Dynamical systems tutorial:
1. Basic concepts

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Dynamical systems: Tutorial

- the word "dynamics"
  - time-varying measures
  - range of a quantity
  - forces causing/accounting for movement => dynamical systems
- dynamical systems are the universal language of science
  - physics, engineering, chemistry, theoretical biology, economics, quantitative sociology, ...

time-variation and rate of change

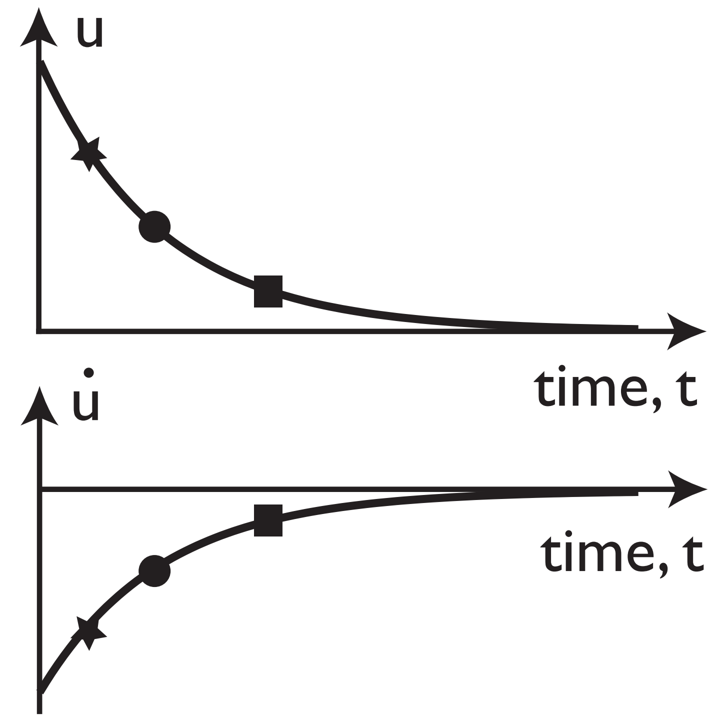
- variable $x(t)$;
- rate of change dx/dt

time-variation and rate of change

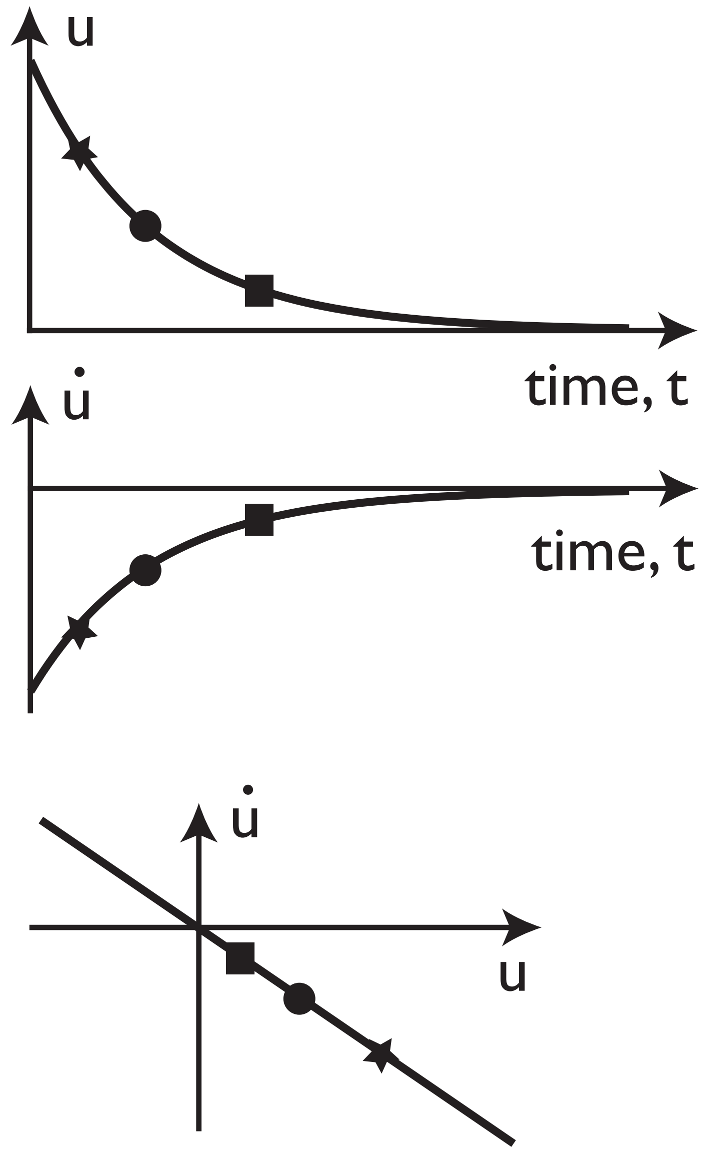
- example:
  - variable $x(t)$ = position
  - rate of change dx/dt = velocity
- example
  - variable $v(t)$ = velocity
  - rate of change ?

time-variation and rate of change

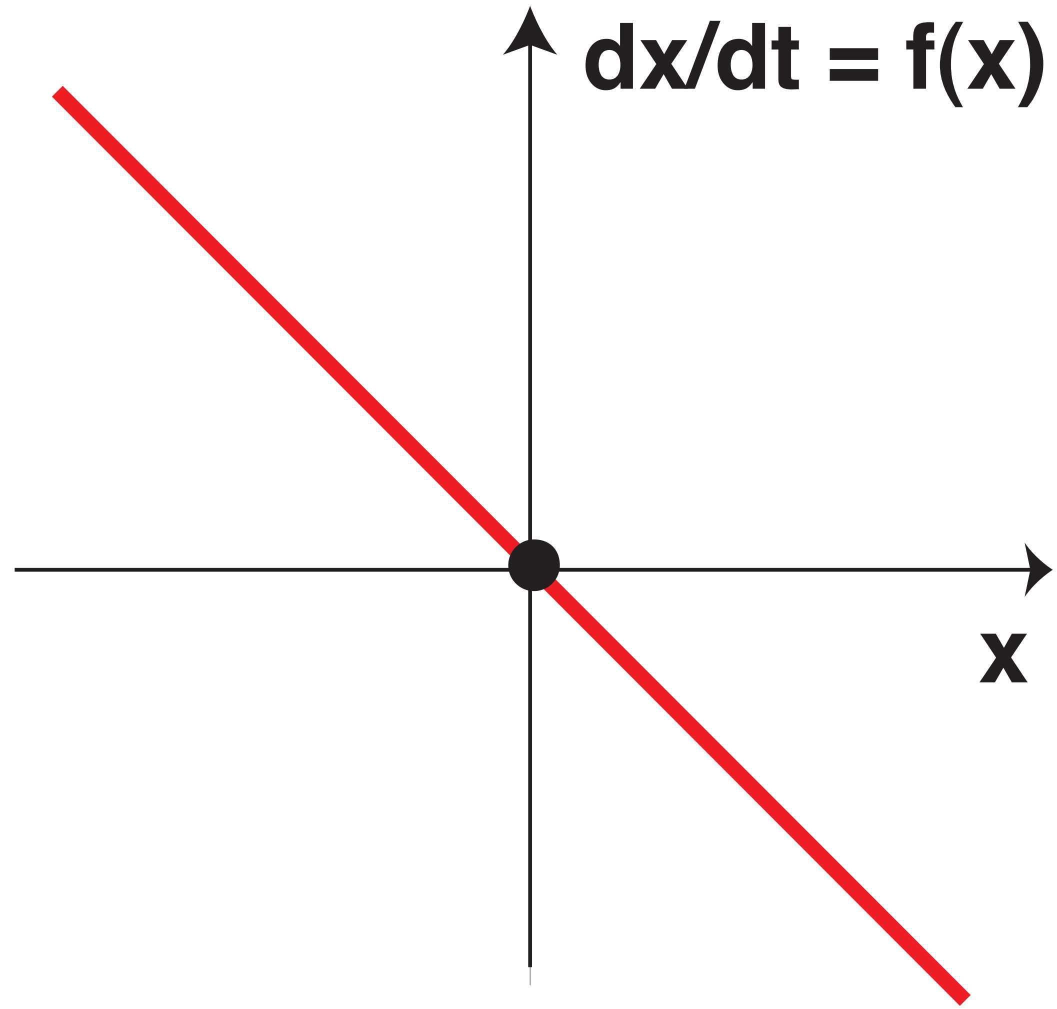
- trajectory: time course of a dynamical variable
- rate of change: slope of the trajectory



dynamical system: relationship between a variable and its rate of change

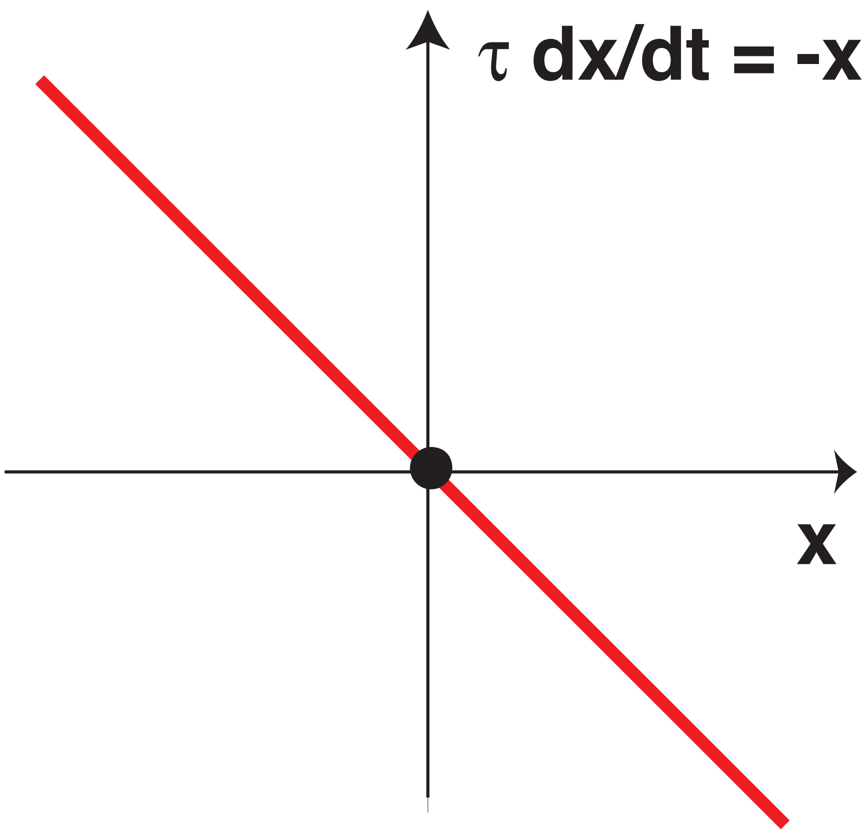


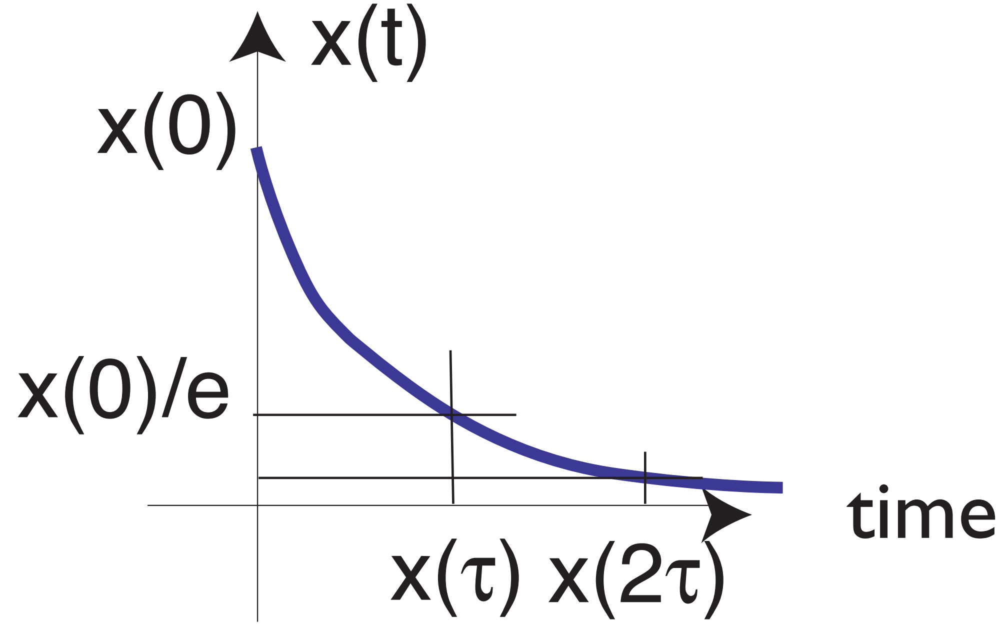
linear dynamical system



solution of linear dynamical systems

$$x(t) = x(0)\exp[-t/\tau]$$





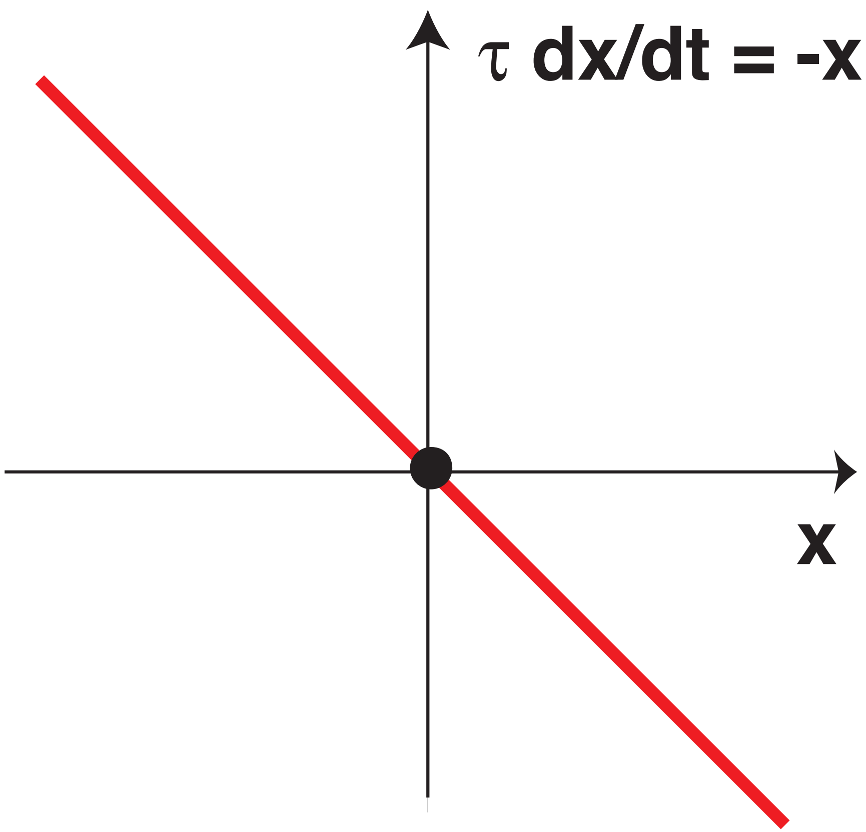
exponential relaxation to attractors

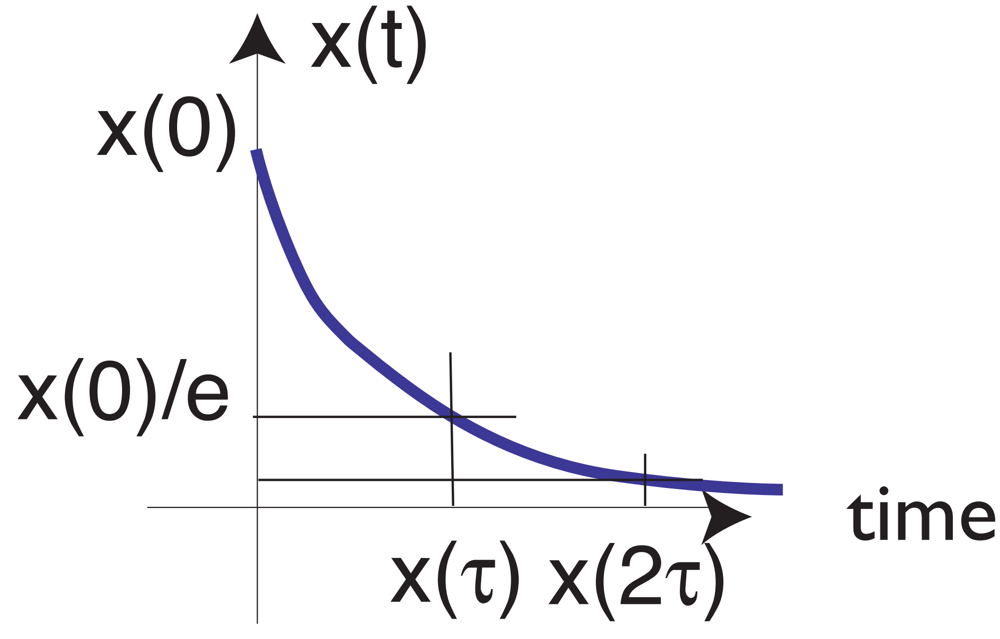
$$x(t) = x(0)\exp[-t/\tau]$$

$$\dot{x}(t) = x(0)\exp[-t/\tau](-1/\tau)$$

$$\tau\dot{x}(t) = -\,x(0)\exp[-t/\tau]$$

$$\tau\dot{x}(t) = -\,x(t)$$

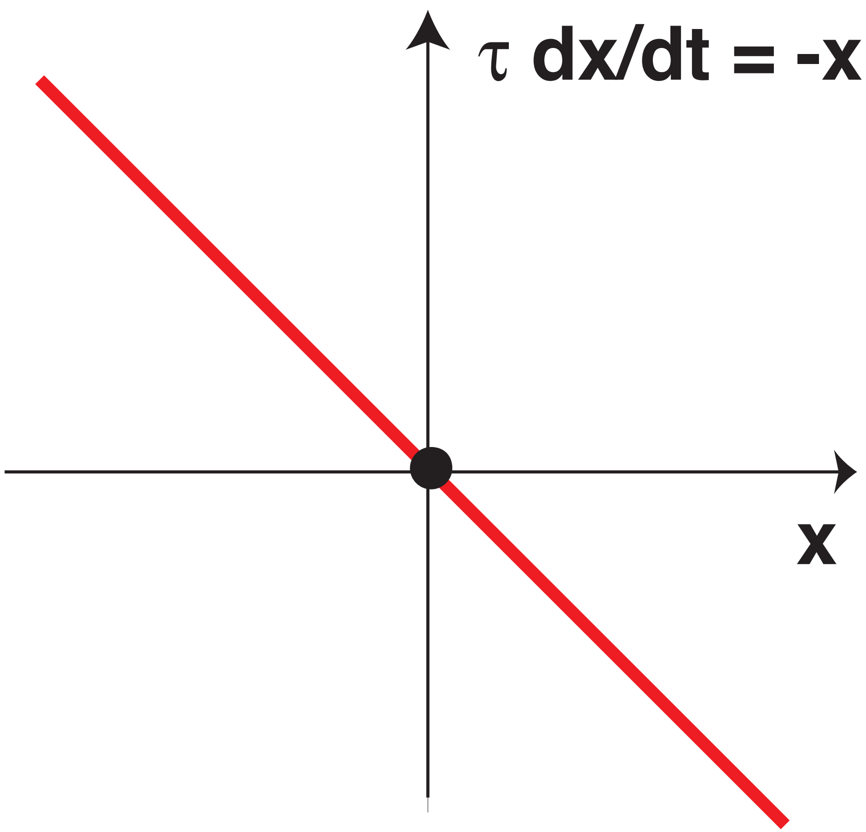


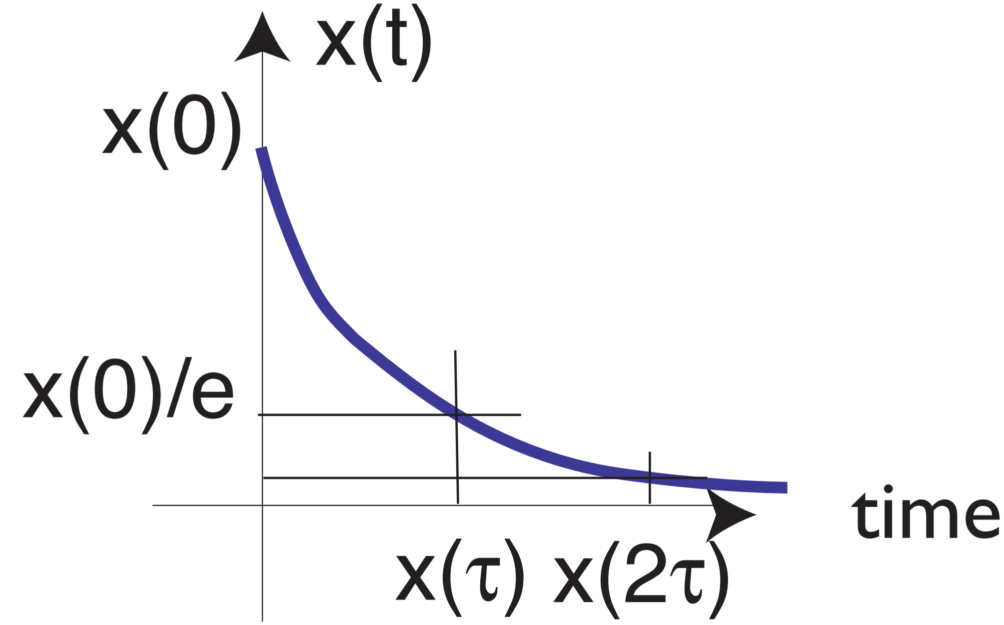


exponential relaxation to attractors

- => time scale

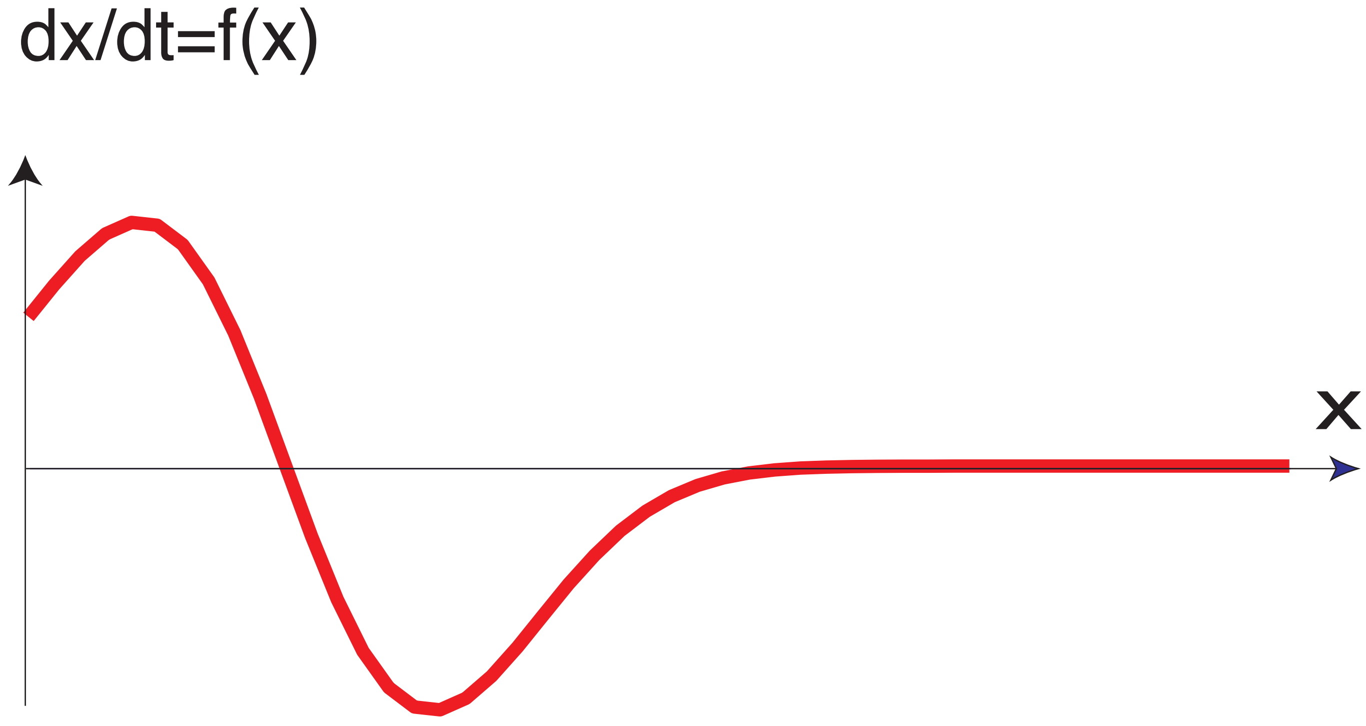
$$x(t) = x(0)\exp[-t/\tau]$$





Dynamical system

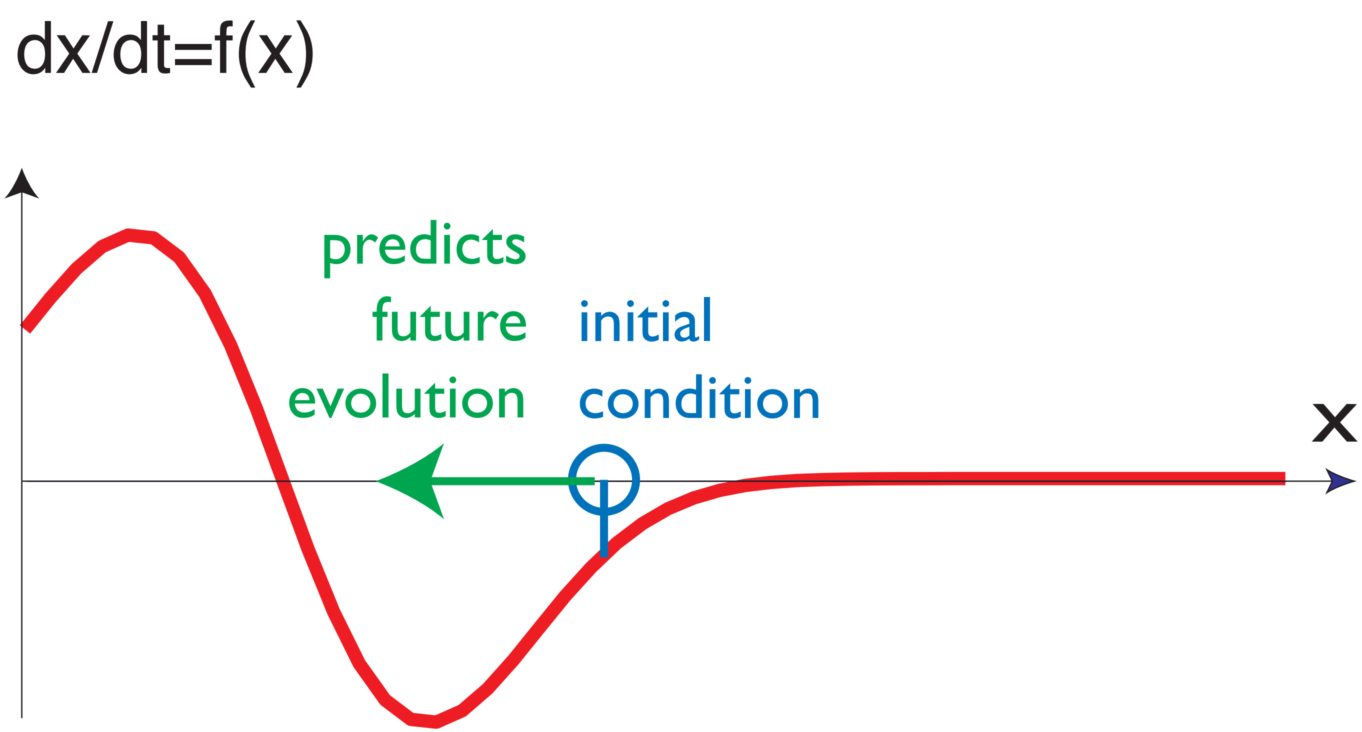
$$\dot{x} = \frac{dx}{dt} = f(x)$$



Dynamical system

$$\dot{x} = \frac{dx}{dt} = f(x)$$

- the present determines the future



Dynamical system

$$\dot{x} = \frac{dx}{dt} = f(x)$$

- x spans the state space (can be vector-valued or even function valued)
- $f(x)$ is the "dynamics" of x (or vector-field)
- $x(t)$ is a solution of the dynamical systems with initial condition $x_0 \iff$ the rate of change of $x(t)$ obeys $\dot{x}(t) = f(x)$ and $x(0) = x_0$

Different forms of dynamical systems

- one-dimensional differential equation: initial value determines the future

$$\dot{x} = f(x)$$

Different forms of dynamical systems

- vector-valued differential equation
- a vector of initial states determines the future: systems of differential equations:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \qquad \text{where} \qquad \mathbf{x} = (x_1, x_2, \ldots, x_n)$$

Different forms of dynamical systems

- continuously many variables x(y) determine the future = an initial function x(y) determines the future
  - partial differential equations
  - integro-differential equations

$$\dot{x}(y,t) = f\left(x(y,t), \frac{\partial x(y,t)}{\partial y}, \ldots\right)$$

$$\dot{x}(y,t) = \int dy' g\left(x(y,t), x(y',t)\right)$$

Different forms of dynamical systems

- a piece of past trajectory determines the future
  - delay differential equations
  - functional differential equations

$$\dot{x}(t) = f(x(t-\tau))$$

$$\dot{x}(t) = \int^{t} dt' f(x(t'))$$

Numerical solutions

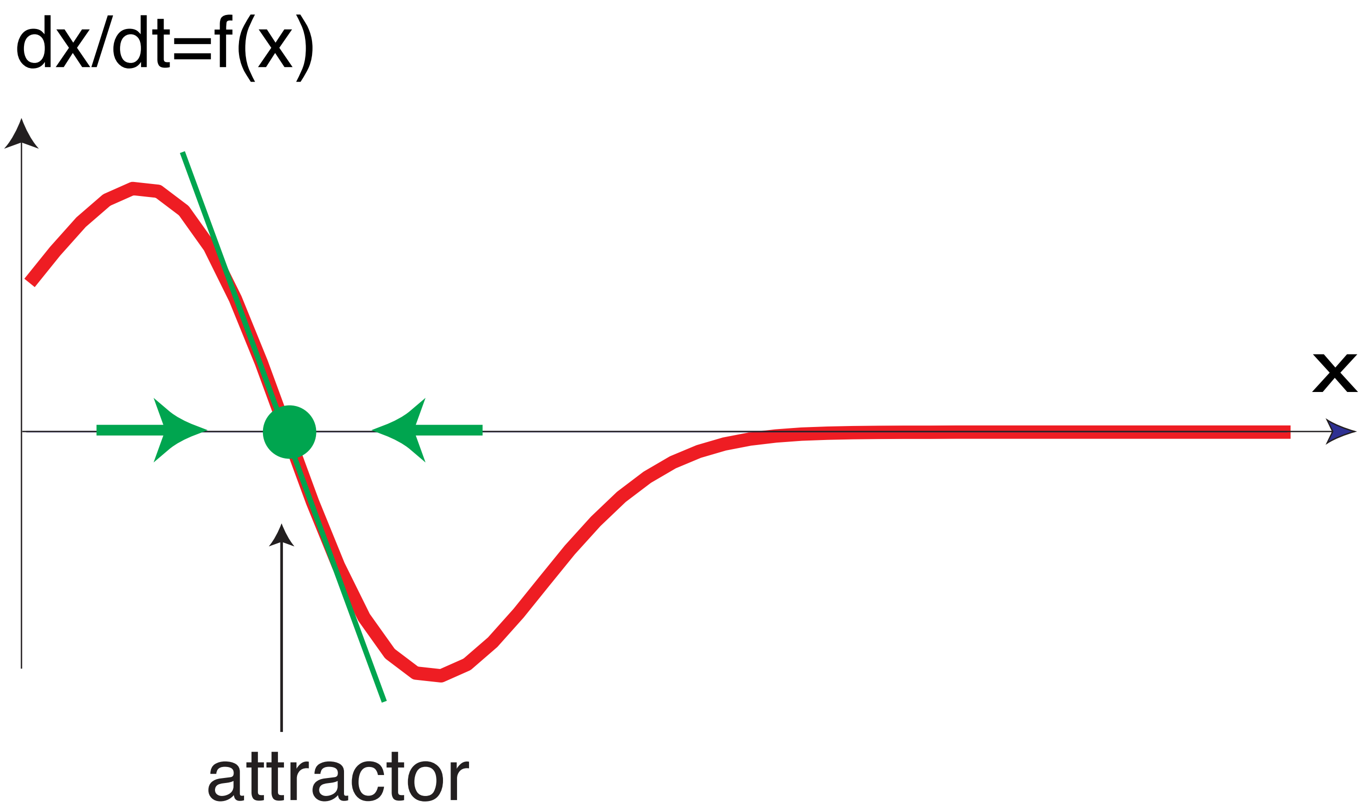
- sample time discretely, t_i, with $i \in \{0, 1, \ldots, N\}$,
- for example: $t_i = i\ \Delta t$
- compute solution, $x(t_i) = x_i$, by iterating through time,
- for example: $x_{i+1} = x_i + \Delta t\ f(x_i)$ (forward Euler)

$$\left[\frac{x_{i+1} - x_i}{\Delta t} \approx \frac{dx}{dt} = f(x) \approx f(x_i)\right]$$

=> code / simulation

attractor

- fixed point, to which neighboring initial conditions converge = attractor



fixed point

- is a constant solution of the dynamical system
- $$\dot{x} = f(x)$$

$$\dot{x} = 0 \Rightarrow f(x_0) = 0$$

stability

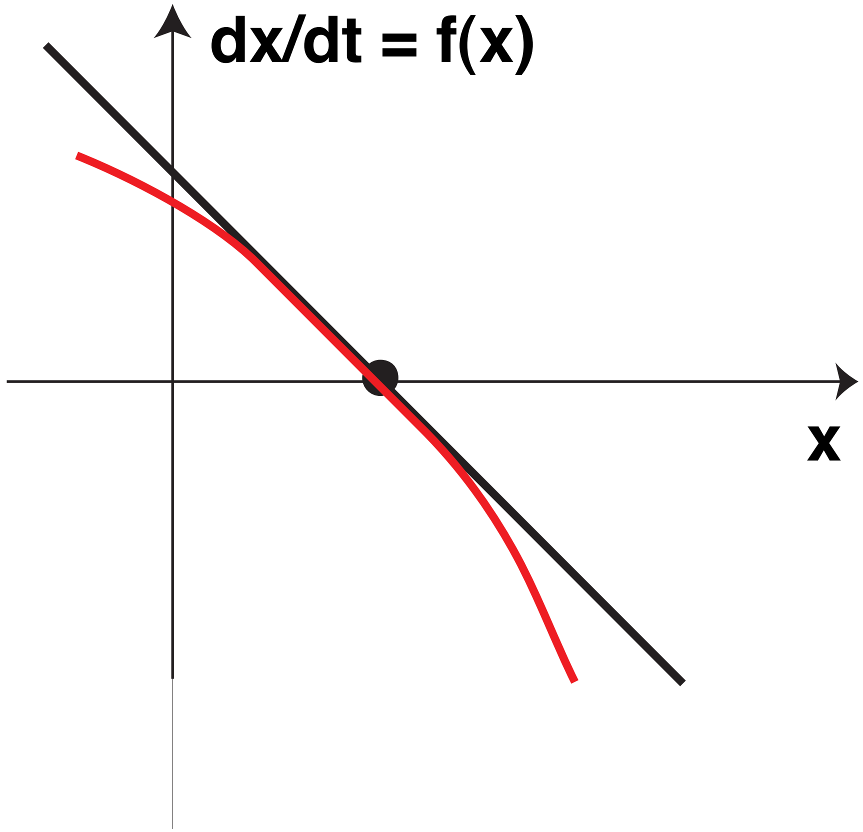
- mathematically really: asymptotic stability
- defined: a fixed point is asymptotically stable, when solutions of the dynamical system that start nearby converge in time to the fixed point

stability

- the mathematical concept of stability (which we do not use) requires only that nearby solutions stay nearby
- Definition: a fixed point is unstable if it is not stable in that more general sense,
    - that is: if nearby solutions do not necessarily stay nearby (may diverge)

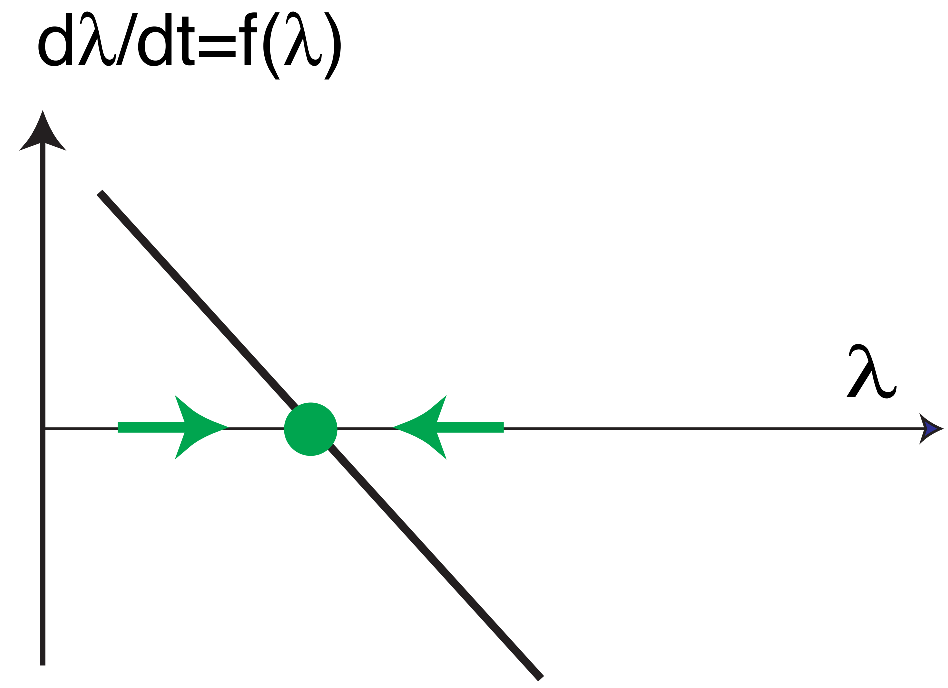
linear approximation near attractor

- non-linearity as a small perturbation/ deformation of linear system
- => non-essential non-linearity



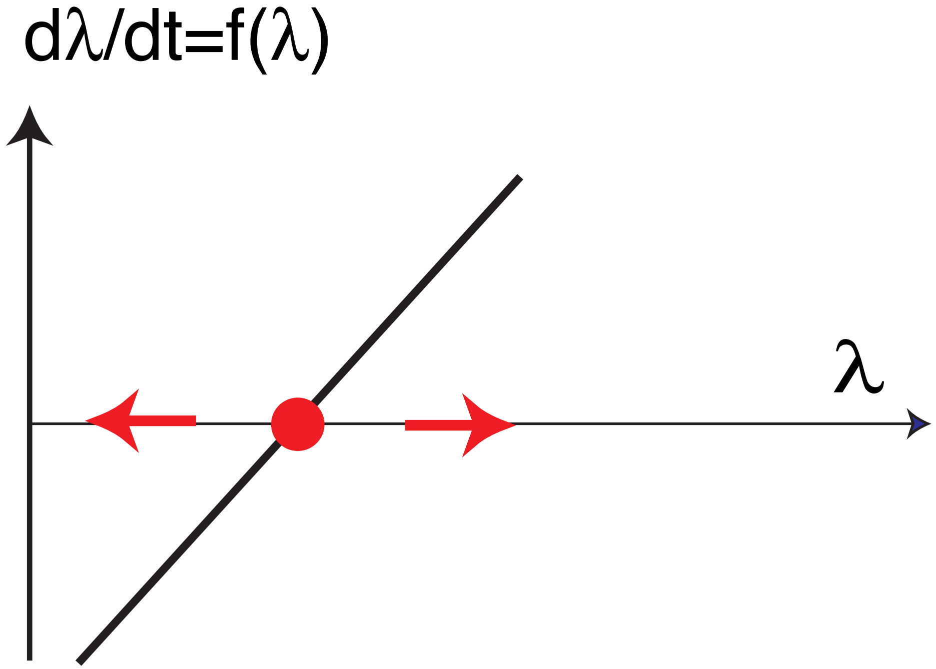
stability in a linear system

- if the slope of the linear system is negative, the fixed point is (asymptotically stable)



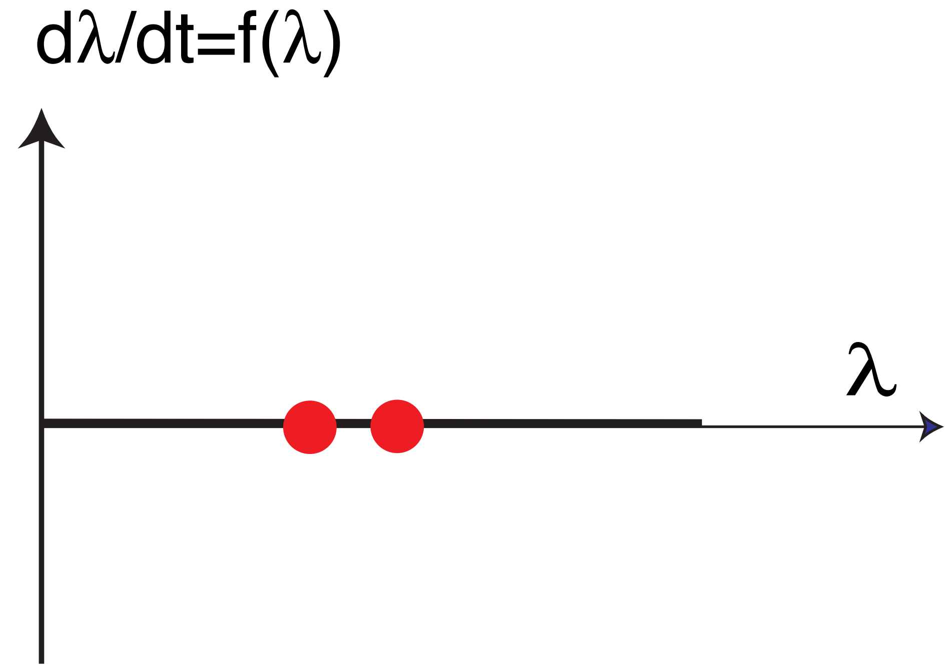
stability in a linear system

- if the slope of the linear system is positive, then the fixed point is unstable



stability in a linear system

- if the slope of the linear system is zero, then the system is indifferent (marginally stable: stable but not asymptotically stable)



stability in linear systems

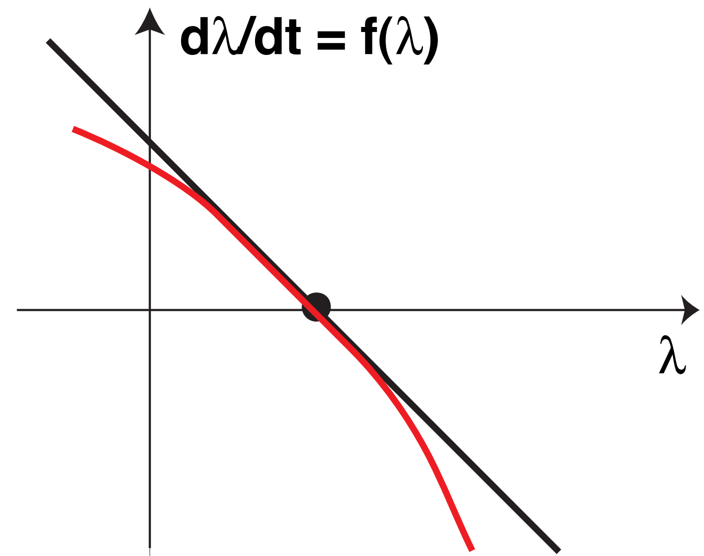
- generalization to multiple dimensions
  - if the real-parts of all Eigenvalues are negative: stable
  - if the real-part of any Eigenvalue is positive: unstable
  - if the real-part of any Eigenvalue is zero: marginally stable in that direction (stability depends on other eigenvalues)

stability in nonlinear systems

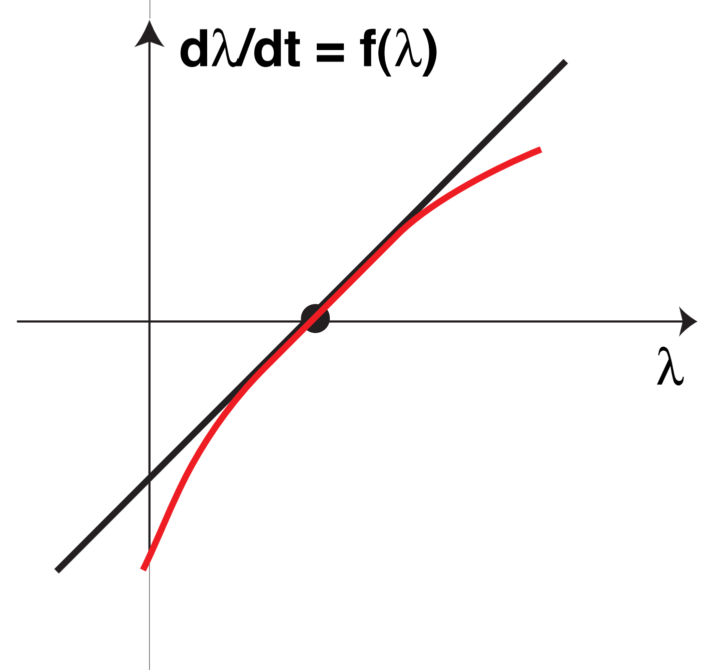
- stability is a local property of the fixed point
- => linear stability theory
  - the eigenvalues of the linearization around the fixed point determine stability
  - all real-parts negative: stable
  - any real-part positive: unstable
  - any real-part zero: undecided: now nonlinearity decides (non-hyberpolic fixed point)

stability in nonlinear systems

- all real-parts negative: stable

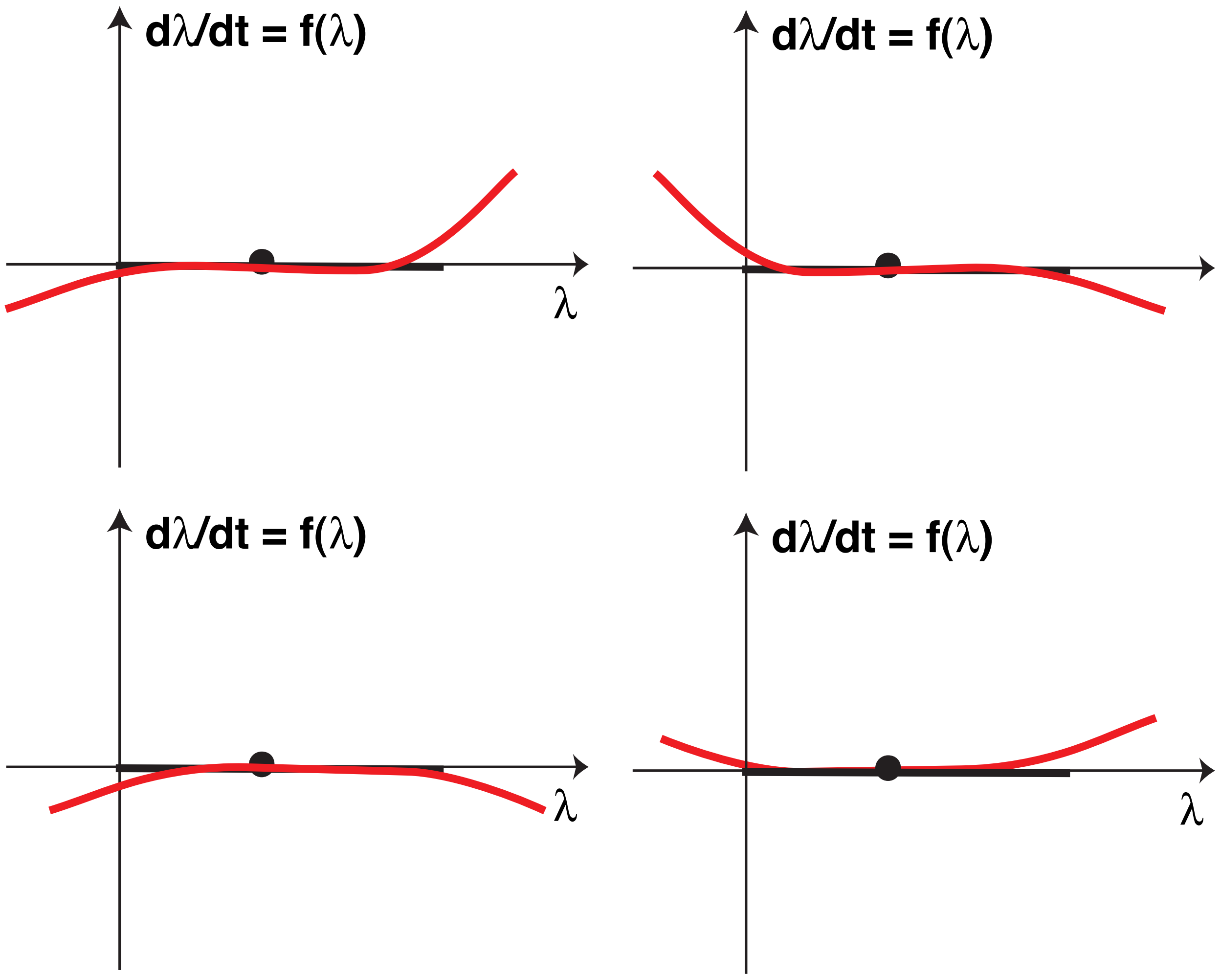


- any real-part positive: unstable



stability in nonlinear systems

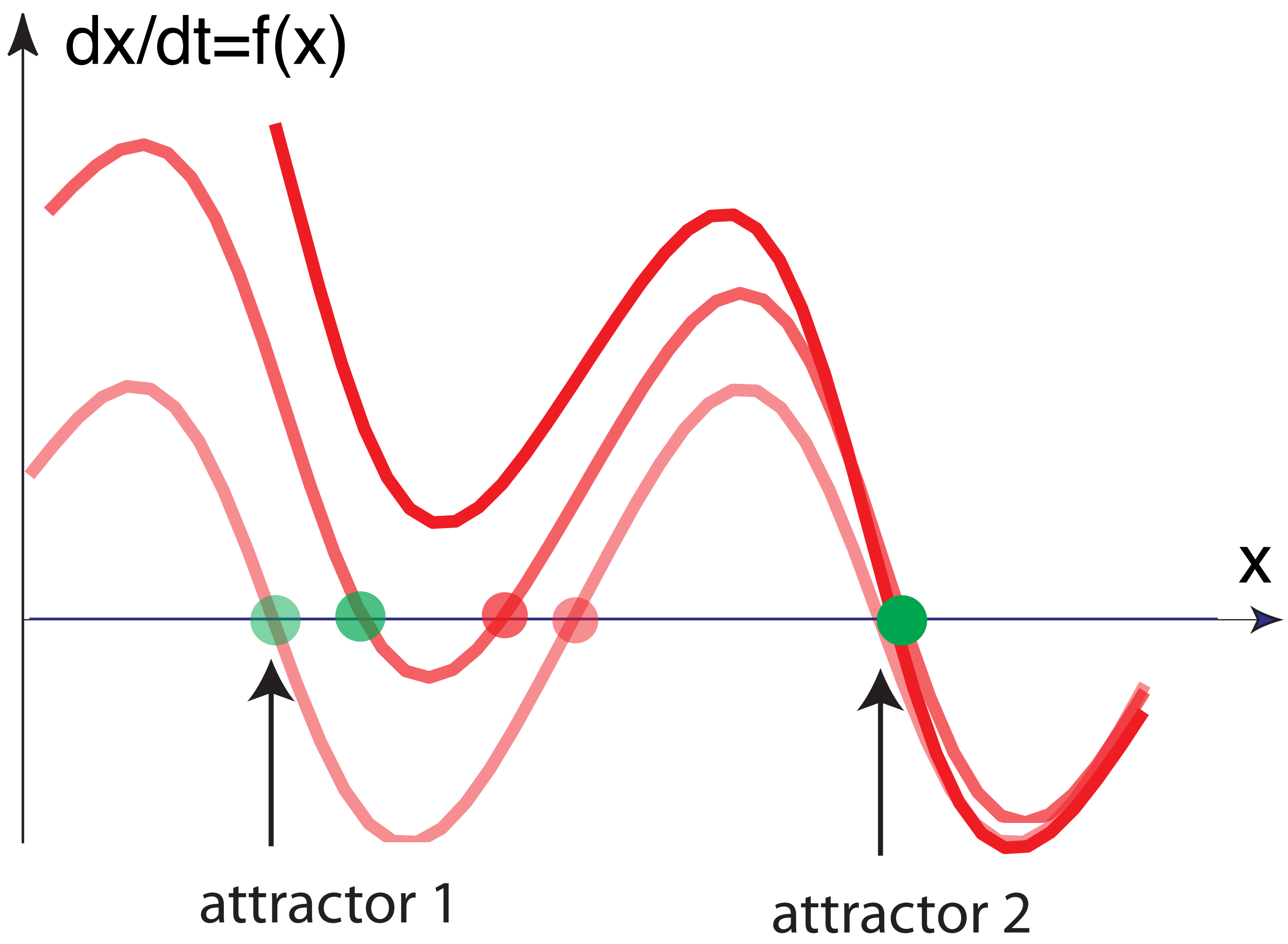
- any real-part zero: undecided: now nonlinearity decides (non-hyberpolic fixed point)



bifurcations

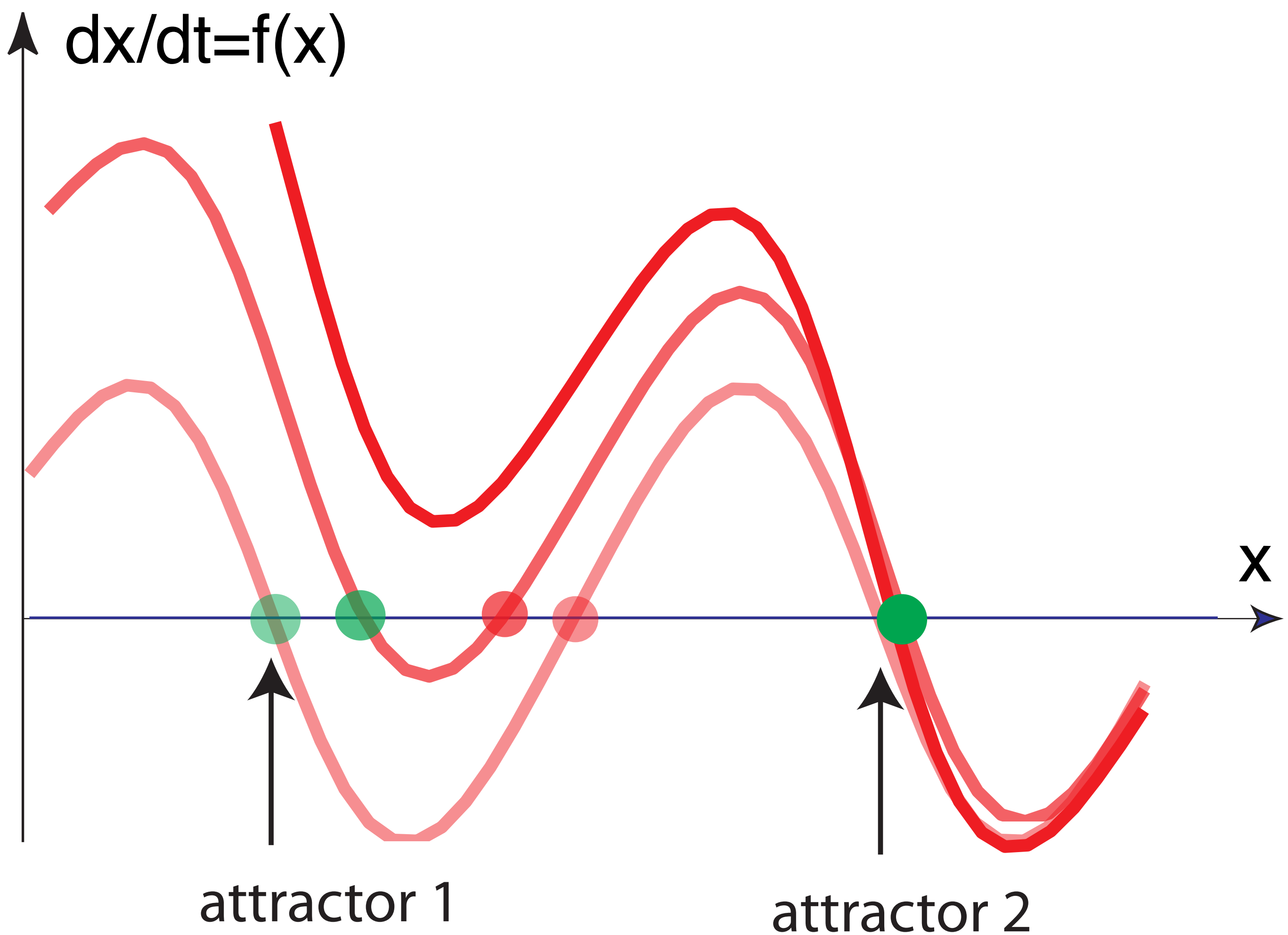
- look now at families of dynamical systems, which depend (smoothly) on parameters
- ask: as the parameters change (smoothly), how do the solutions change (smoothly?)
  - smoothly: topological equivalence of the dynamical systems at neighboring parameter values
  - bifurcation: dynamical systems NOT topological equivalent as parameter changes infinitesimally

bifurcation



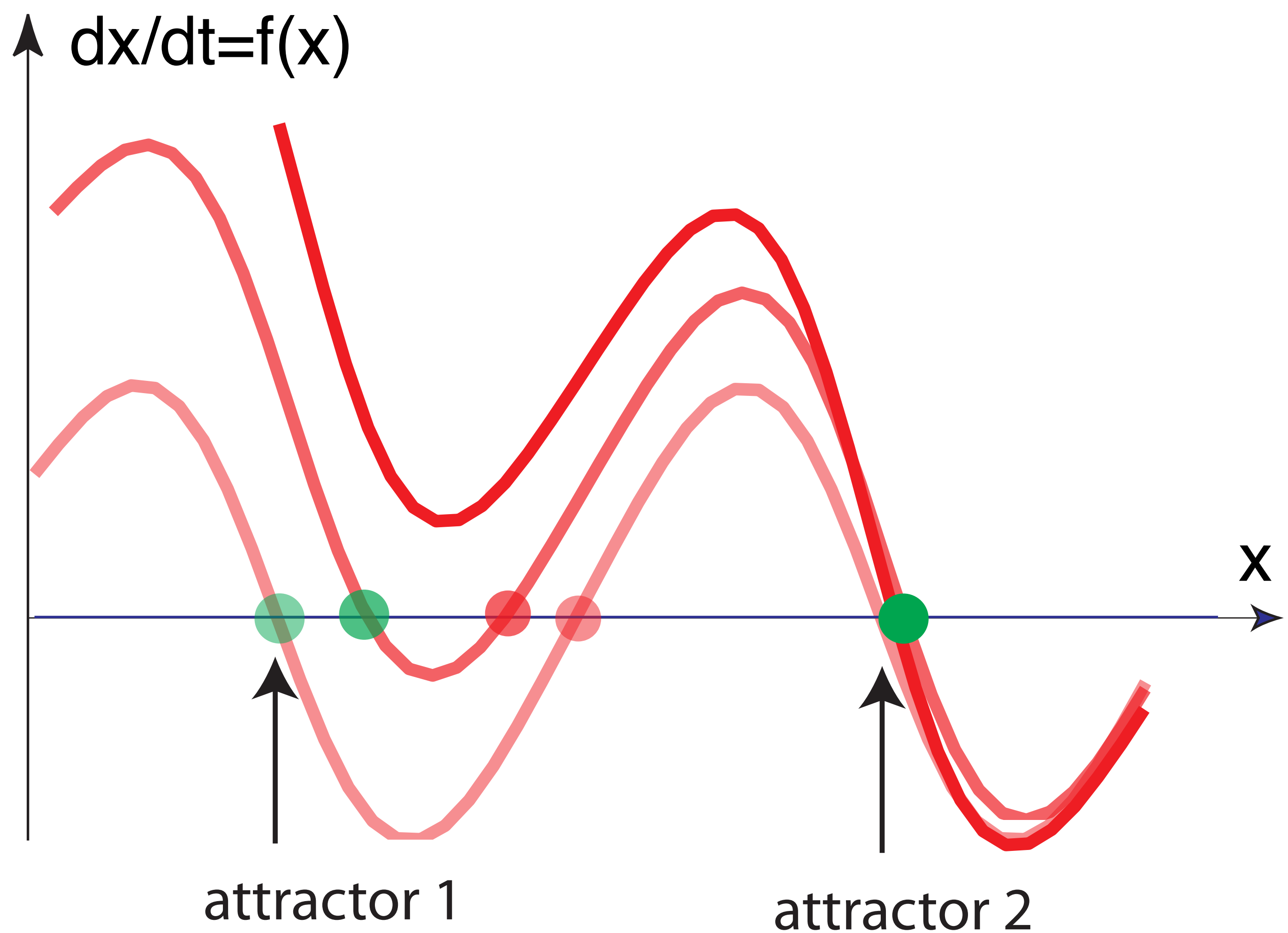
bifurcation

- bifurcation=qualitative change of dynamics (change in number, nature, or stability of fixed points) as the dynamics changes smoothly

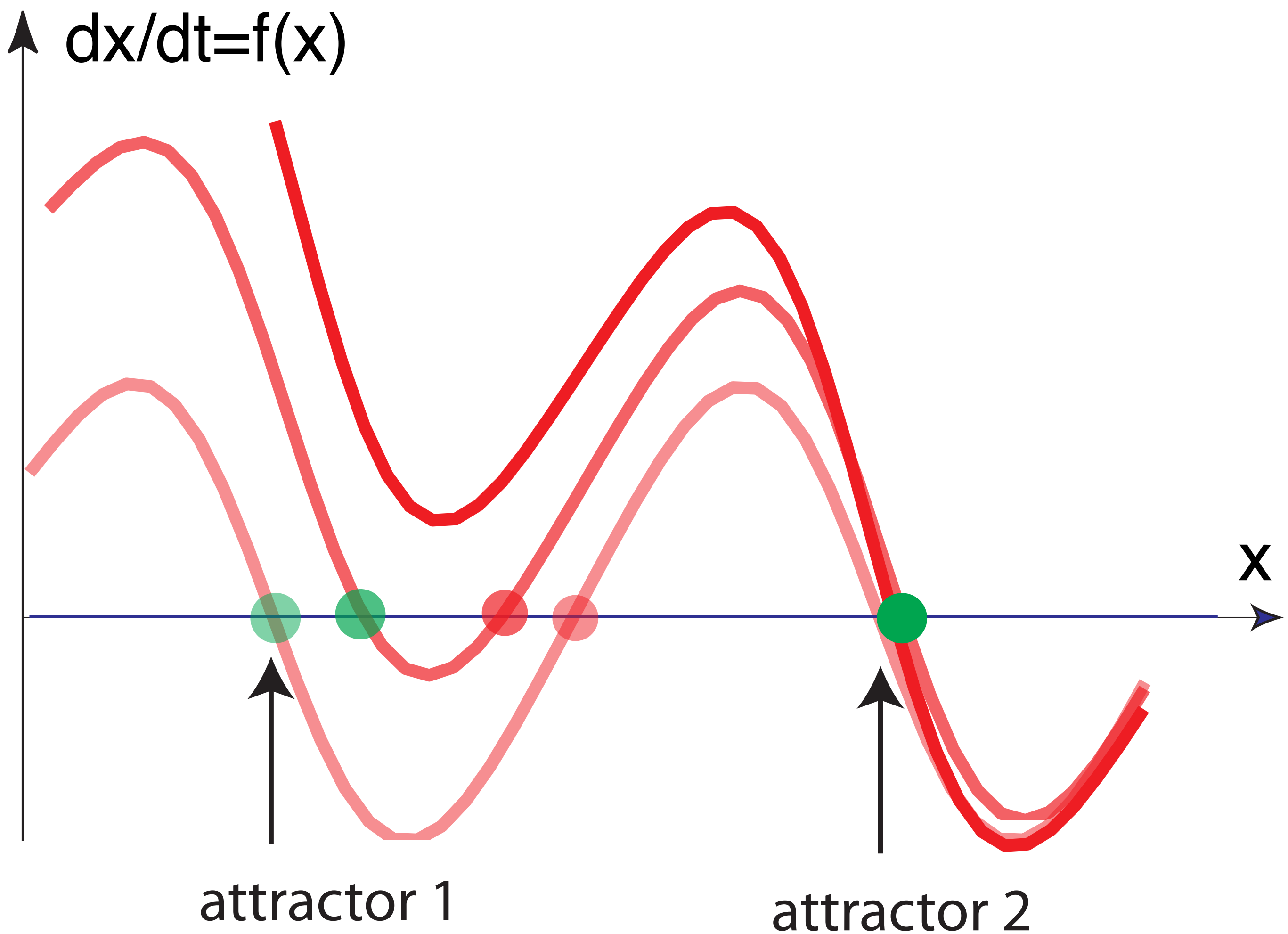


tangent bifurcation

- the simplest bifurcation (co-dimension 0): an attractor collides with a repellor and the two annihilate

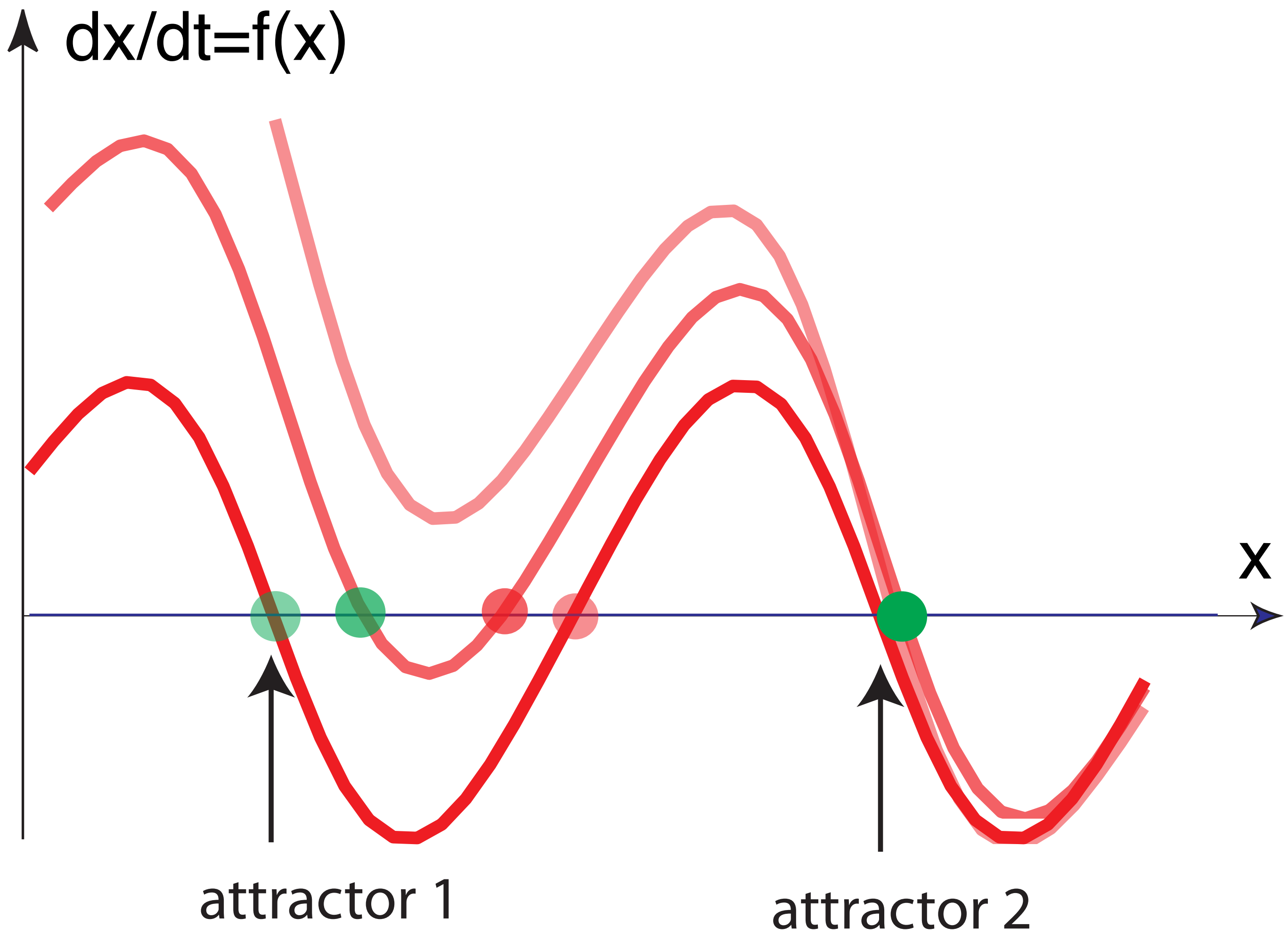


local bifurcation



reverse bifurcation

- changing the dynamics in the opposite direction



bifurcations are instabilities

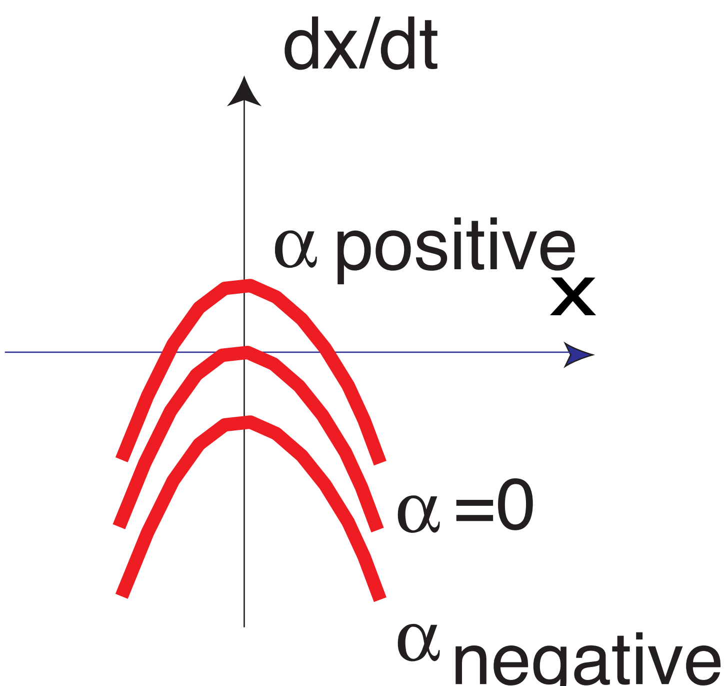
- that is, an attractor becomes unstable before disappearing
- (or the attractor appears with reduced stability)
- formally: a zero-real part is a necessary condition for a bifurcation to occur

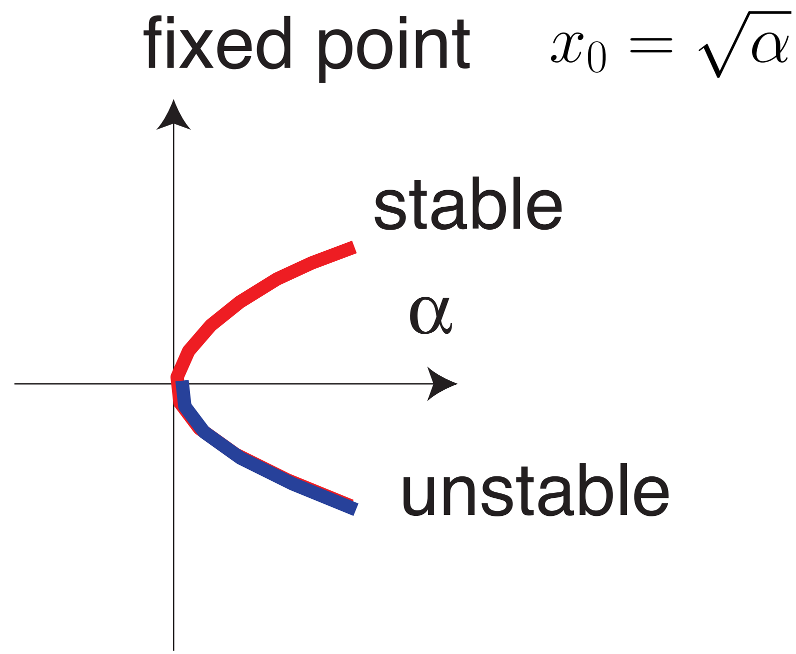
tangent bifurcation

- normal form of tangent bifurcation

  $$\dot{x} = \alpha - x^2$$

- (=simplest polynomial equation whose flow is topologically equivalent to the bifurcation)



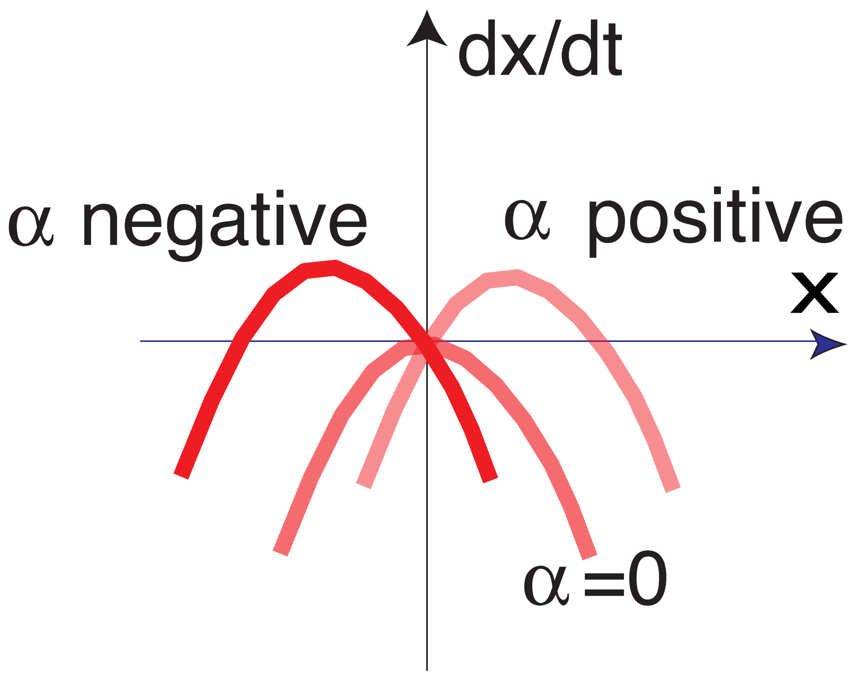


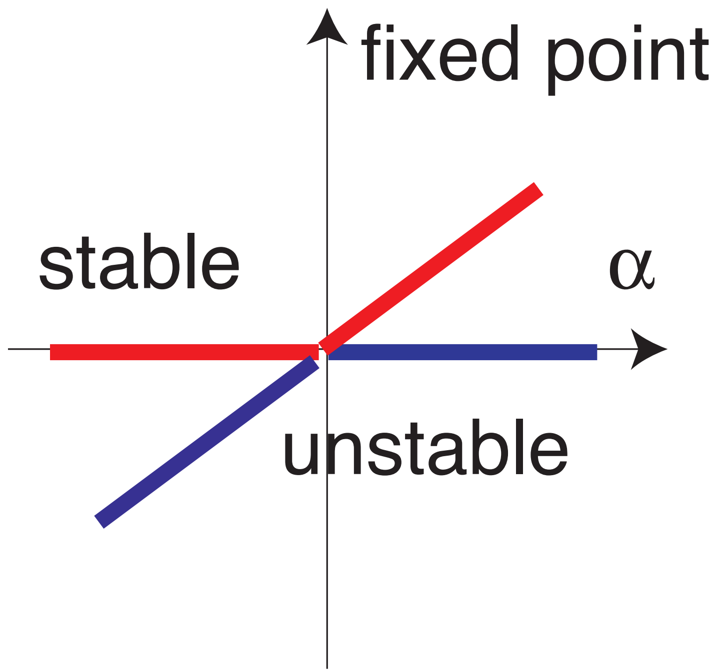
Hopf theorem

- when a single (or pair of complex conjugate) eigenvalue crosses the imaginary axis, one of four bifurcations occur
  - tangent bifurcation
  - transcritical bifurcation
  - pitchfork bifurcation
  - Hopf bifurcation

transcritical bifurcation

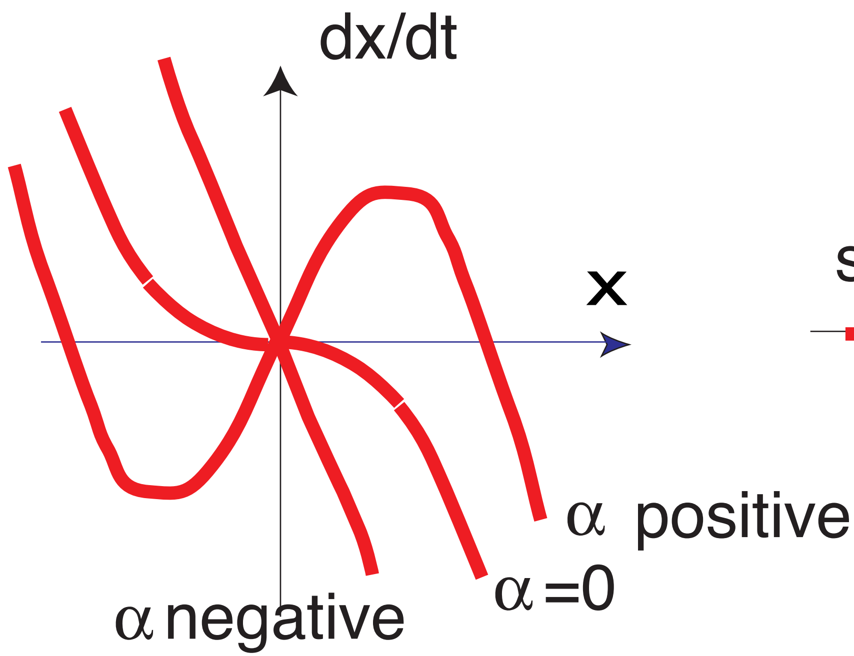
- normal form $$\dot{x} = \alpha x - x^2$$

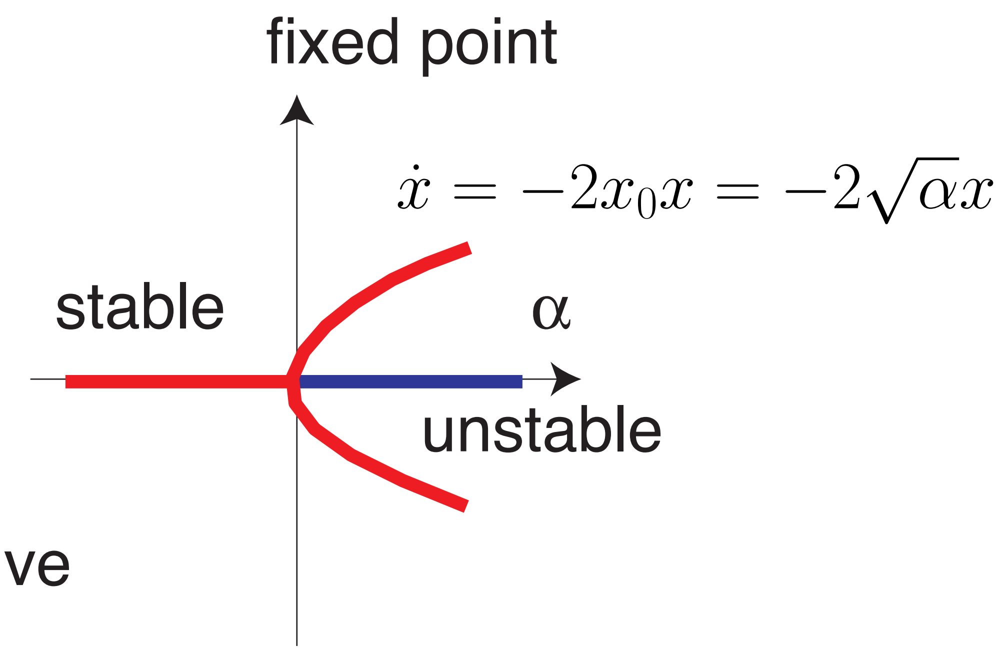




pitchfork bifurcation

- normal form $\dot{x} = \alpha x - x^3$

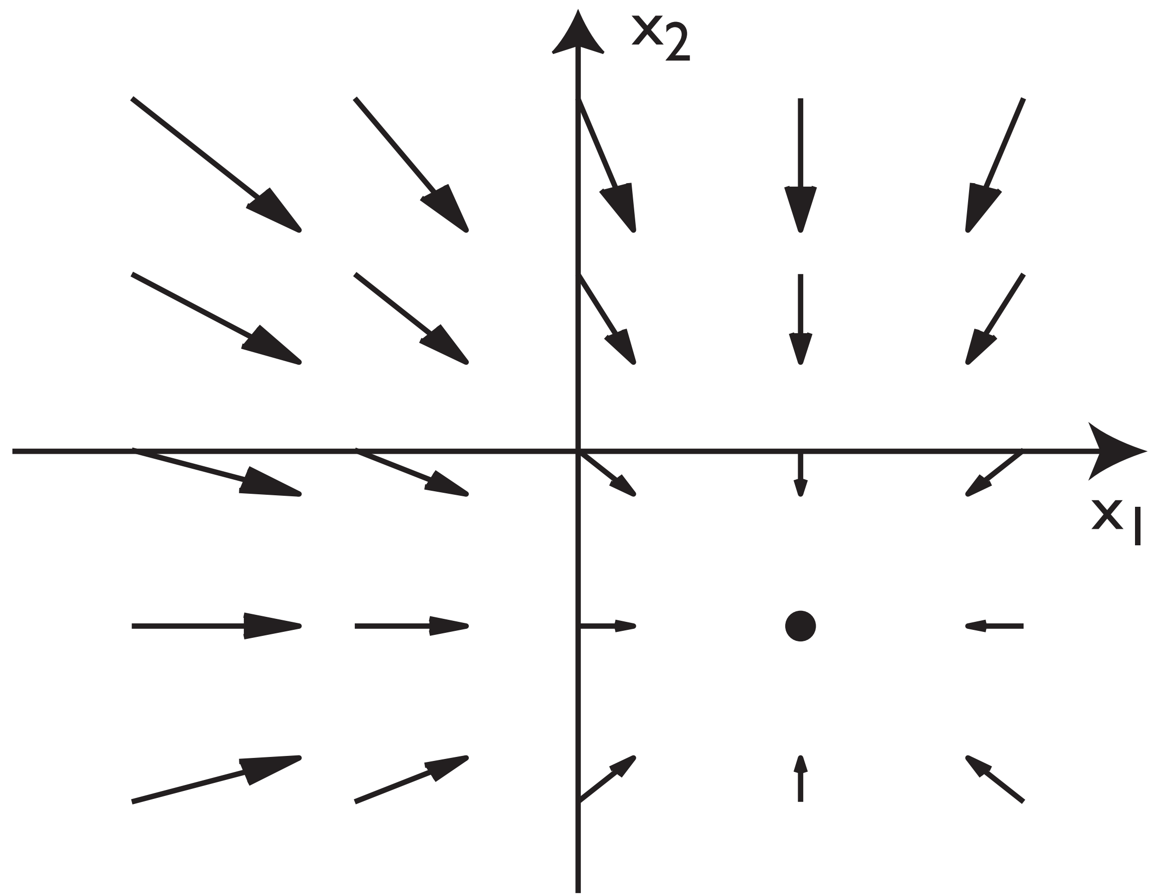




Hopf: need higher dimensions

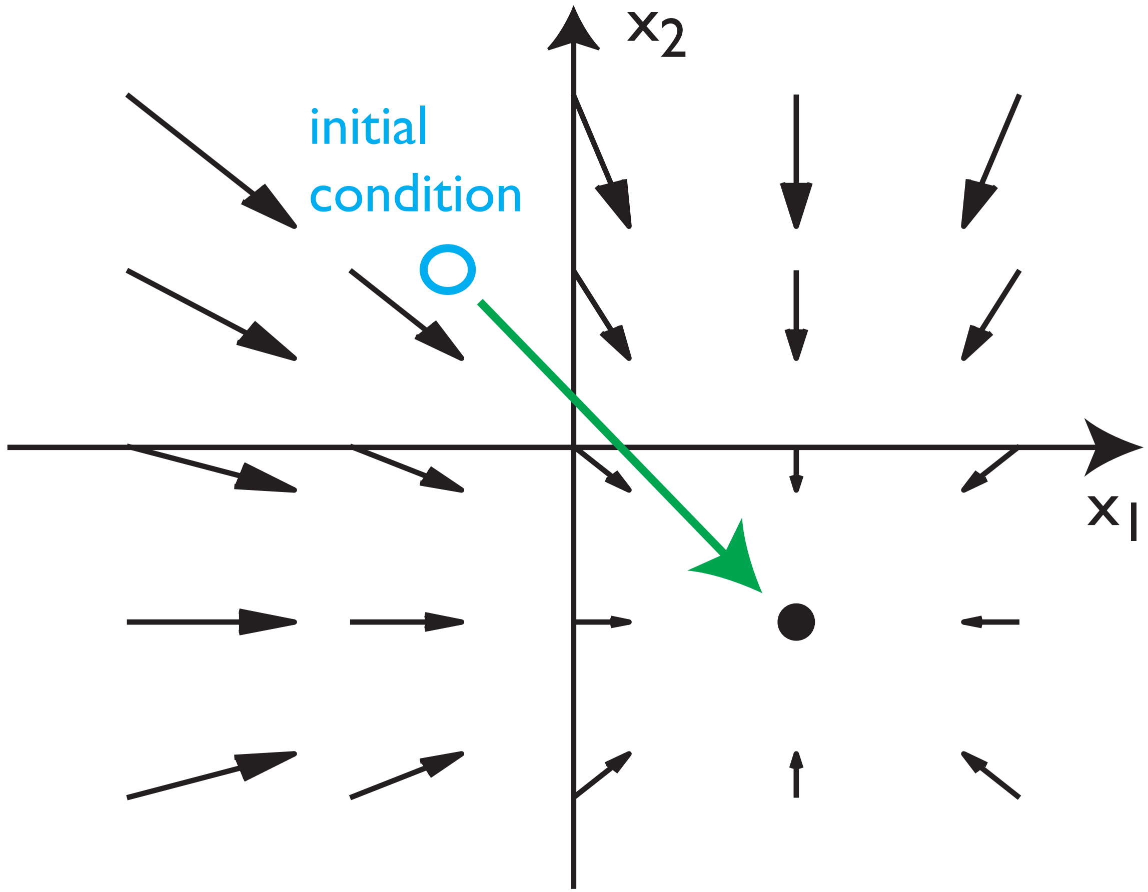
2D dynamical system: vector-field

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$



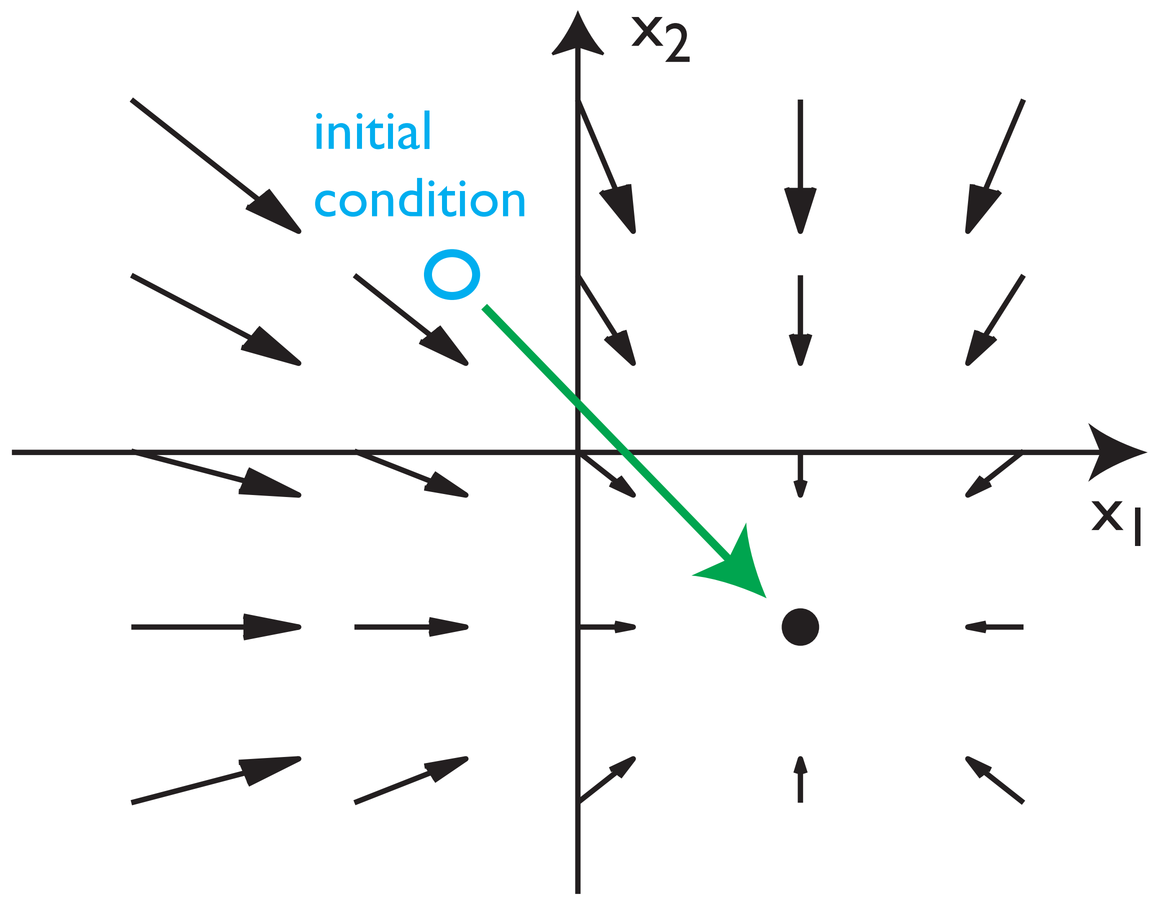
vector-field

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$



fixed point, stability, attractor

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

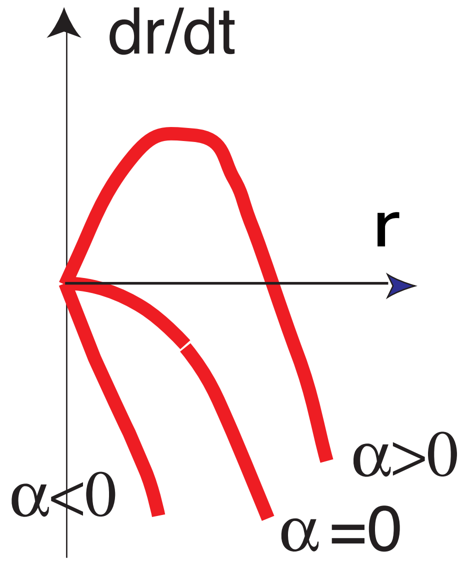


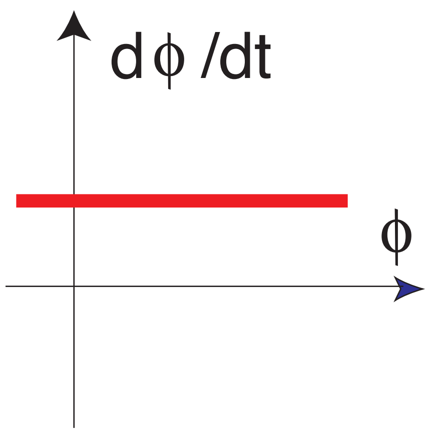
Hopf bifurcation

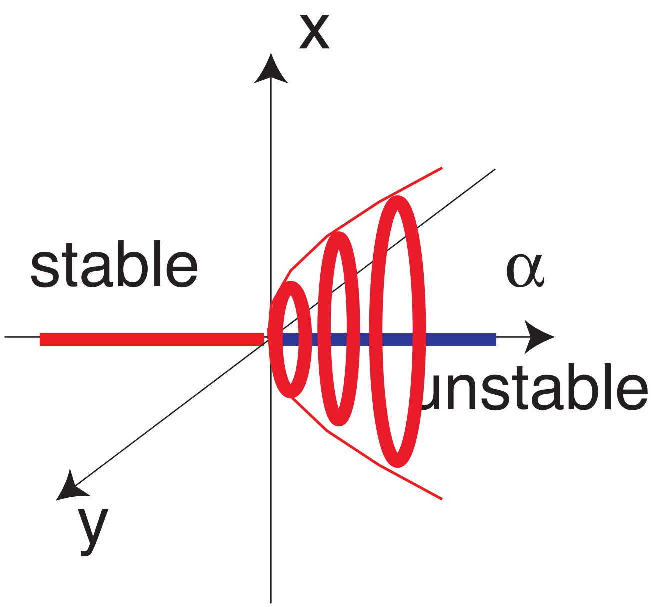
- normal form

$$\dot{r} = \alpha r - r^3$$

$$\dot{\phi} = \omega$$







forward dynamics

- given known equation, determined fixed points / limit cycles and their stability
- more generally: determine invariant solutions (stable, unstable and center manifolds)

inverse dynamics

- given solution, find the equation…
- this is the problem faced in design of behavioral dynamics…

inverse dynamics: design

- in the design of behavioral dynamics… you may be given:
- attractor solutions/stable states
- and how they change as a function of parameters/ conditions
- => identify the class of dynamical systems using the 4 elementary bifurcations
- and use normal form to provide an exemplary representative of the equivalence class of dynamics