

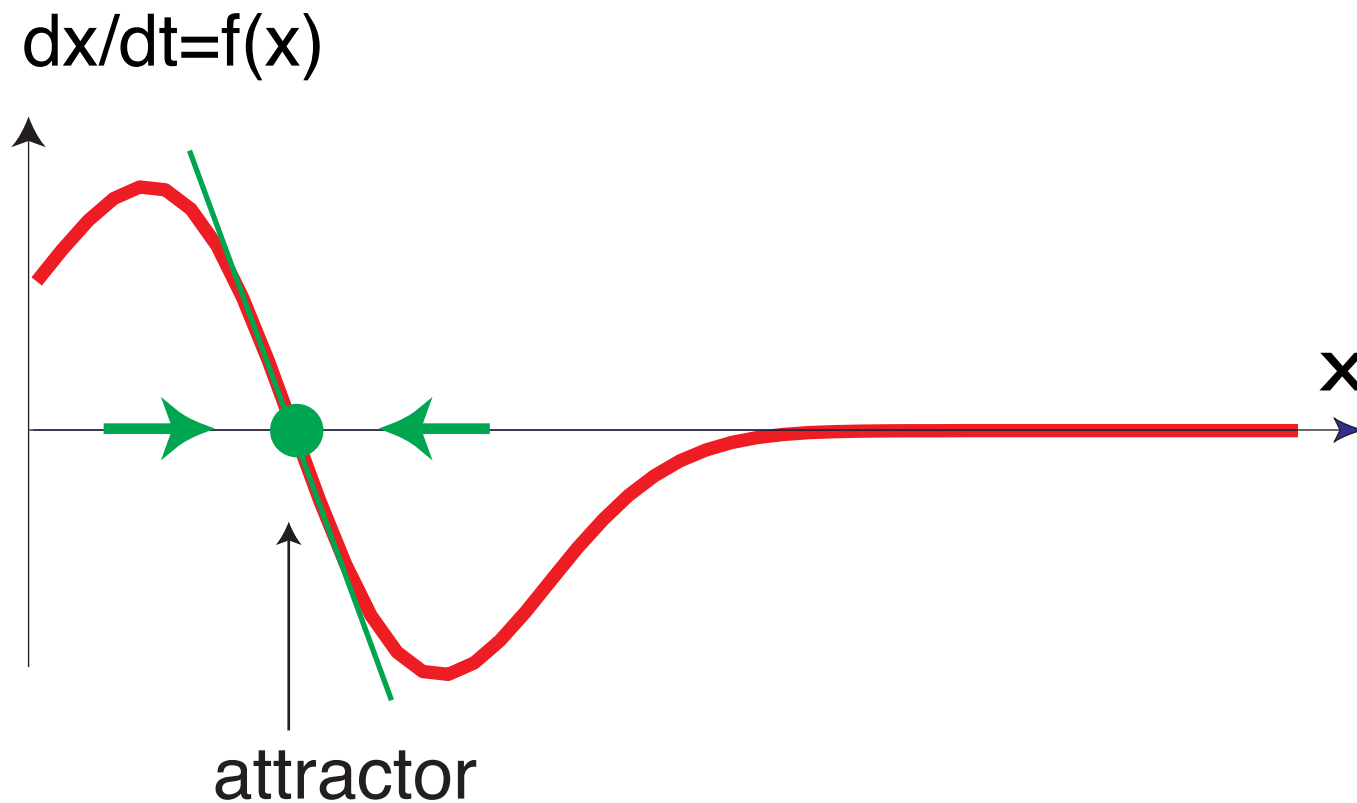
Dynamical systems tutorial:

3. Attractors and instabilities

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attractor

- **fixed point**, to which neighboring initial conditions converge = **attractor**



fixed point

■ is a constant solution of the dynamical system

■
$$\dot{x} = f(x)$$

$$\dot{x} = 0 \Rightarrow f(x_0) = 0$$

stability

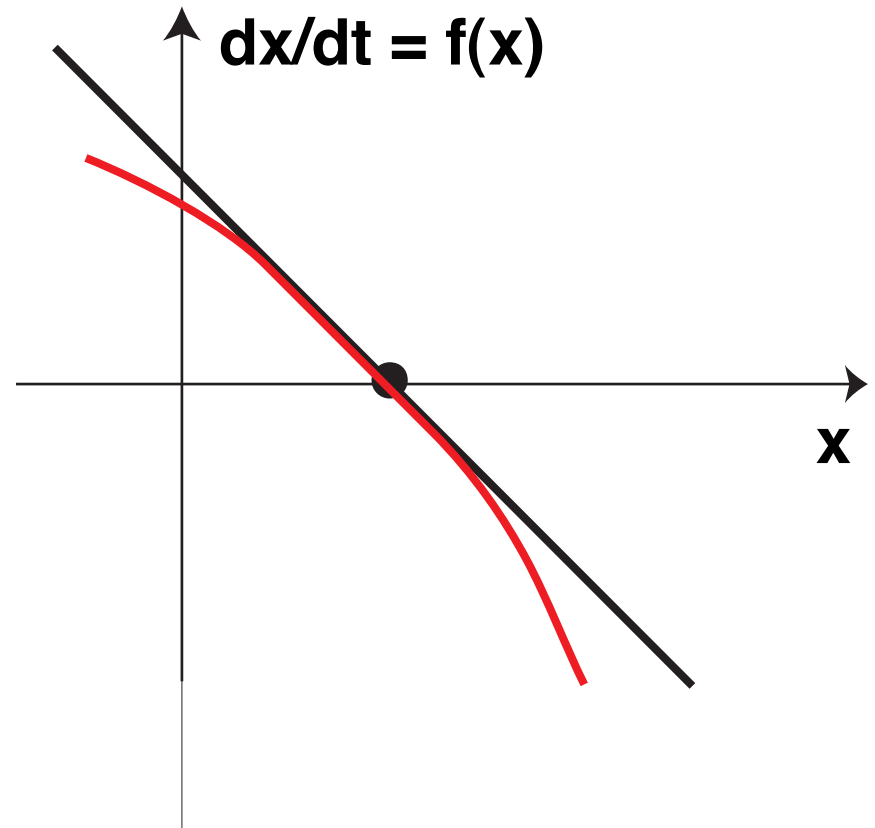
- mathematically really: **asymptotic stability**
- defined: a fixed point is asymptotically stable, when solutions of the dynamical system that start nearby converge in time to the fixed point

stability

- the mathematical concept of stability (which we do not use) requires only that nearby solutions stay nearby
- Definition: a fixed point is **unstable** if it is not stable in that more general sense,
 - that is: if nearby solutions do not necessarily stay nearby (may diverge)

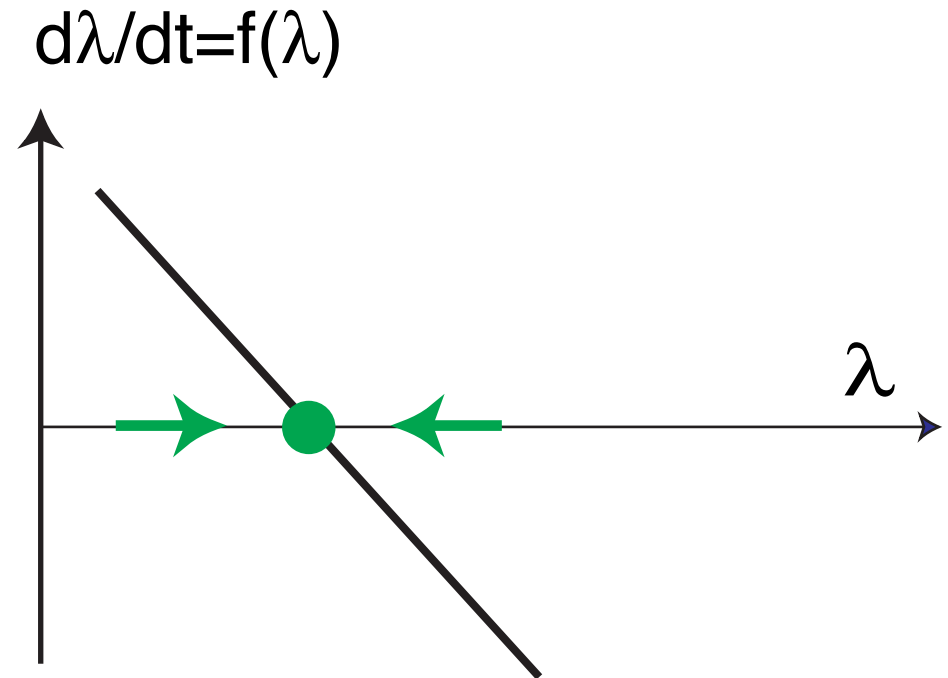
linear approximation near attractor

- non-linearity as a small perturbation/ deformation of linear system
- \Rightarrow non-essential non-linearity



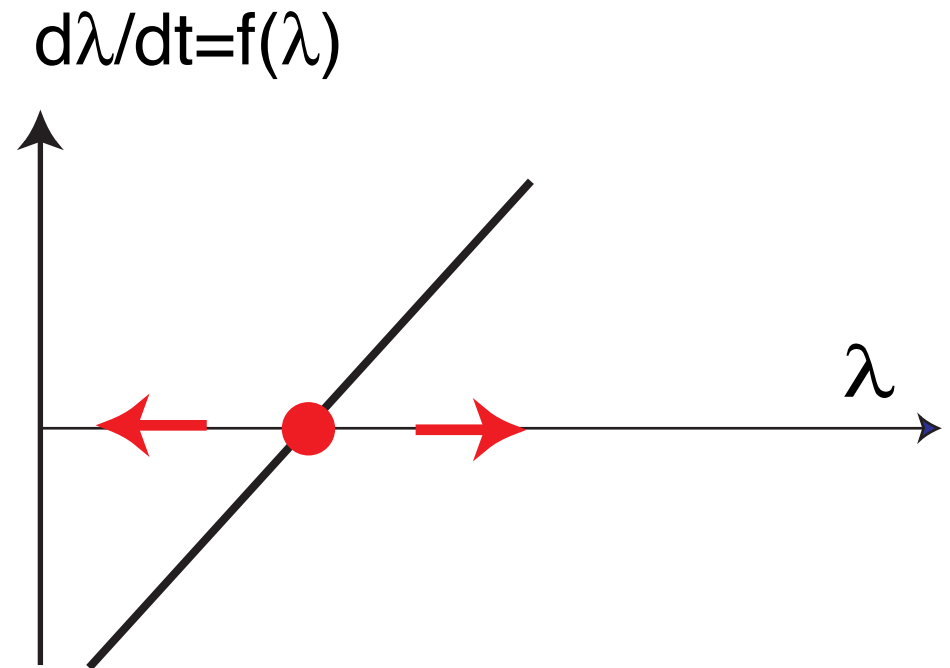
stability in a linear system

- if the slope of the linear system is negative, the fixed point is (asymptotically stable)



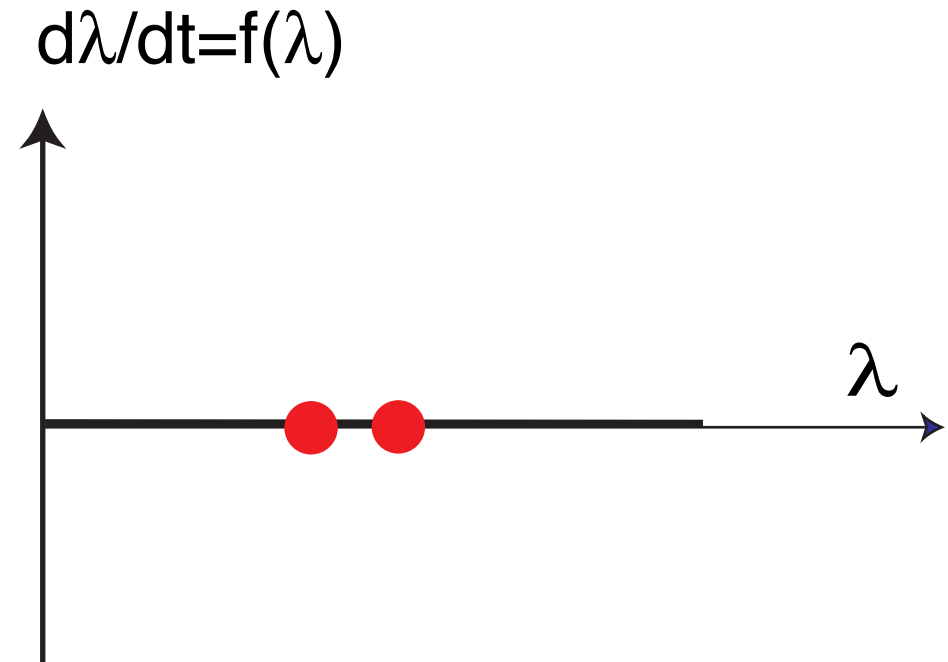
stability in a linear system

- if the slope of the linear system is positive, then the fixed point is unstable



stability in a linear system

- if the slope of the linear system is zero, then the system is indifferent (marginally stable: stable but not asymptotically stable)



stability in linear systems

■ generalization to multiple dimensions

- if the real-parts of all Eigenvalues are negative: stable
- if the real-part of any Eigenvalue is positive: unstable
- if the real-part of any Eigenvalue is zero: marginally stable in that direction (stability depends on other eigenvalues)

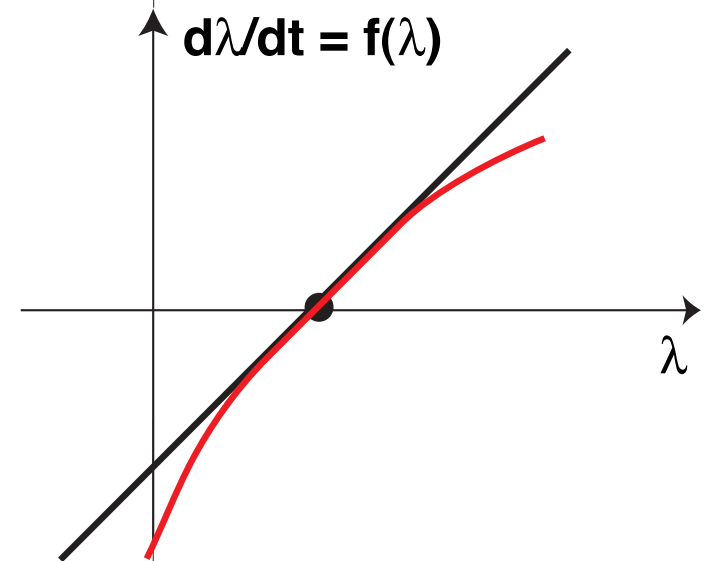
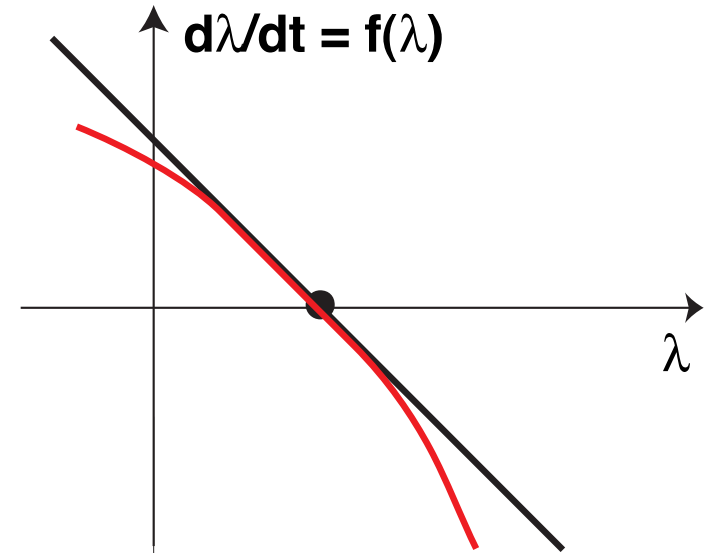
stability in nonlinear systems

- stability is a local property of the fixed point
- \Rightarrow linear stability theory
 - the eigenvalues of the linearization around the fixed point determine stability
 - all real-parts negative: stable
 - any real-part positive: unstable
 - any real-part zero: undecided: now nonlinearity decides (non-hyperbolic fixed point)

stability in nonlinear systems

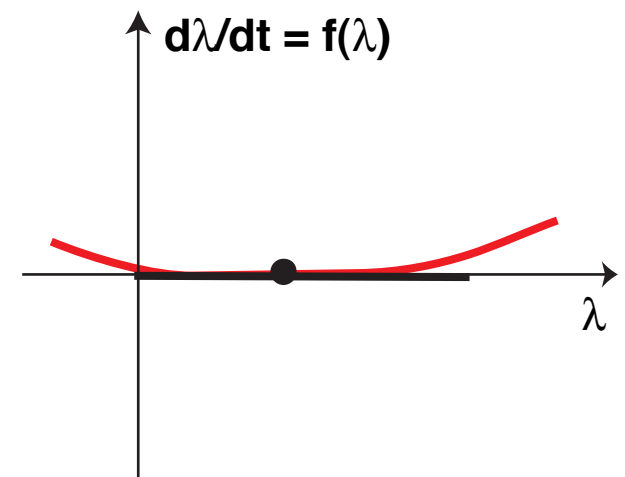
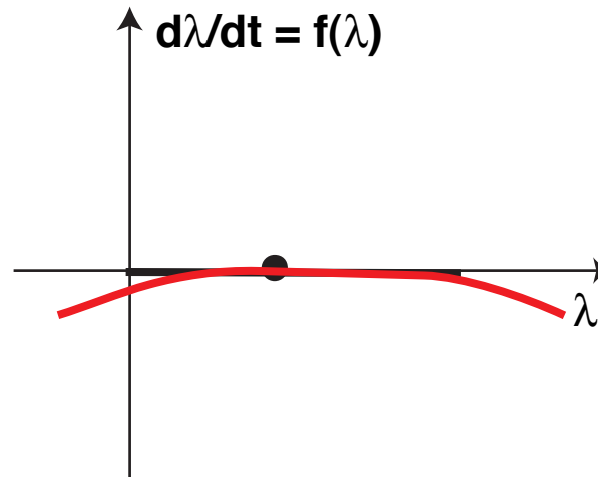
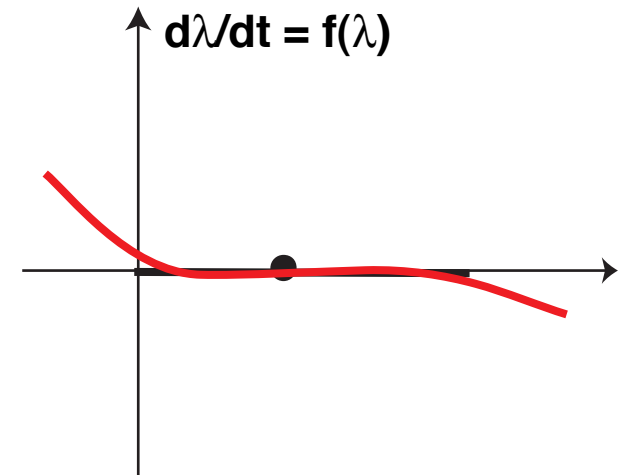
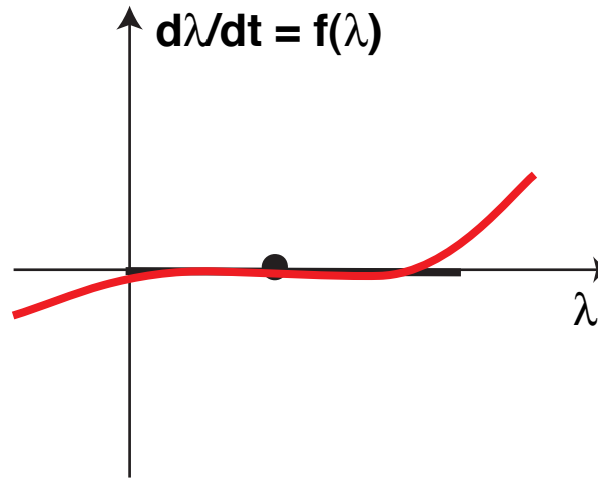
■ all real-parts negative: stable

■ any real-part positive: unstable



stability in nonlinear systems

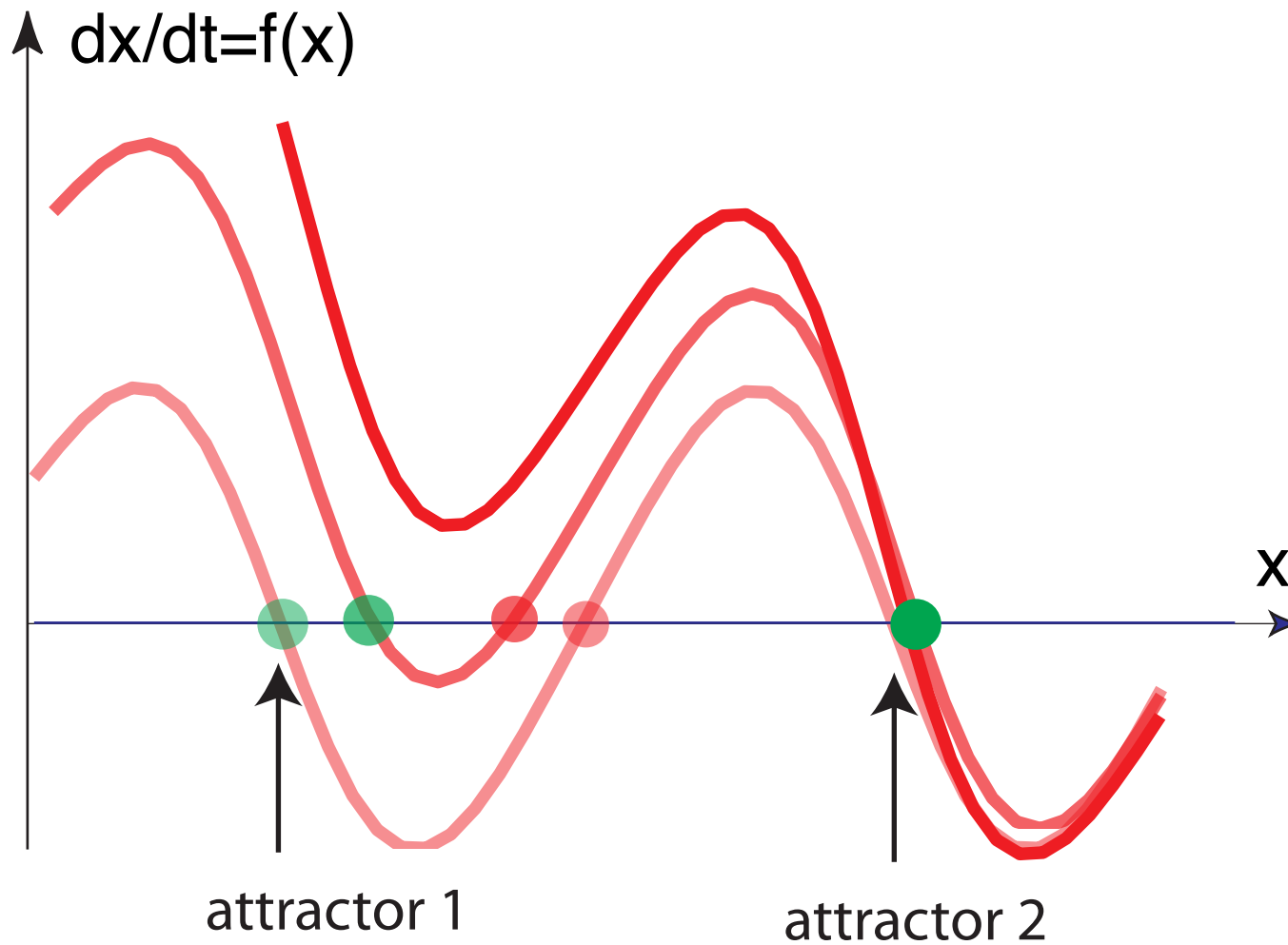
any real-part zero:
undecided: now
nonlinearity decides
(non-hyperbolic fixed
point)



bifurcations

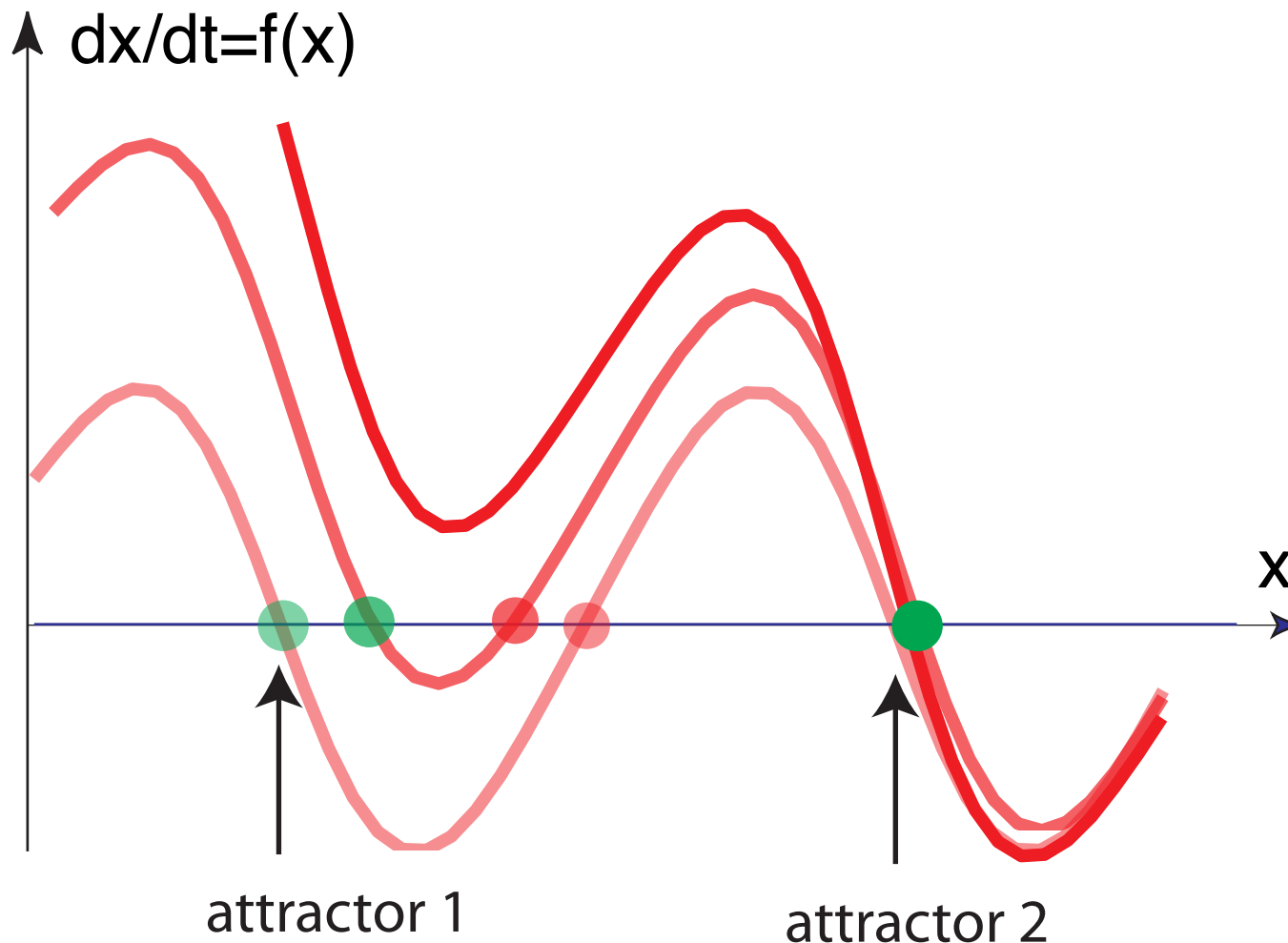
- look now at families of dynamical systems, which depend (smoothly) on parameters
- ask: as the parameters change (smoothly), how do the solutions change (smoothly?)
 - smoothly: topological equivalence of the dynamical systems at neighboring parameter values
 - bifurcation: dynamical systems NOT topological equivalent as parameter changes infinitesimally

bifurcation



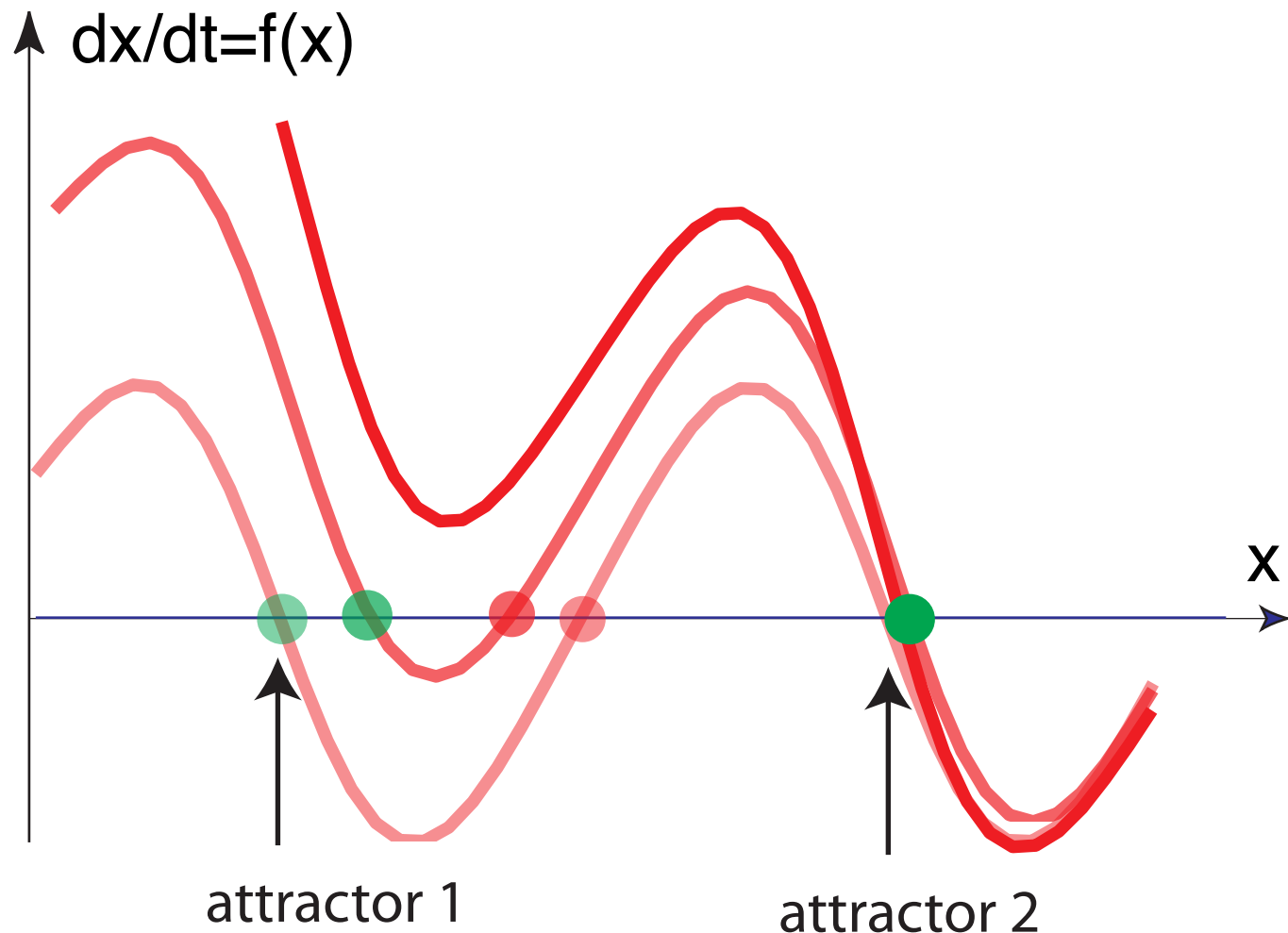
bifurcation

- bifurcation=qualitative change of dynamics (change in number, nature, or stability of fixed points) as the dynamics changes smoothly

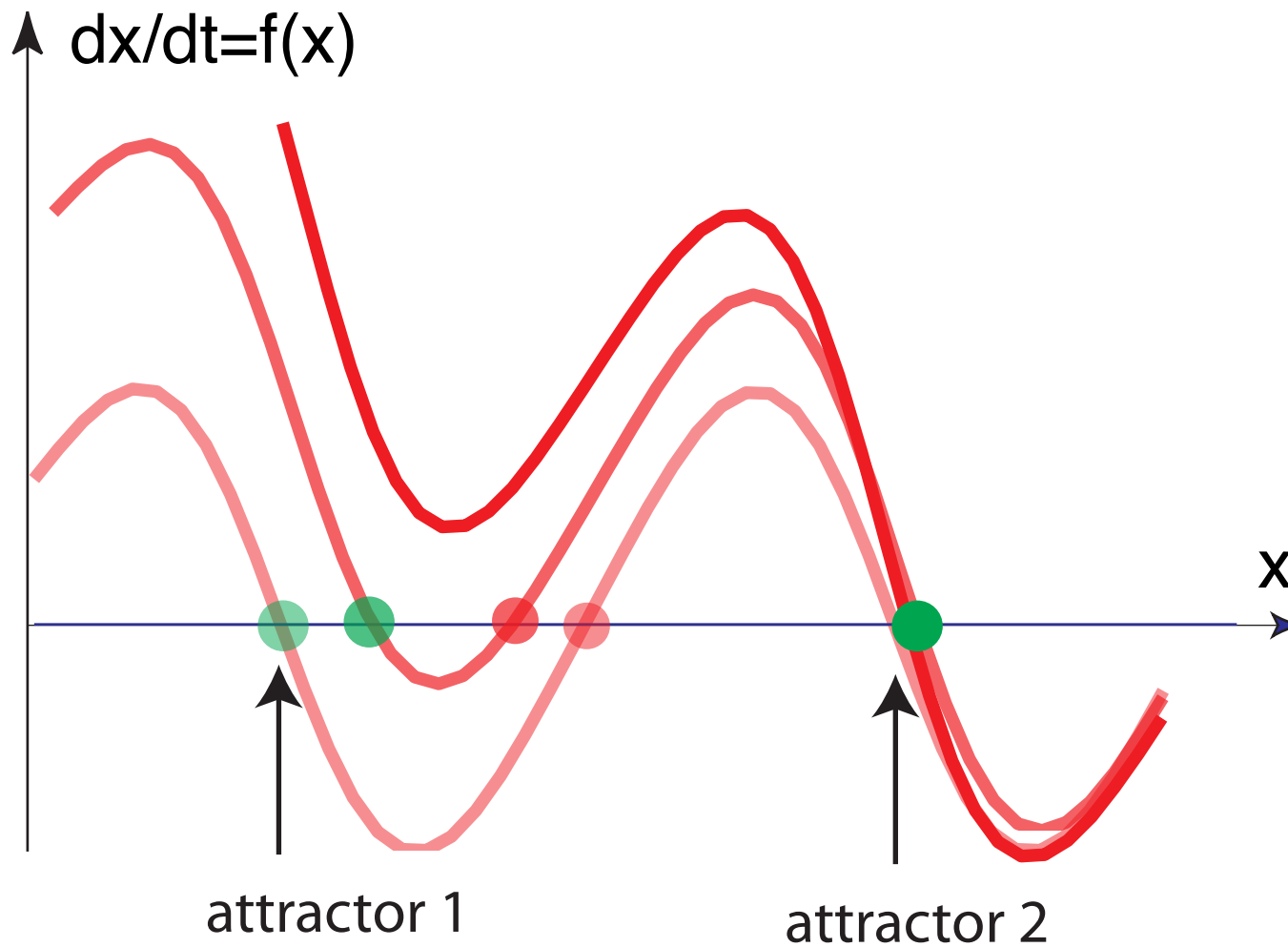


tangent bifurcation

- the simplest bifurcation (co-dimension 0): an attractor collides with a repeller and the two annihilate

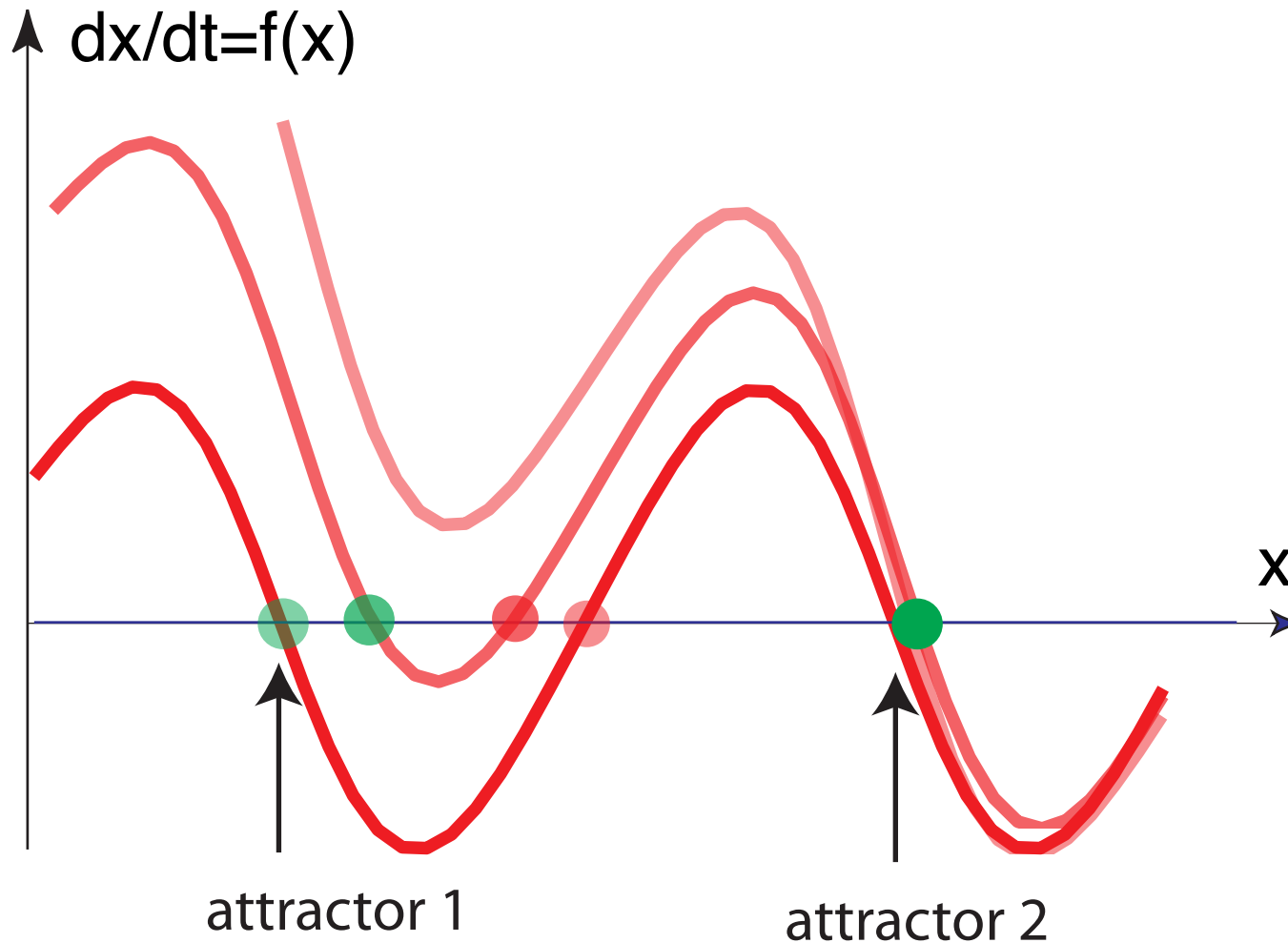


local bifurcation



reverse bifurcation

■ changing the dynamics in the opposite direction



bifurcations are instabilities

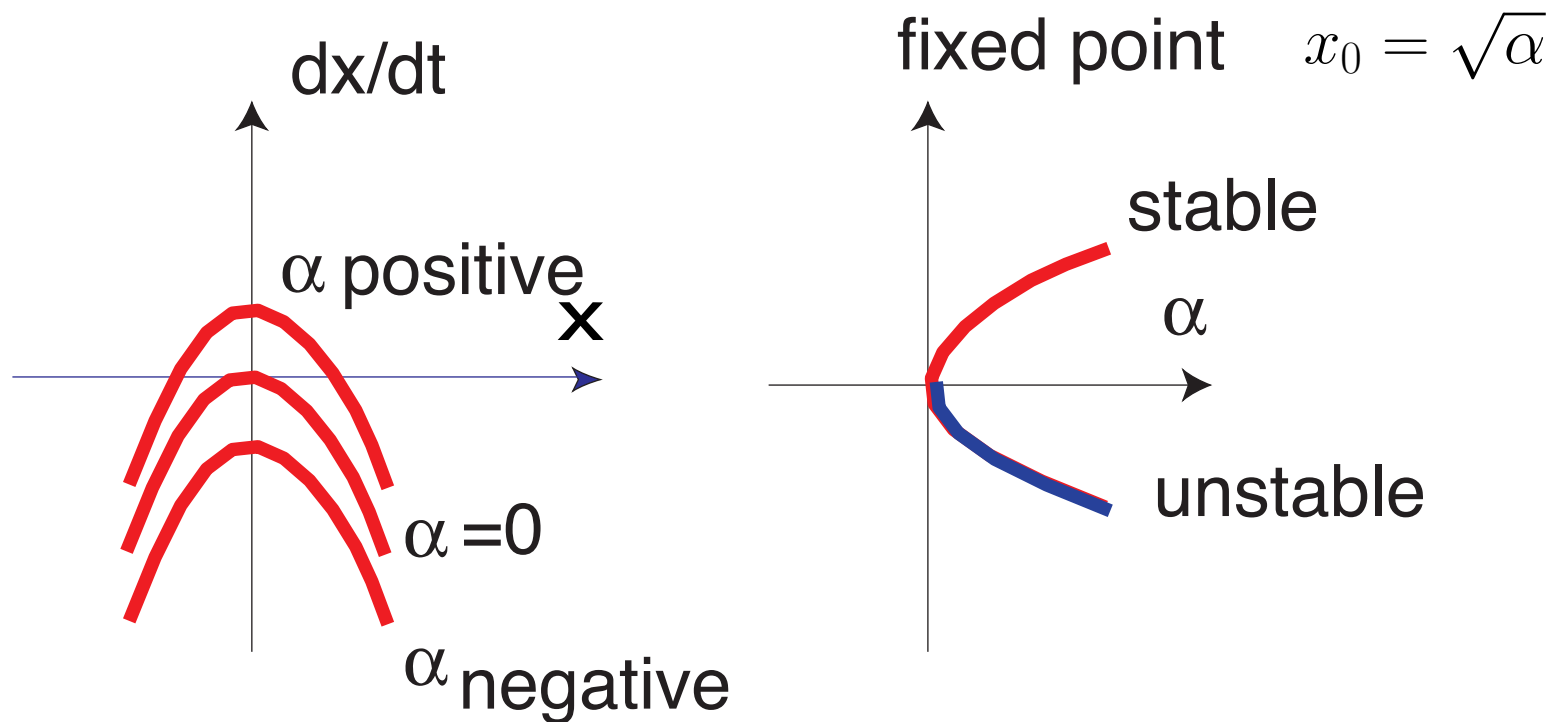
- that is, an attractor becomes unstable before disappearing
- (or the attractor appears with reduced stability)
- formally: a zero-real part is a necessary condition for a bifurcation to occur

tangent bifurcation

- normal form of tangent bifurcation

$$\dot{x} = \alpha - x^2$$

- (=simplest polynomial equation whose flow is topologically equivalent to the bifurcation)



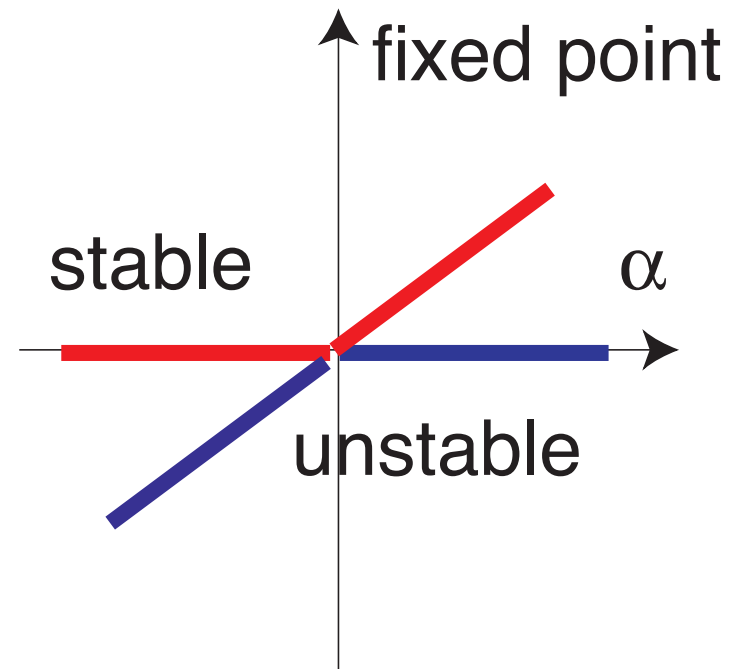
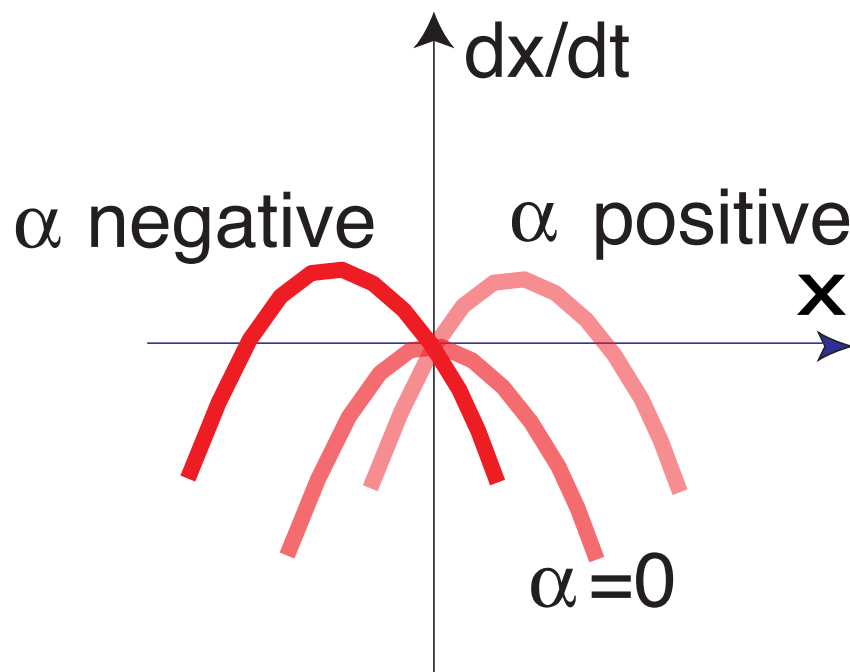
Hopf theorem

- when a single (or pair of complex conjugate) eigenvalue crosses the imaginary axis, one of four bifurcations occur
 - tangent bifurcation
 - transcritical bifurcation
 - pitchfork bifurcation
 - Hopf bifurcation

transcritical bifurcation

■ normal form

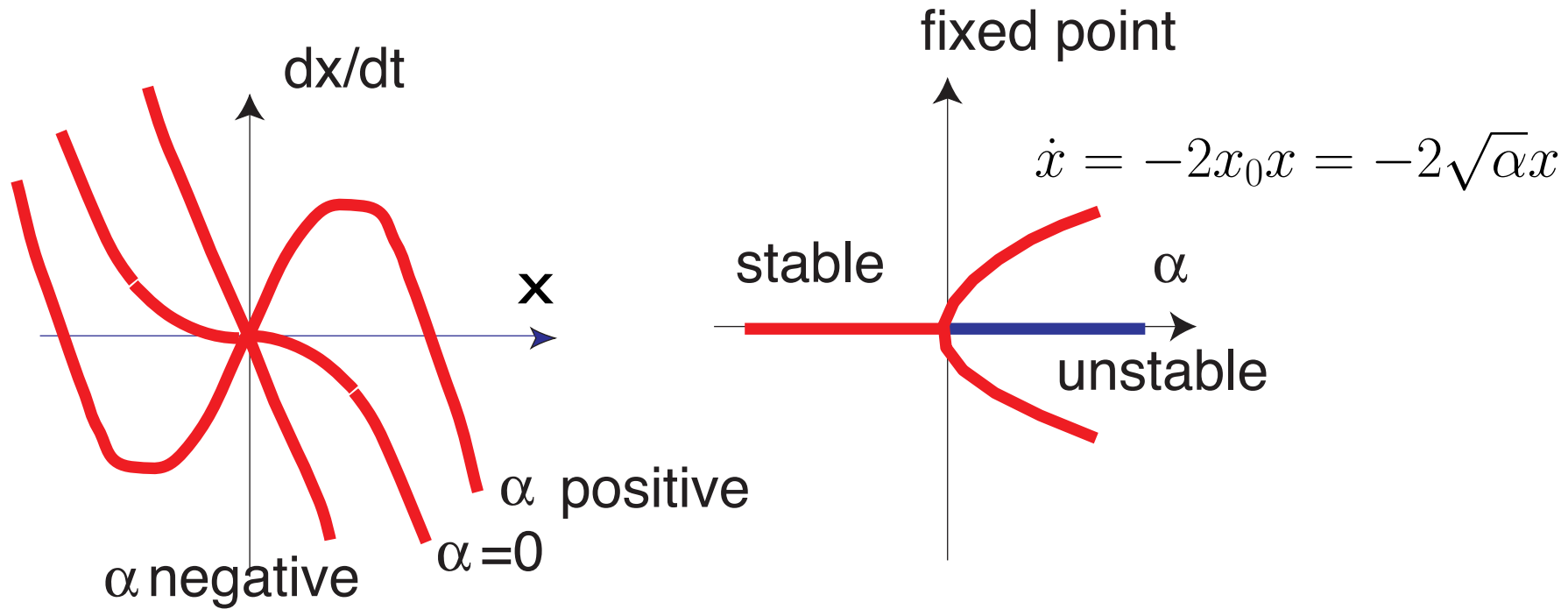
$$\dot{x} = \alpha x - x^2$$



pitchfork bifurcation

■ normal form

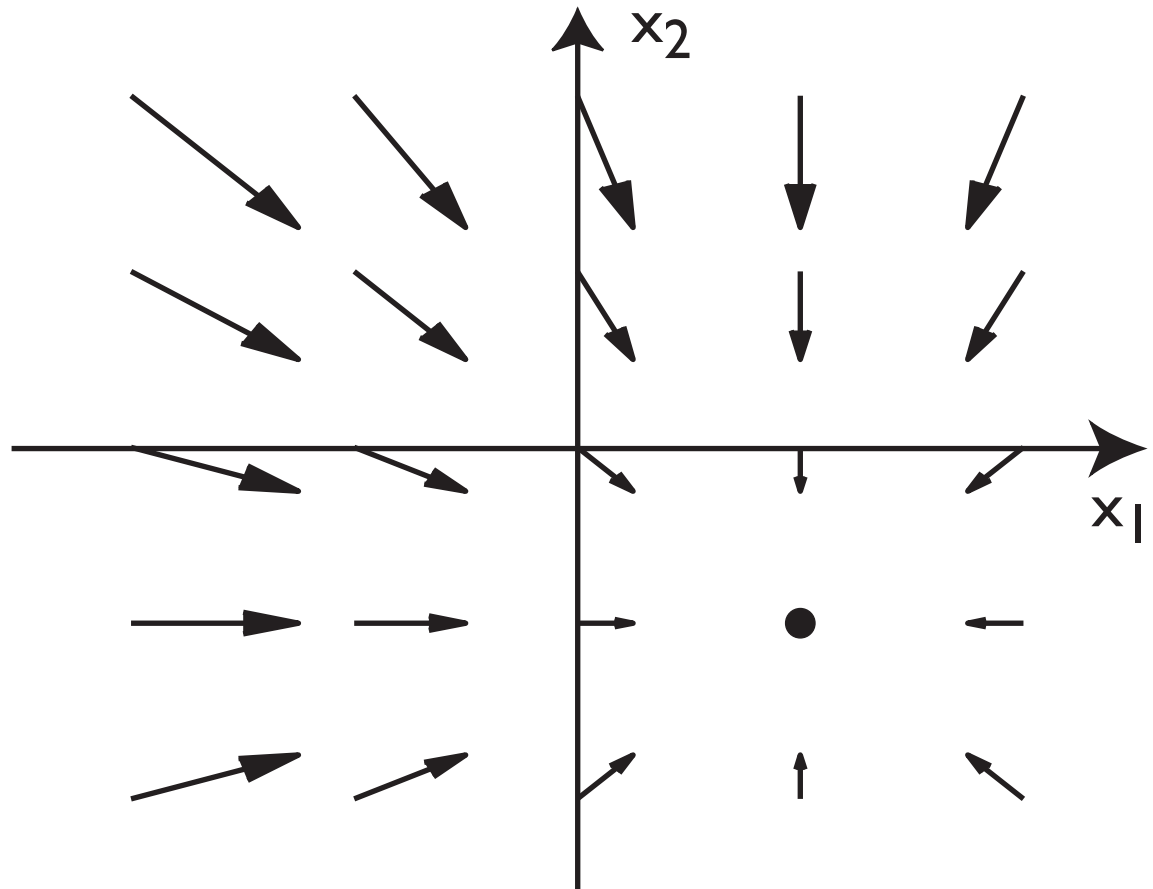
$$\dot{x} = \alpha x - x^3$$



Hopf: need higher dimensions

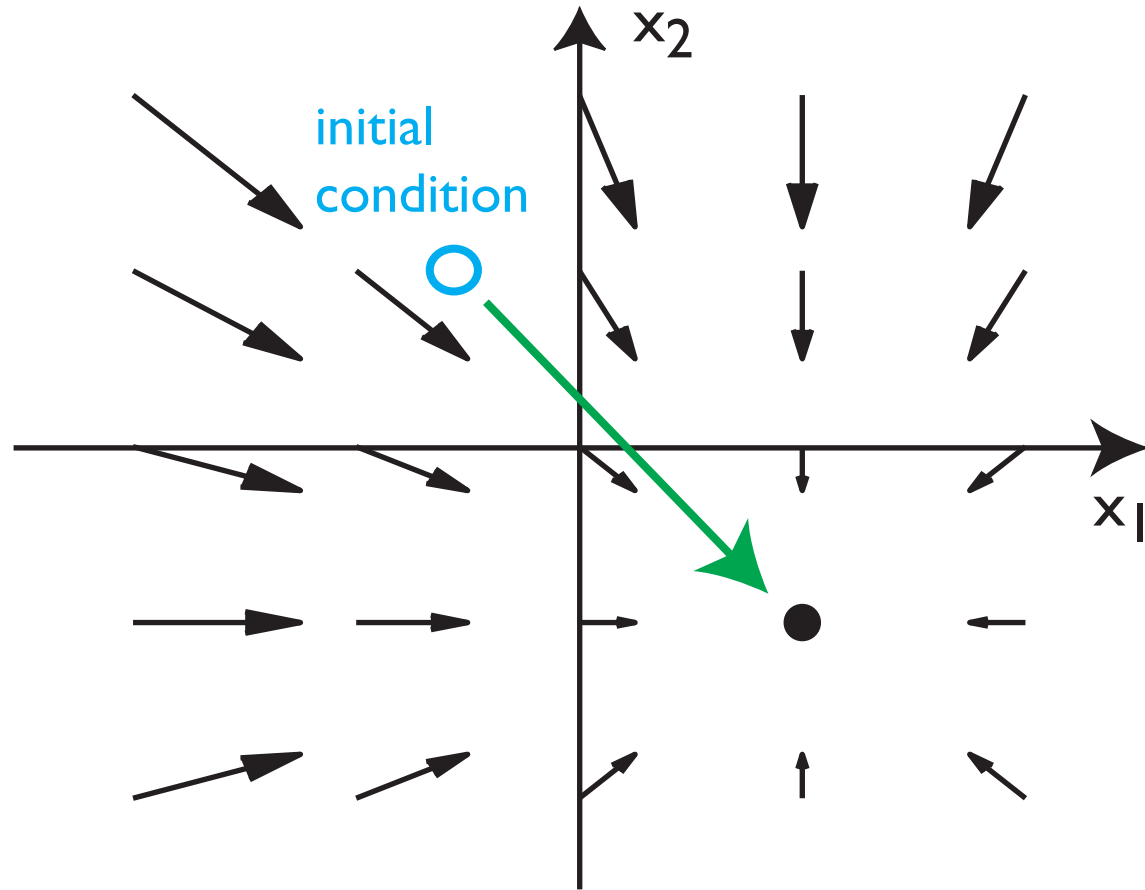
2D dynamical system: vector-field

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$



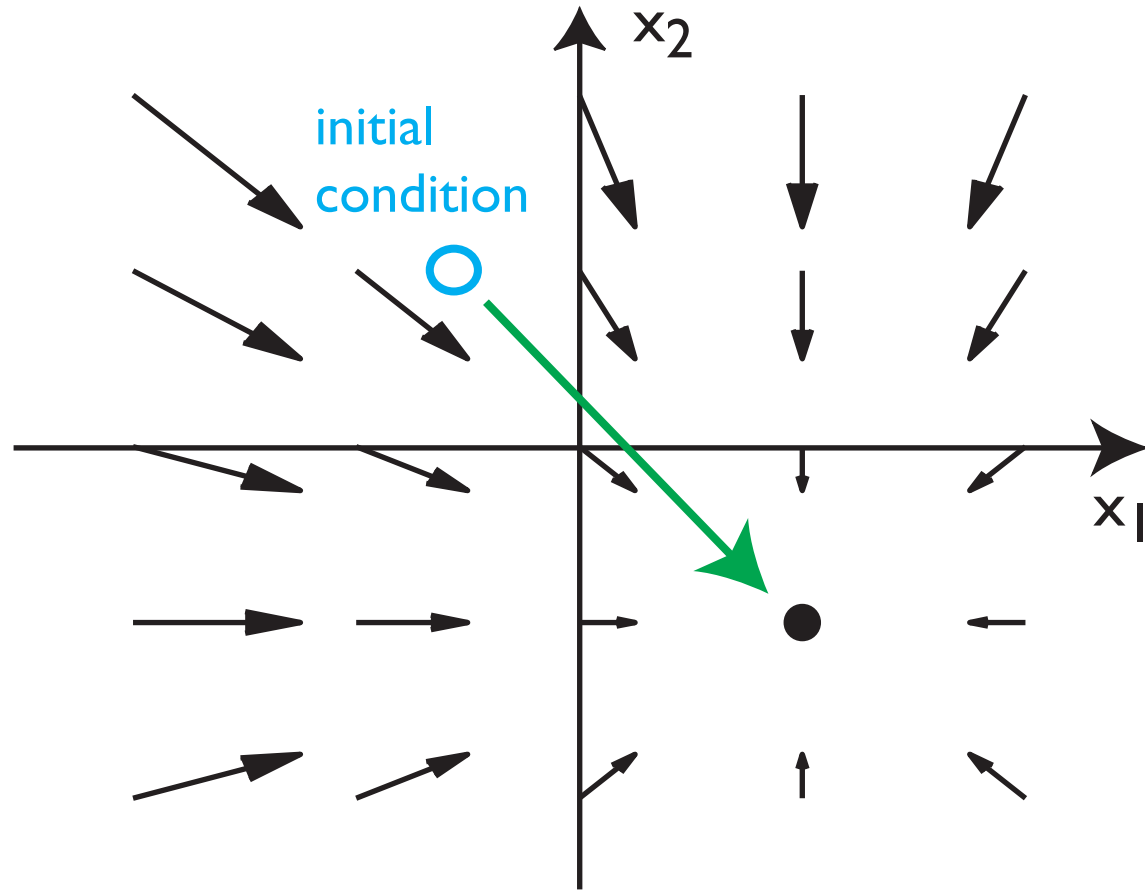
vector-field

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$



fixed point, stability, attractor

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

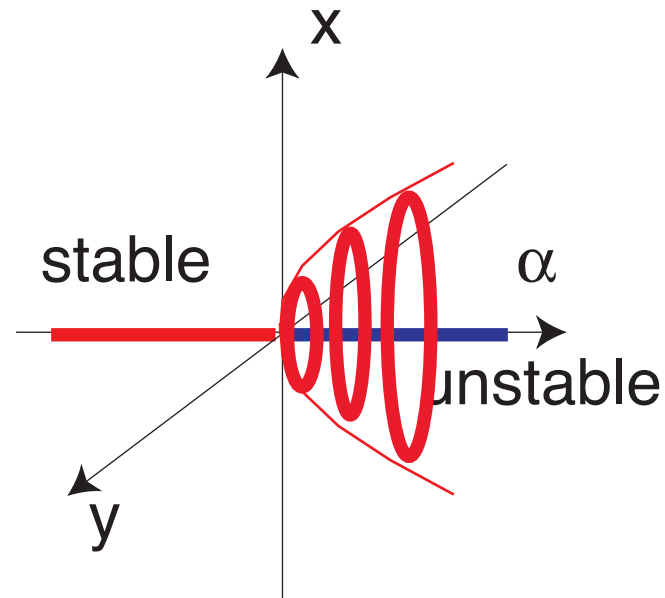
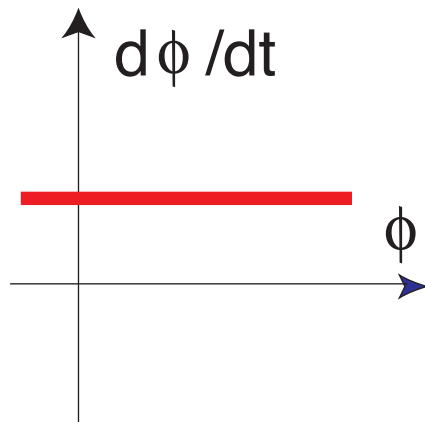
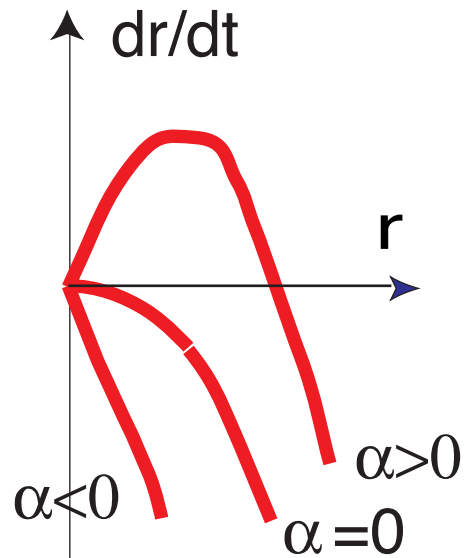


Hopf bifurcation

$$\dot{r} = \alpha r - r^3$$

$$\dot{\phi} = \omega$$

■ normal form



forward dynamics

- given known equation, determined fixed points / limit cycles and their stability
- more generally: determine invariant solutions (stable, unstable and center manifolds)

inverse dynamics

- given solution, find the equation...
- this is the problem faced in design of behavioral dynamics...

inverse dynamics: design

- in the design of behavioral dynamics... you may be given:
- attractor solutions/stable states
- and how they change as a function of parameters/conditions
- => identify the class of dynamical systems using the 4 elementary bifurcations
- and use normal form to provide an exemplary representative of the equivalence class of dynamics