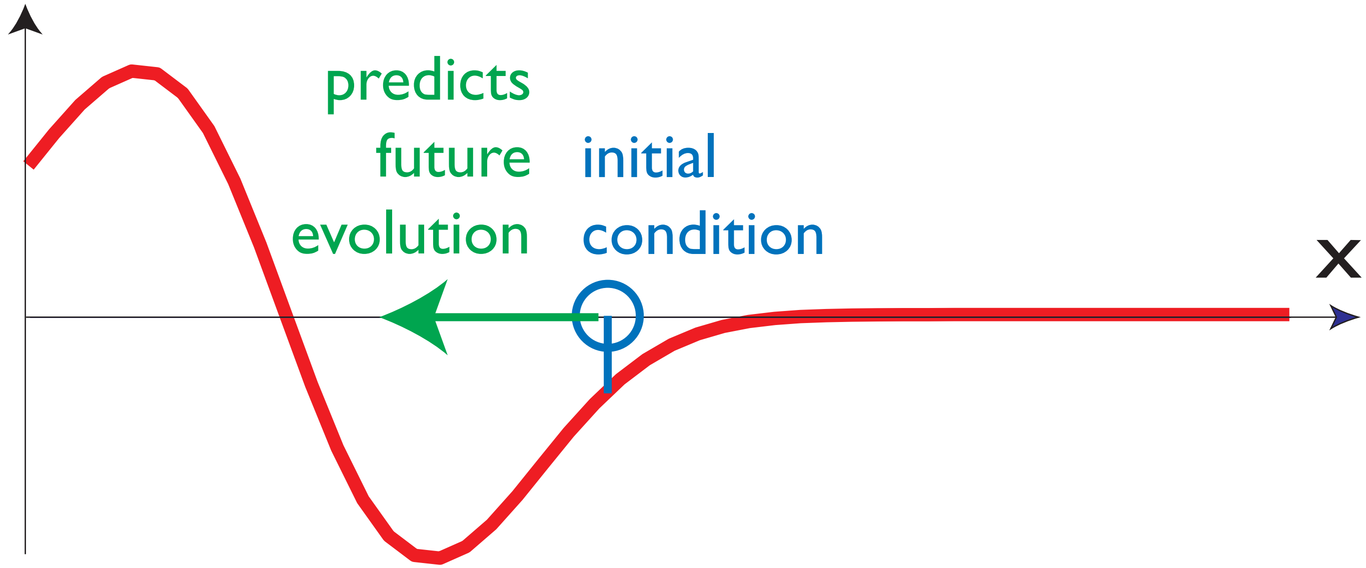
(nonlinear) dynamical system



dynamical system

- present determines the future
  - given initial condition
  - predict evolution (or predict the past)

$dx/dt=f(x)$



dynamical systems

- x: spans the state space (or phase space)
- f(x): is the “dynamics” of x (or vector-field)
- x(t) is a solution of the dynamical systems to the initial condition x_0
  - if its rate of change = f(x)
  - and x(0)=x_0

Dynamical systems

- differential equation $\dot{x} = f(x)$ in one dimension
- => an initial value of x determines the future

Dynamical systems

- system of differential equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$
- => a vector of initial states, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ determines the future

Dynamical systems

- partial differential equations

$$\dot{x}(y,t) = f\left(x(y,t), \frac{\partial x(y,t)}{\partial y}, \ldots \right)$$

- integro-differential equations

$$\dot{x}(y,t) = \int dy' f\left(y, y', x(y',t)\right)$$

- => continuously many initial values=initial function $x(y)$ determine the future

Dynamical systems

- delay differential equations $\dot{x}(t) = f(x(t-\tau))$
- functional differential equations
$$\dot{x}(t) = \int^{t} dt' f(x(t'))$$
- => a past piece of trajectory determines the future

Dynamical systems

- iteration equation in discrete time (map)
  $x_{n+1} = g(x_n)$
- every dynamical system in continuous time => dynamical system in discrete time (Poincaré)
- a dynamical system in discrete time can be lifted to a dynamical system in continuous time (but not uniquely)

Resources

- free online textbook by Scheinermann
  - https://github.com/scheinerman/InvitationToDynamicalSystems
  - send him a postcard (as instructed there)
  - really nice book for beginners…
  - focus on the time-continuous part..

numerics

- sample time discretely
- compute solution by iterating through time
- valid approximation for small time steps…

forward Euler

- $t_i = i\Delta t$ so that $x_i = x(t_i)$
- $\dot{x} = dx/dt \approx \Delta x/\Delta t$ where $\Delta x = x_{i+1} - x_i$
- $\dot{x} = f(x)$ => $x_{i+1} = x_i + \Delta t\ f(x_i)$
- … valid for small Δt
- is the "worst" approximation scheme (needs smallest time step to achieve given precision…)
- but useful for real-time embedded (and for stochastic systems)

modern numerics

- Runge-Kutte: error scales with step size to a power (e.g. 4)
- adaptive step size..
- built-into numerical packages… e.g. ode45 in Matlab

- => simulation

qualitative theory of dynamical systems

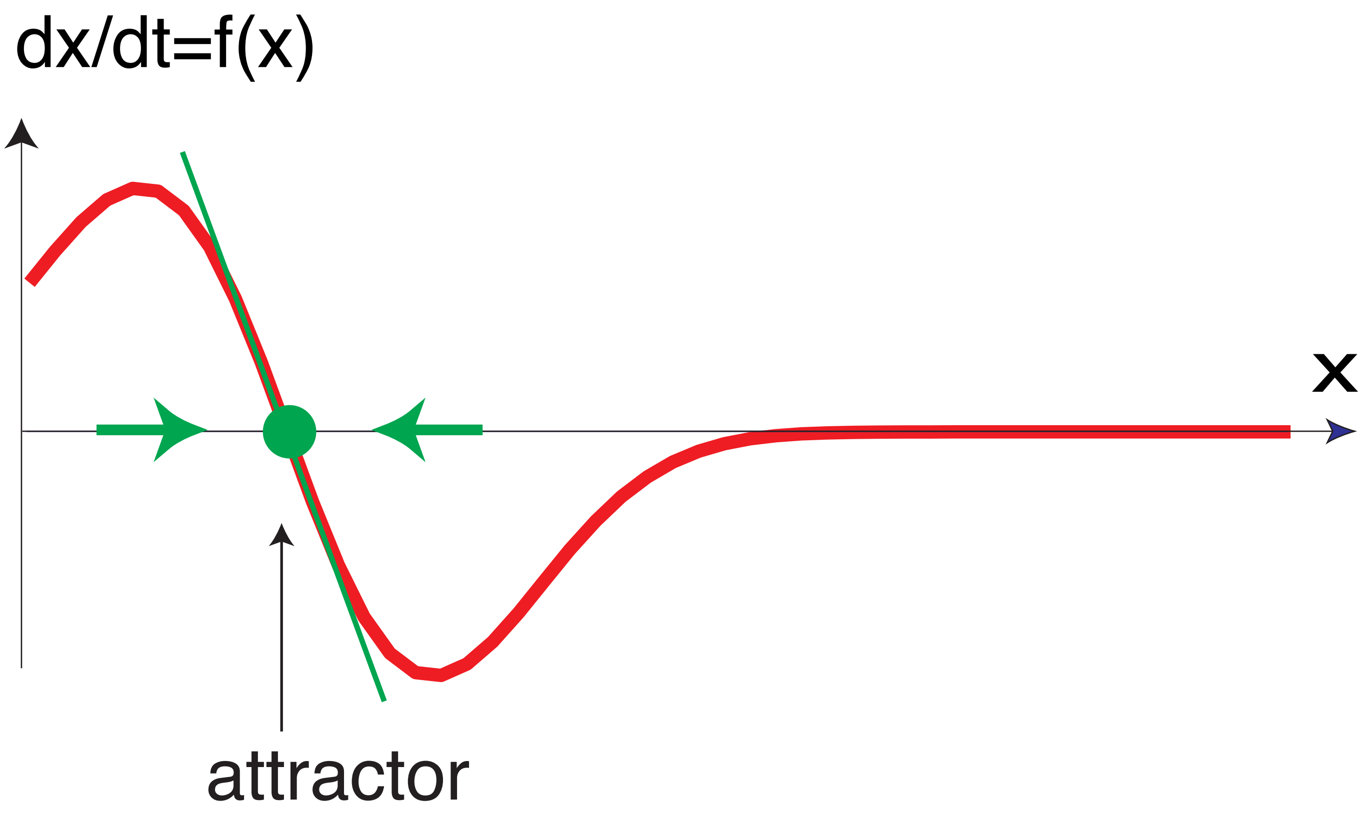
- good source:
    - Lawrence Perko: Differential Equations and Dynamical Systems, Springer 2001 (4th edition)

qualitative theory of dynamical systems

- the goals is to characterize the ensemble of solutions of the dynamical system (or a family of such)
- = the flow
- … use special invariant solutions to do that… fixed points, their stable/unstable manifolds…

attractor

- fixed point, to which neighboring initial conditions converge = attractor



fixed point

- is a constant solution of the dynamical system

$$\dot{x} = f(x)$$

$$\dot{x} = 0 \Rightarrow f(x_0) = 0$$

stability

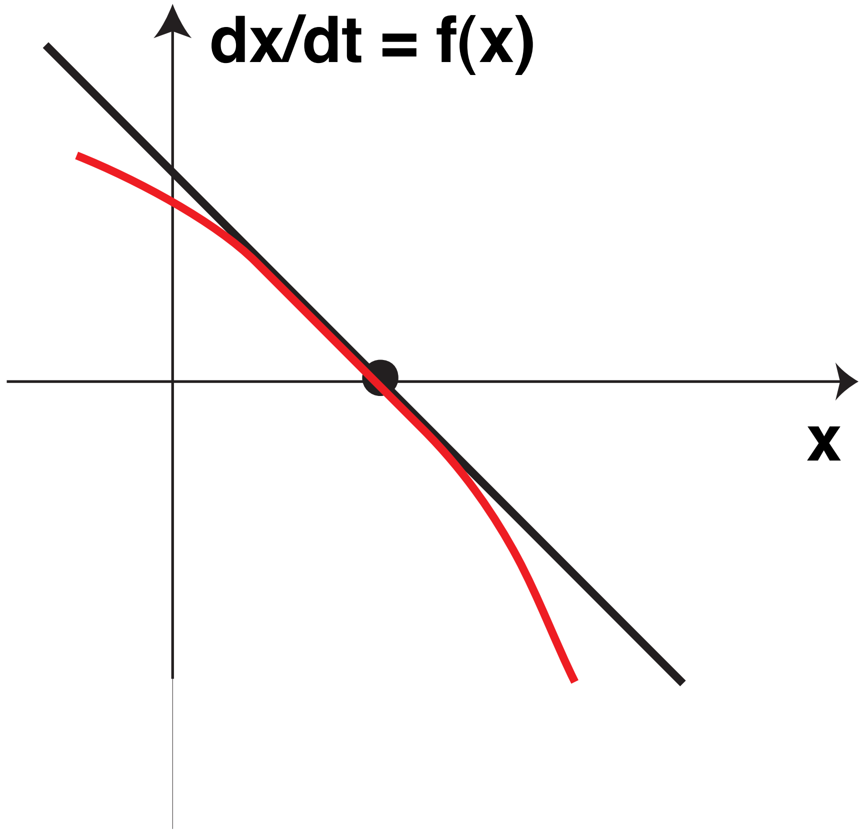
- mathematically really: asymptotic stability
- defined: a fixed point is asymptotically stable, when solutions of the dynamical system that start nearby converge in time to the fixed point

stability

- the mathematical concept of stability (which we do not use) requires only that nearby solutions stay nearby
- defined: a fixed point is unstable if it is not stable in that more general sense,
  - that is: if nearby solutions do not necessarily stay nearby (may diverge)

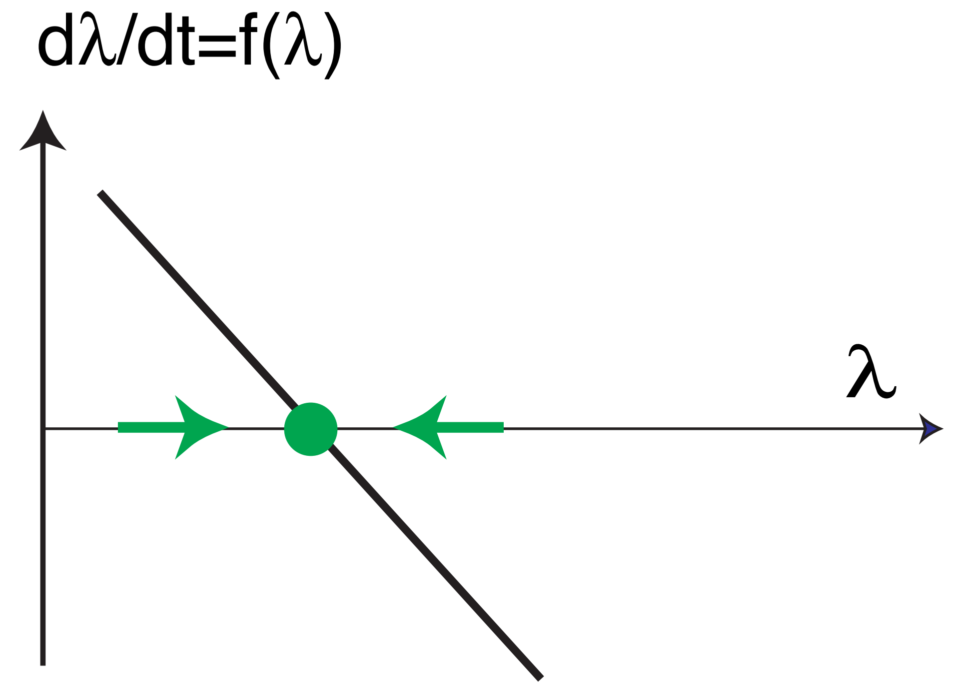
linear approximation near attractor

- non-linearity as a small perturbation/ deformation of linear system
- => non-essential non-linearity



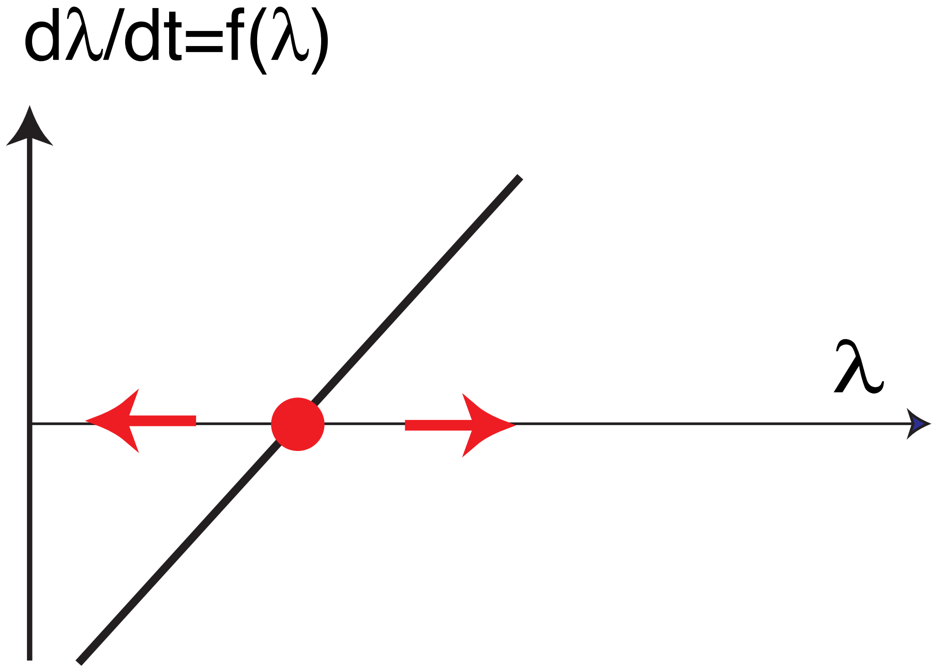
stability in a linear system

- if the slope of the linear system is negative, the fixed point is (asymptotically stable)



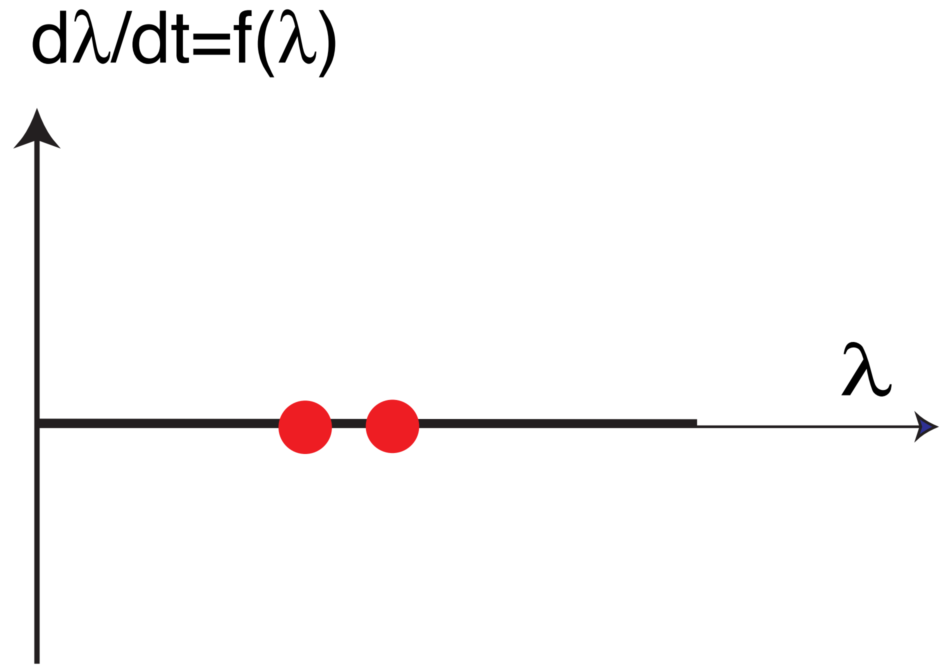
stability in a linear system

- if the slope of the linear system is positive, then the fixed point is unstable



stability in a linear system

- if the slope of the linear system is zero, then the system is indifferent (marginally stable: stable but not asymptotically stable)



stability in linear systems

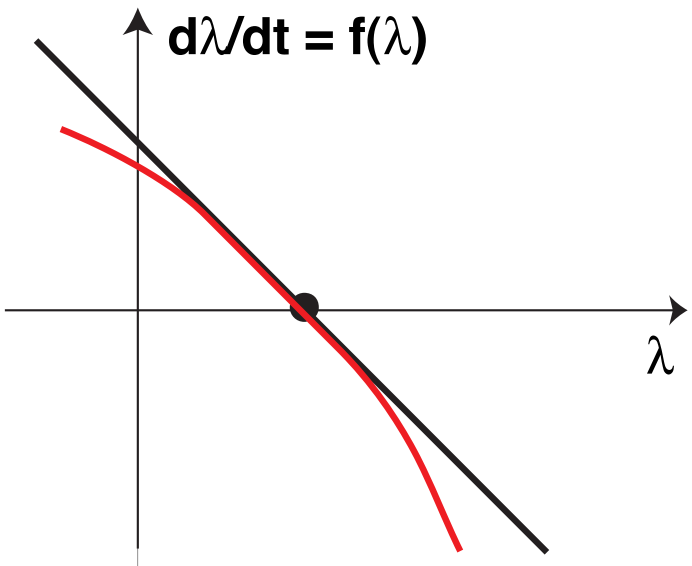
- generalization to multiple dimensions
  - if the real-parts of all Eigenvalues are negative: stable
  - if the real-part of any Eigenvalue is positive: unstable
  - if the real-part of any Eigenvalue is zero: marginally stable in that direction (stability depends on other eigenvalues)

stability in nonlinear systems

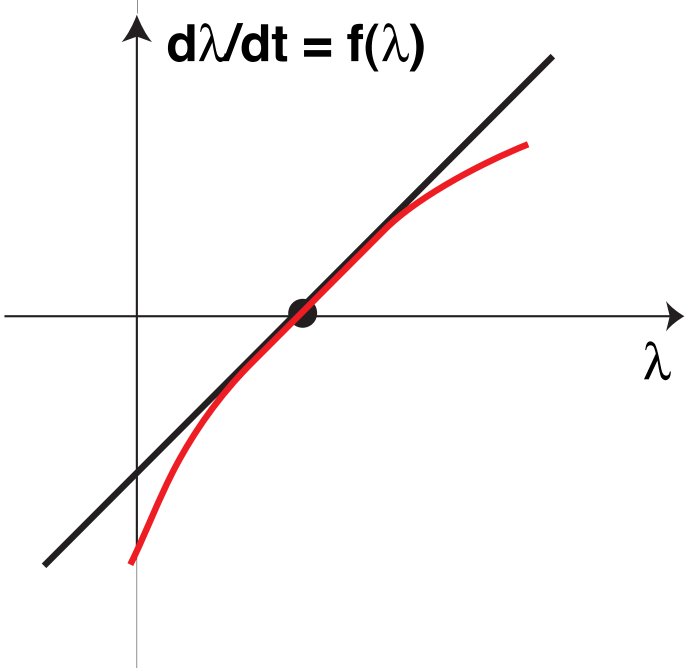
- stability is a local property of the fixed point
- => linear stability theory
  - the eigenvalues of the linearization around the fixed point determine stability
  - all real-parts negative: stable
  - any real-part positive: unstable
  - any real-part zero: undecided: now nonlinearity decides (non-hyberpolic fixed point)

stability in nonlinear systems

- all real-parts negative: stable

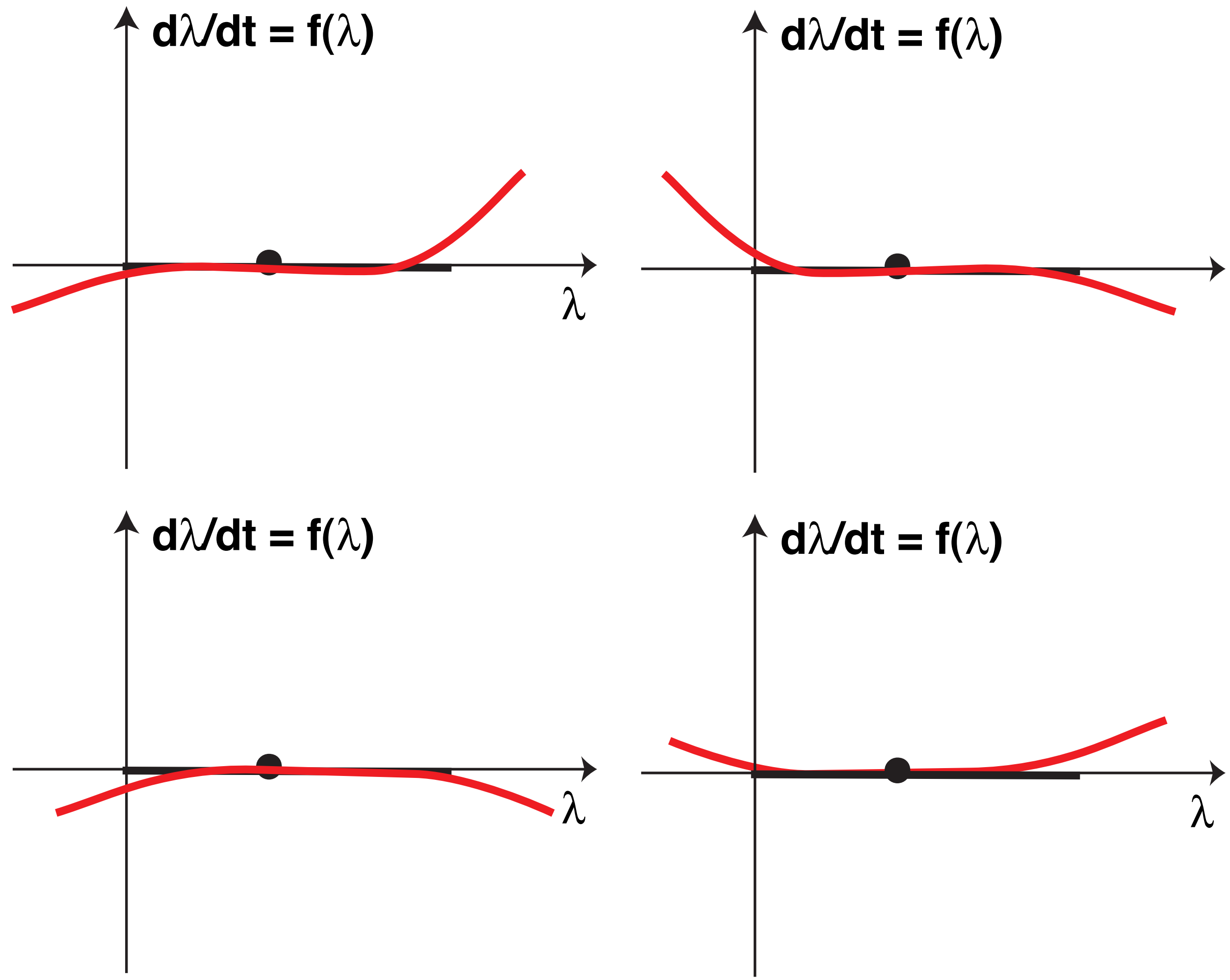


- any real-part positive: unstable



stability in nonlinear systems

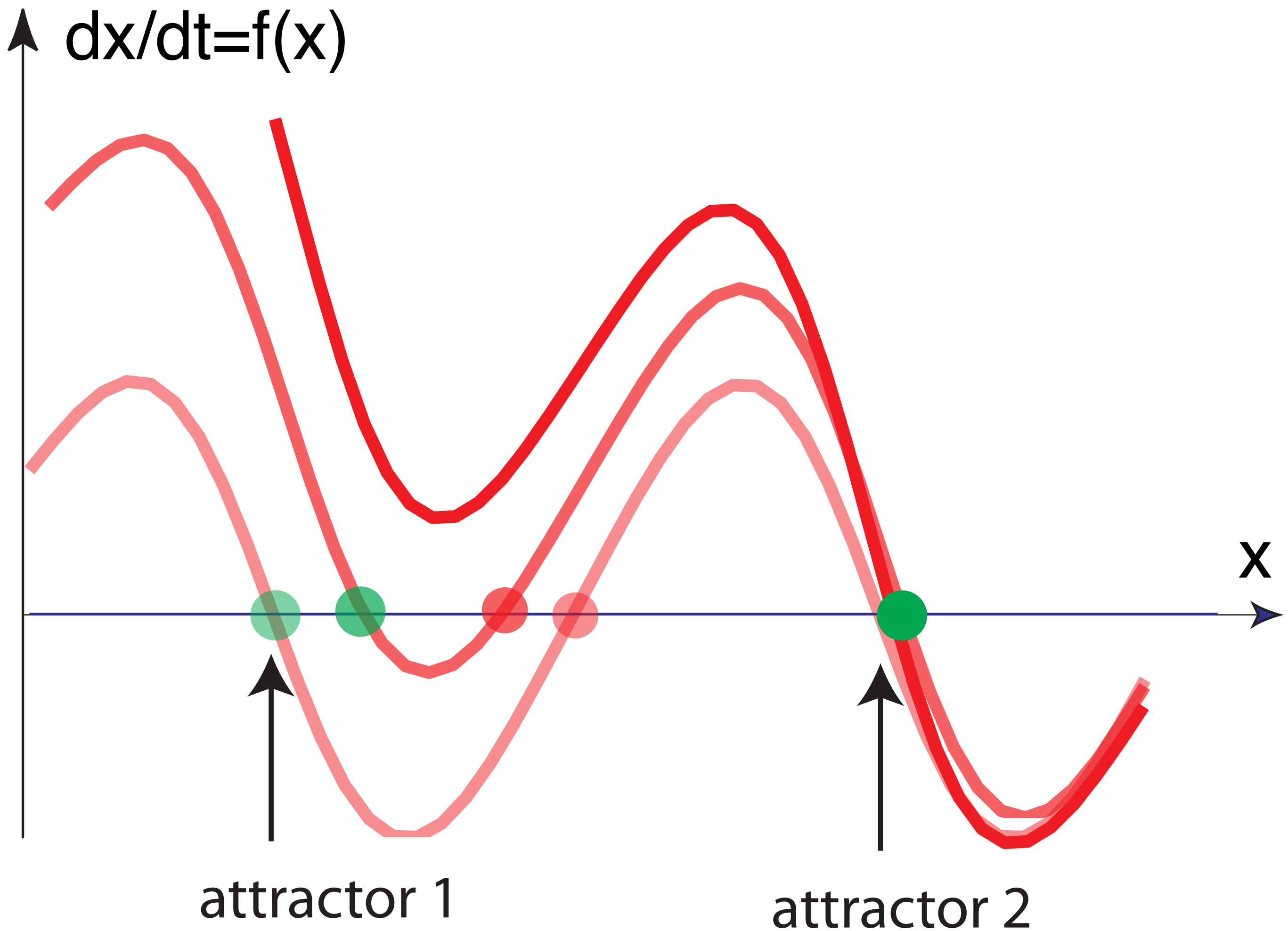
- any real-part zero: undecided: now nonlinearity decides (non-hyberpolic fixed point)



bifurcations

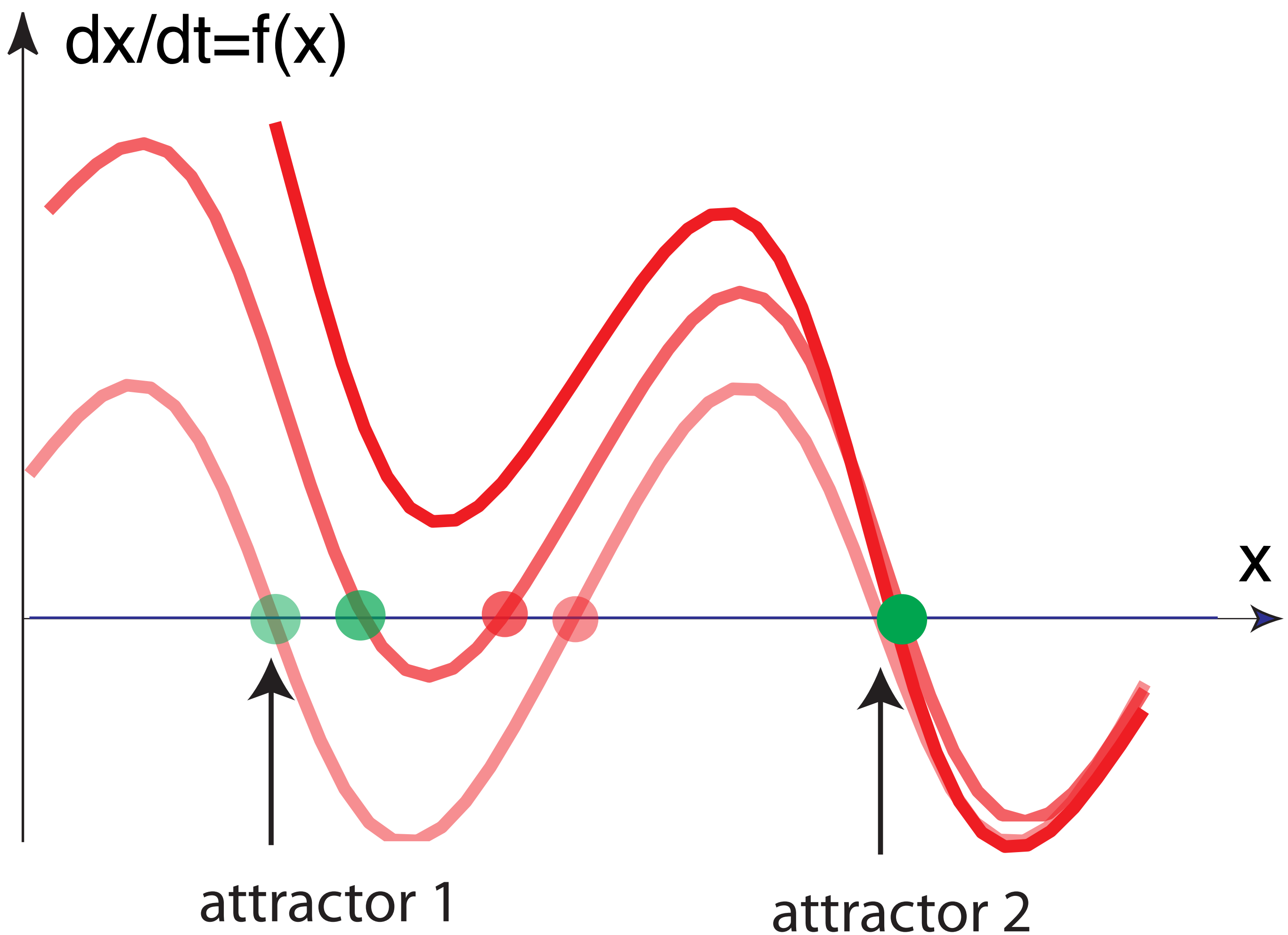
- look now at families of dynamical systems, which depend (smoothly) on parameters
- ask: as the parameters change (smoothly), how do the solutions change (smoothly?)
  - smoothly: topological equivalence of the dynamical systems at neighboring parameter values
  - bifurcation: dynamical systems NOT topological equivalent as parameter changes infinitesimally

bifurcation



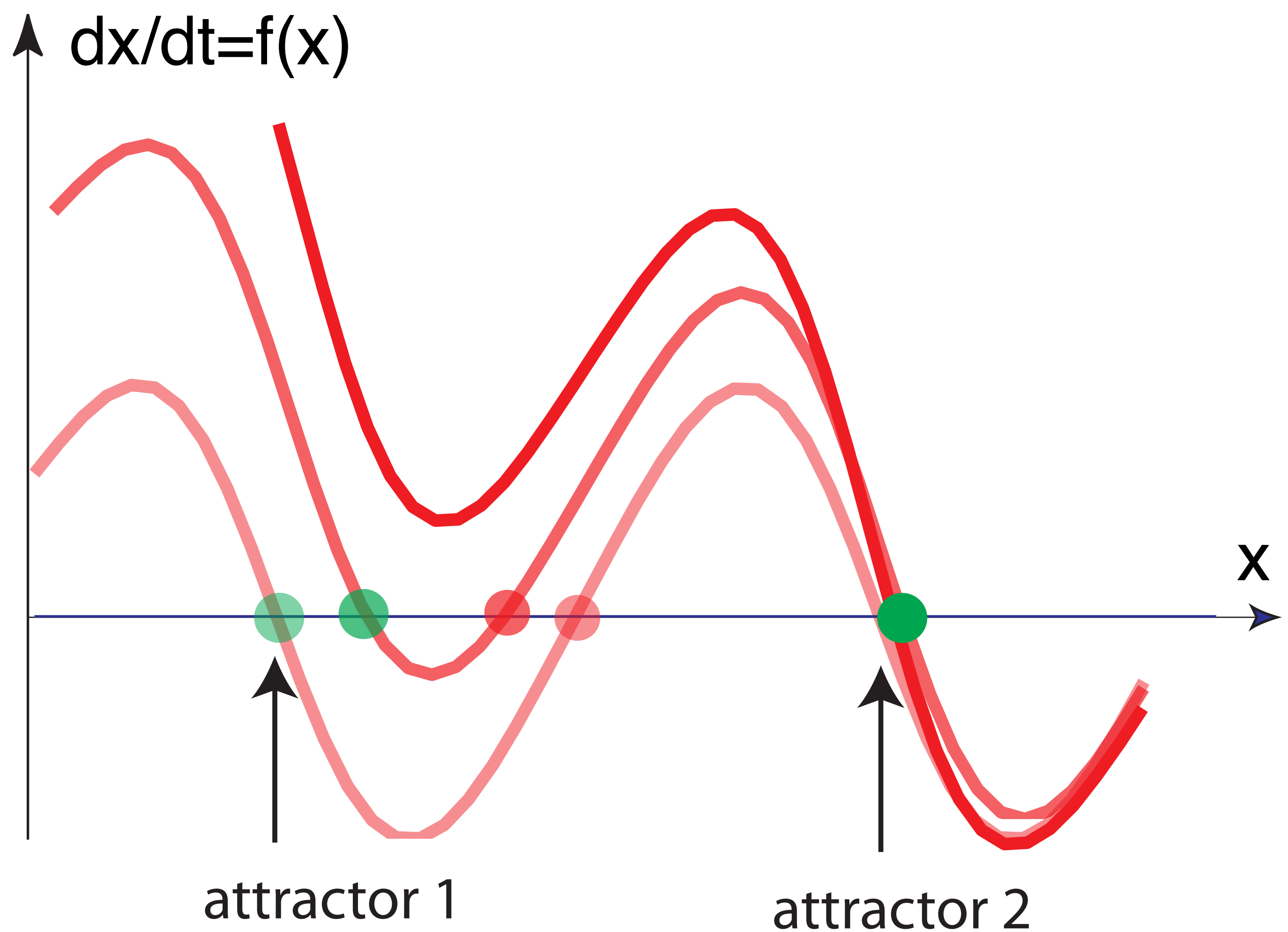
bifurcation

- bifurcation=qualitative change of dynamics (change in number, nature, or stability of fixed points) as the dynamics changes smoothly

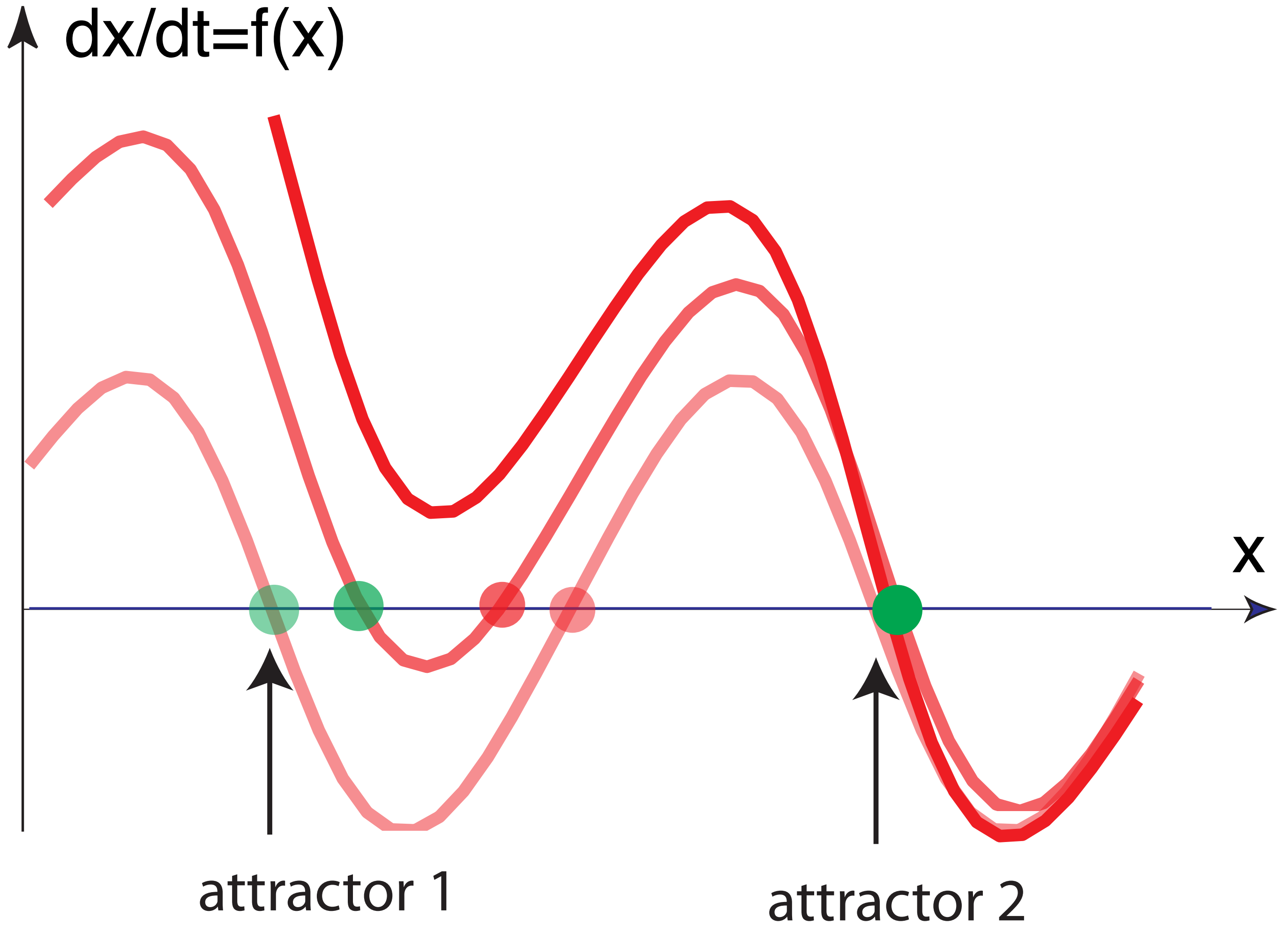


tangent bifurcation

- the simplest bifurcation (co-dimension 0): an attractor collides with a repellor and the two annihilate

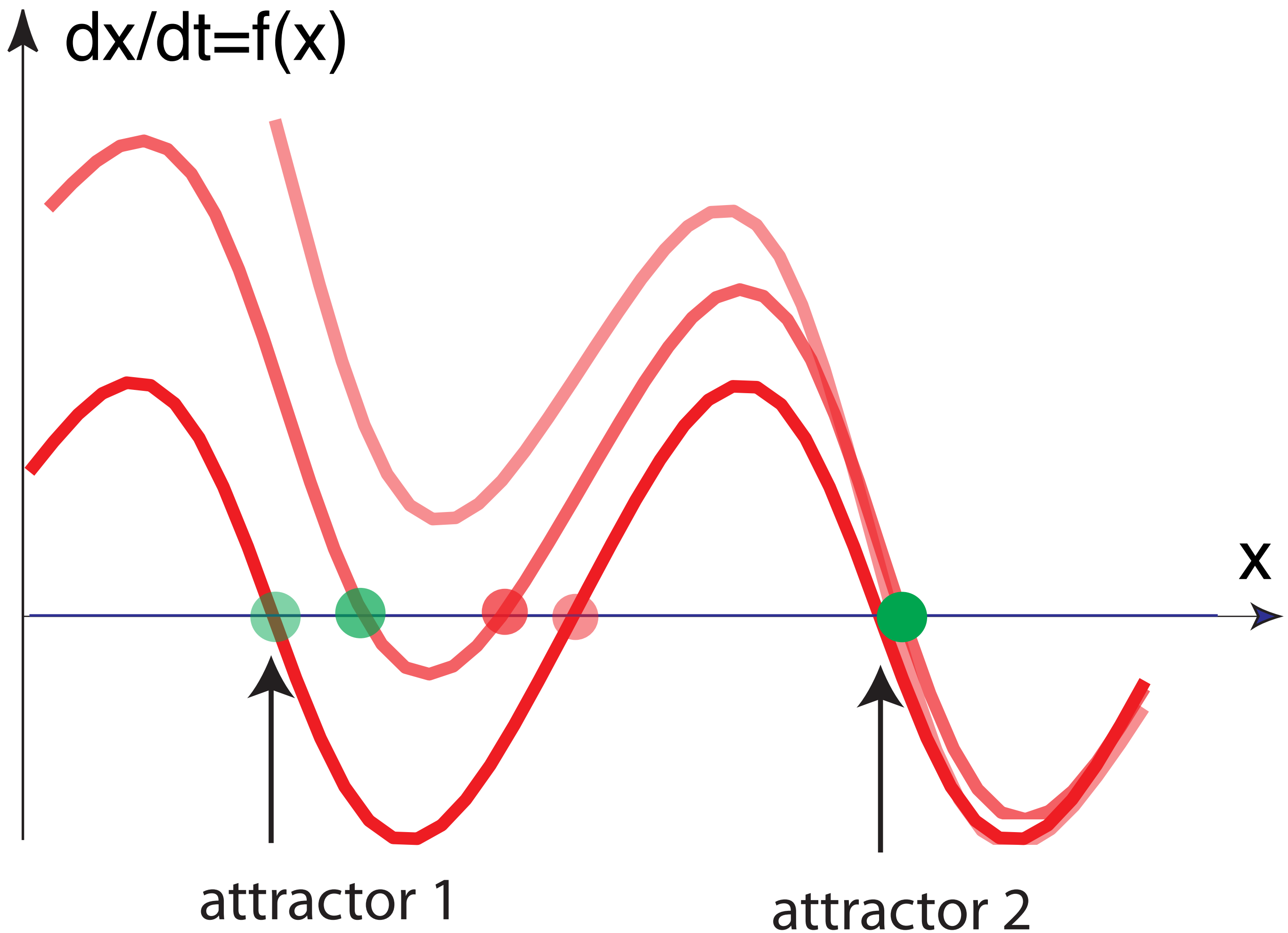


local bifurcation



reverse bifurcation

- changing the dynamics in the opposite direction



bifurcations are instabilities

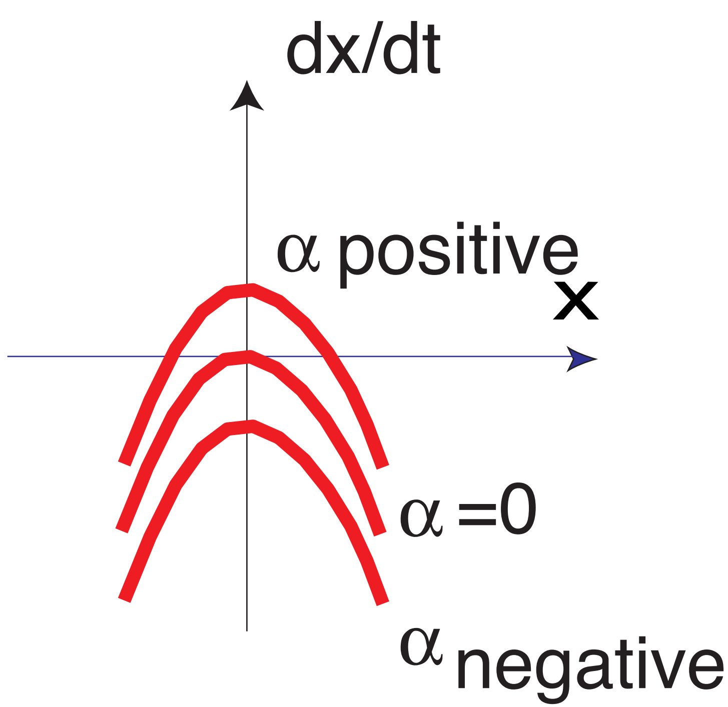
- that is, an attractor becomes unstable before disappearing
- (or the attractor appears with reduced stability)
- formally: a zero-real part is a necessary condition for a bifurcation to occur

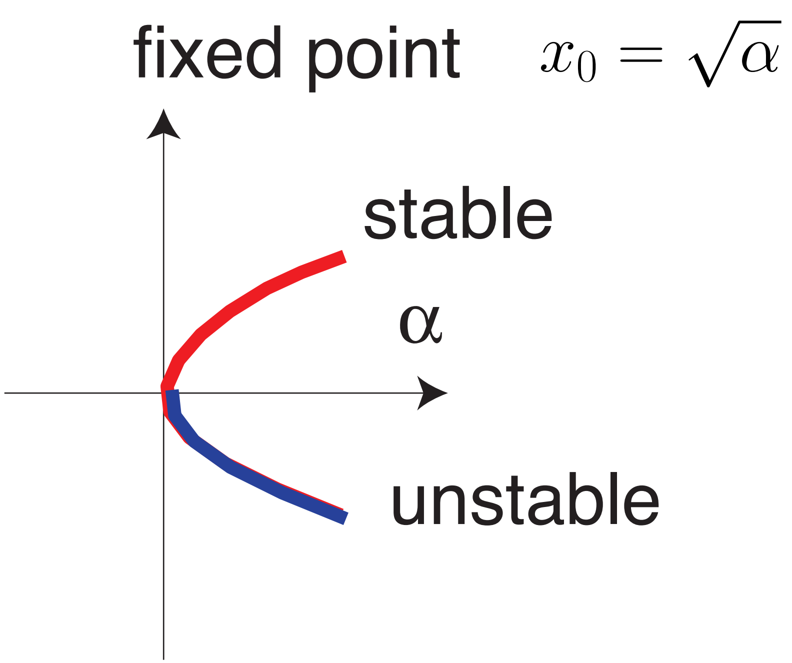
tangent bifurcation

- normal form of tangent bifurcation

$$\dot{x} = \alpha - x^2$$

- (=simplest polynomial equation whose flow is topologically equivalent to the bifurcation)



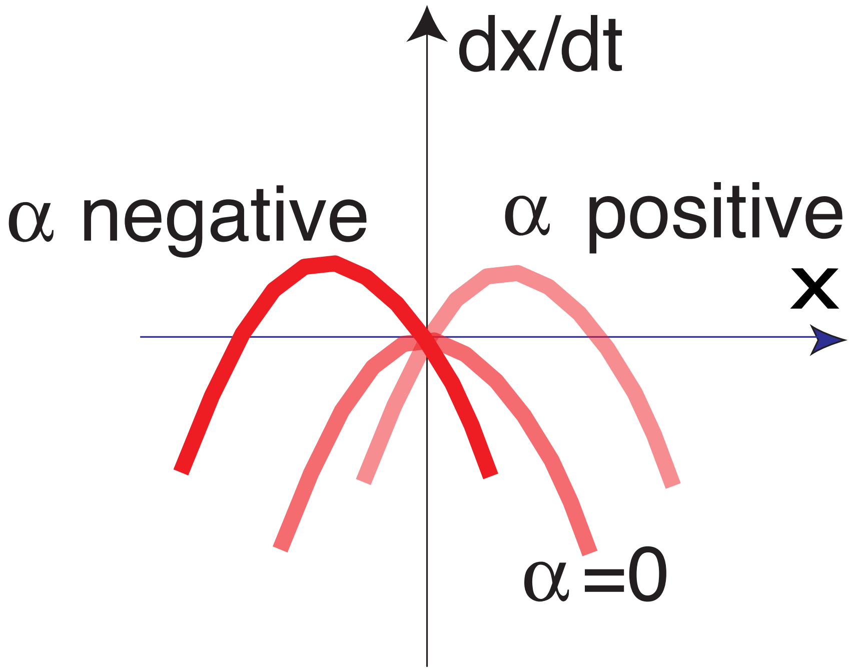


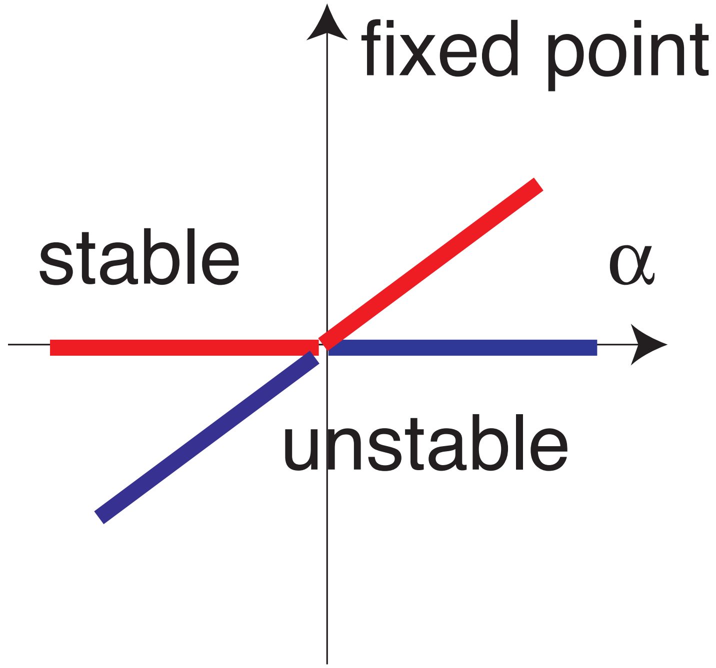
Hopf theorem

- when a single (or pair of complex conjugate) eigenvalue crosses the imaginary axis, one of four bifurcations occur
  - tangent bifurcation
  - transcritical bifurcation
  - pitchfork bifurcation
  - Hopf bifurcation

transcritical bifurcation

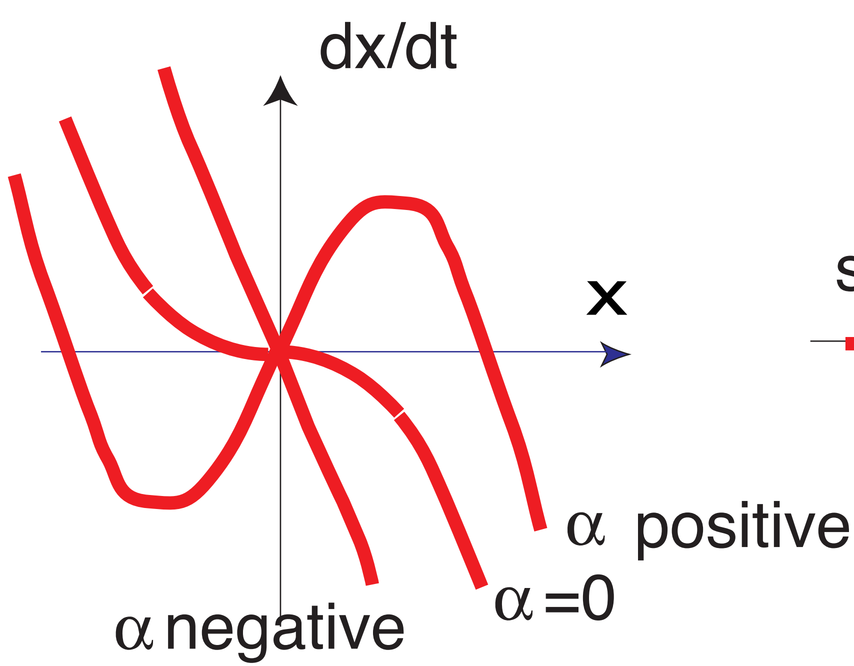
- normal form $\dot{x} = \alpha x - x^2$

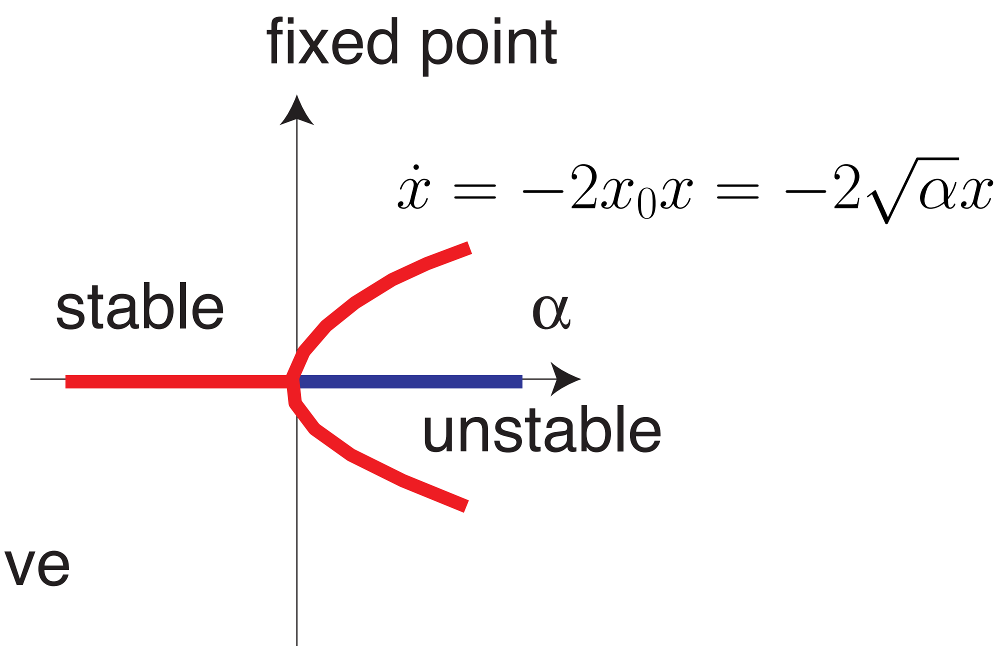




pitchfork bifurcation

- normal form $$\dot{x} = \alpha x - x^3$$

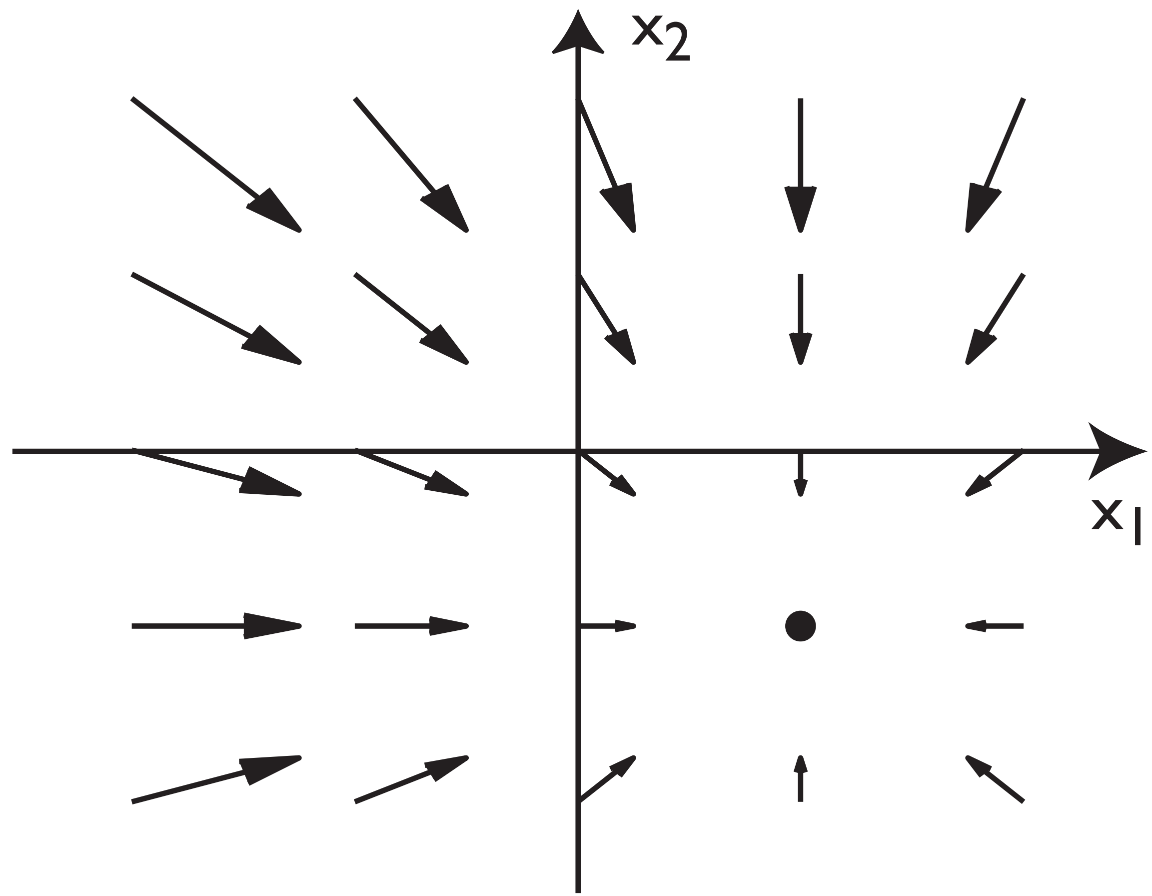




Hopf: need higher dimensions

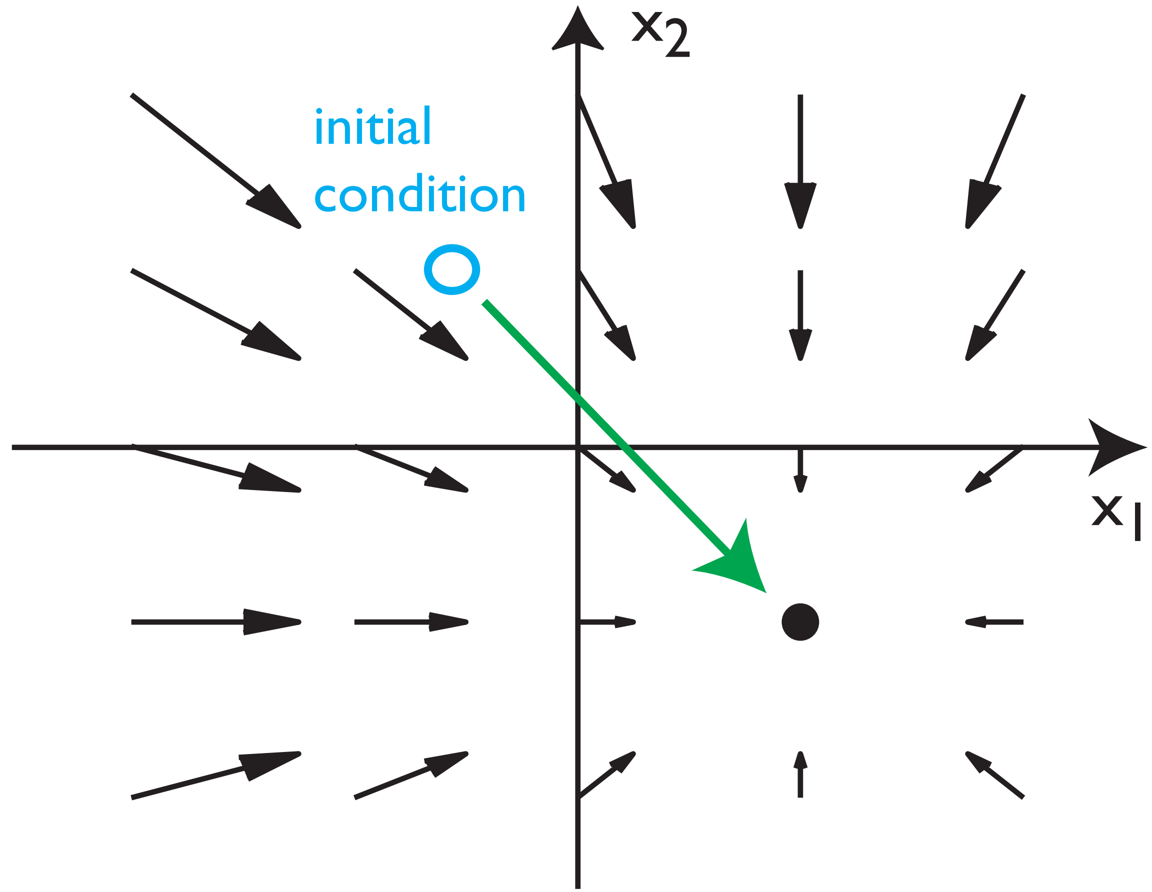
2D dynamical system: vector-field

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$



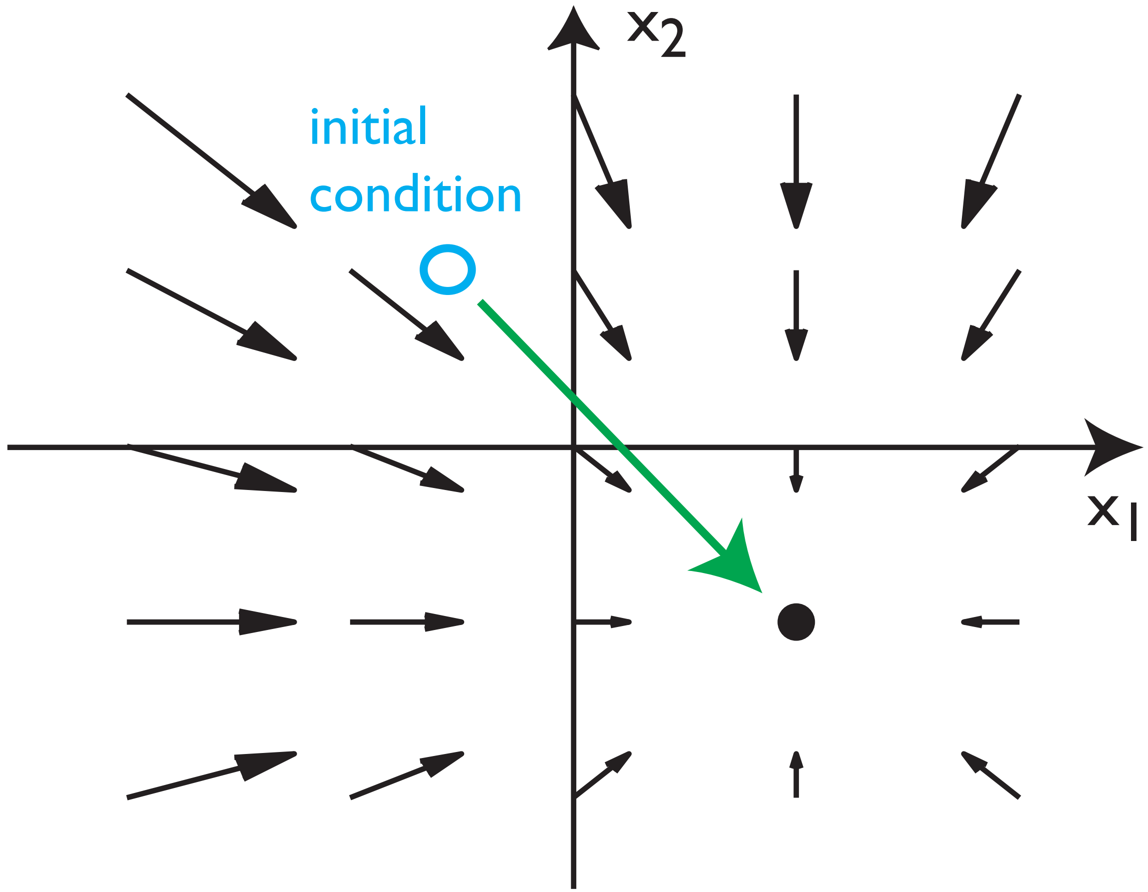
vector-field

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$



fixed point, stability, attractor

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

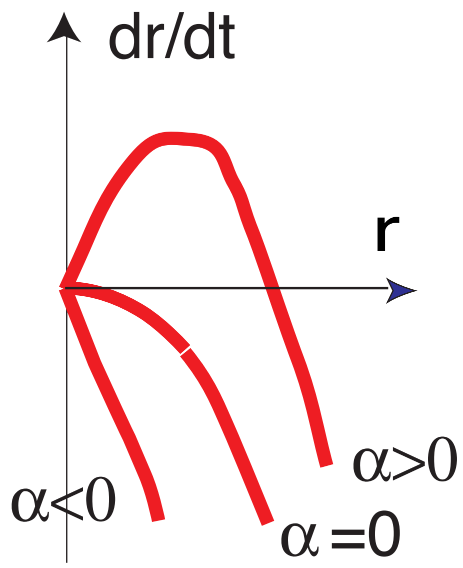


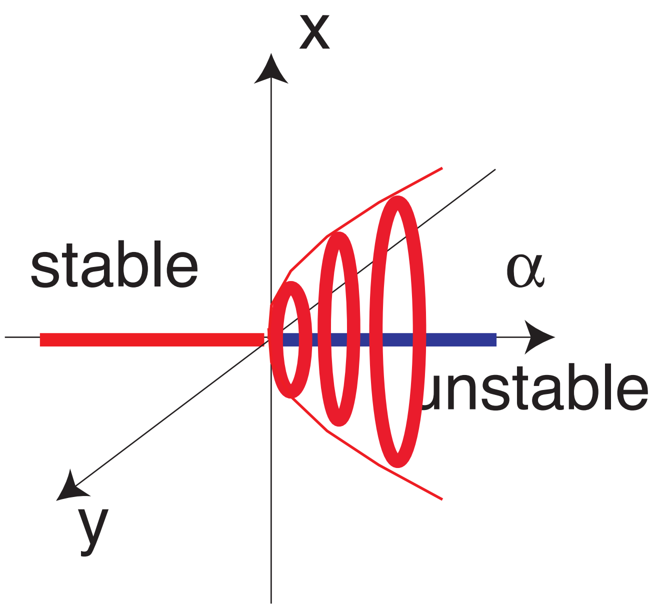
Hopf bifurcation

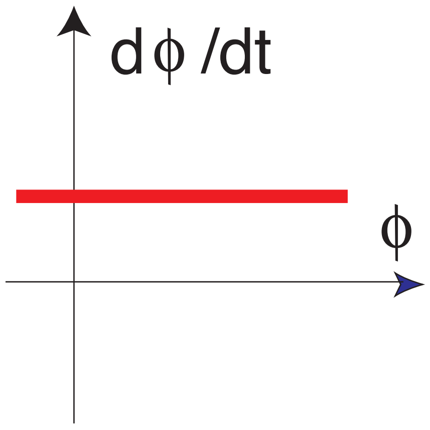
- normal form

$$\dot{r} = \alpha r - r^3$$

$$\dot{\phi} = \omega$$







forward dynamics

- given known equation, determined fixed points / limit cycles and their stability
- more generally: determine invariant solutions (stable, unstable and center manifolds)

inverse dynamics

- given solution, find the equation…
- this is the problem faced in design of behavioral dynamics…

inverse dynamics: design

- in the design of behavioral dynamics… you may be given:
- attractor solutions/stable states
- and how they change as a function of parameters/ conditions
- => identify the class of dynamical systems using the 4 elementary bifurcations
- and use normal form to provide an exemplary representative of the equivalence class of dynamics