Dynamical systems tutorial:

I. Basic concepts

Gregor Schöner, INI, RUB
Dynamical systems: Tutorial

- the word “dynamics”
  - time-varying measures
  - range of a quantity
  - forces causing/accounting for movement => dynamical systems

- dynamical systems are the universal language of science
  - physics, engineering, chemistry, theoretical biology, economics, quantitative sociology, ...
time-variation and rate of change

- variable $x(t)$;
- rate of change $\frac{dx}{dt}$
time-variation and rate of change

example:

- variable \( x(t) = \) position
- rate of change \( \frac{dx}{dt} = \) velocity

example

- variable \( v(t) = \) velocity
- rate of change ?
time-variation and rate of change

- **trajectory:** time course of a dynamical variable
- **rate of change:** slope of the trajectory
dynamical system: relationship between a variable and its rate of change
linear dynamical system

\[ \frac{dx}{dt} = f(x) \]
solution of linear dynamical systems

\[ \tau \frac{dx}{dt} = -x \]

\[ x(t) = x(0) \exp\left[-\frac{t}{\tau}\right] \]
exponential relaxation to attractors

\[ x(t) = x(0) \exp\left[-\frac{t}{\tau}\right] \]

\[ \tau \frac{dx}{dt} = -x \]

\[ \dot{x}(t) = x(0) \exp\left[-\frac{t}{\tau}\right] \left(-\frac{1}{\tau}\right) \]

\[ \tau \dot{x}(t) = -x(0) \exp\left[-\frac{t}{\tau}\right] \]

\[ \tau \dot{x}(t) = -x(t) \]
exponential relaxation to attractors

\[ x(t) = x(0) \exp[-t/\tau] \]

\[ \tau \frac{dx}{dt} = -x \]
Dynamical system

\[ \dot{x} = \frac{dx}{dt} = f(x) \]

dx/dt = f(x)
Dynamical system

\[ \dot{x} = \frac{dx}{dt} = f(x) \]

The present determines the future

dx/dt = f(x)
Dynamical system

\[ \dot{x} = \frac{dx}{dt} = f(x) \]

- \(x\) spans the state space (can be vector-valued or even function valued)
- \(f(x)\) is the “dynamics” of \(x\) (or vector-field)
- \(x(t)\) is a solution of the dynamical systems with initial condition \(x_0 \iff \) the rate of change of \(x(t)\) obeys \(\dot{x}(t) = f(x)\) and \(x(0) = x_0\)
Different forms of dynamical systems

One-dimensional differential equation:
initial value determines the future

\[ \dot{x} = f(x) \]
Different forms of dynamical systems

- vector-valued differential equation
- a vector of initial states determines the future: systems of differential equations:

\[ \dot{x} = f(x) \quad \text{where} \quad x = (x_1, x_2, \ldots, x_n) \]
Different forms of dynamical systems

- Continuously many variables \( x(y) \) determine the future.
  
- An initial function \( x(y) \) determines the future.

- Partial differential equations
  \[
  \dot{x}(y, t) = f \left( x(y, t), \frac{\partial x(y, t)}{\partial y}, \ldots \right)
  \]

- Integro-differential equations
  \[
  \dot{x}(y, t) = \int dy' g \left( x(y, t), x(y', t) \right)
  \]
Different forms of dynamical systems

- a piece of past trajectory determines the future
- delay differential equations
- functional differential equations

\[ \dot{x}(t) = f(x(t - \tau)) \]

\[ \dot{x}(t) = \int_{t'}^{t} dt' f(x(t')) \]
Numerical solutions

- Sample time discretely, \( t_i \), with 
  \[ i \in \{0,1,\ldots,N\} \],

- For example: \( t_i = i \Delta t \)

- Compute solution, \( x(t_i) = x_i \), by iterating through time,

- For example: \( x_{i+1} = x_i + \Delta t f(x_i) \) (forward Euler)

\[
\left[ \frac{x_{i+1} - x_i}{\Delta t} \approx \frac{dx}{dt} = f(x) \approx f(x_i) \right]
\]
=> code / simulation
attractor

\textbf{fixed point}, to which neighboring initial conditions converge = \textbf{attractor}

\[ \frac{dx}{dt} = f(x) \]
A fixed point is a constant solution of the dynamical system

\[ \dot{x} = f(x) \]

\[ \dot{x} = 0 \Rightarrow f(x_0) = 0 \]
Mathematically really: **asymptotic stability**

defined: a fixed point is asymptotically stable, when solutions of the dynamical system that start nearby converge in time to the fixed point
the mathematical concept of stability (which we do not use) requires only that nearby solutions stay nearby

Definition: a fixed point is unstable if it is not stable in that more general sense,

that is: if nearby solutions do not necessarily stay nearby (may diverge)
linear approximation near attractor

$\frac{dx}{dt} = f(x)$

- non-linearity as a small perturbation/deformation of linear system
- $\Rightarrow$ non-essential non-linearity
stability in a linear system

- if the slope of the linear system is negative, the fixed point is (asymptotically stable)
stability in a linear system

If the slope of the linear system is positive, then the fixed point is unstable.

\[ \frac{d\lambda}{dt} = f(\lambda) \]
stability in a linear system

if the slope of the linear system is zero, then the system is indifferent (marginally stable: stable but not asymptotically stable)
stability in linear systems

- generalization to multiple dimensions
  - if the real-parts of all Eigenvalues are negative: stable
  - if the real-part of any Eigenvalue is positive: unstable
  - if the real-part of any Eigenvalue is zero: marginally stable in that direction (stability depends on other eigenvalues)
stability in nonlinear systems

- stability is a local property of the fixed point

=> linear stability theory

- the eigenvalues of the linearization around the fixed point determine stability
  - all real-parts negative: stable
  - any real-part positive: unstable
  - any real-part zero: undecided: now nonlinearity decides (non-hyperpolic fixed point)
stability in nonlinear systems

- all real-parts negative: stable
- any real-part positive: unstable
stability in nonlinear systems

- any real-part zero: undecided; now nonlinearity decides (non-hyperbolic fixed point)
bifurcations

- Look now at families of dynamical systems, which depend (smoothly) on parameters.

- Ask: As the parameters change (smoothly), how do the solutions change (smoothly?)

  - Smoothly: Topological equivalence of the dynamical systems at neighboring parameter values.

  - Bifurcation: Dynamical systems NOT topological equivalent as parameter changes infinitesimally.
bifurcation

dx/dt = f(x)

attractor 1

attractor 2
bifurcation

- **bifurcation** = qualitative change of dynamics (change in number, nature, or stability of fixed points) as the dynamics changes smoothly

\[ \frac{dx}{dt} = f(x) \]

- **attractor 1**
- **attractor 2**
tangent bifurcation

the simplest bifurcation (co-dimension 0): an attractor collides with a repellor and the two annihilate

\[ \frac{dx}{dt} = f(x) \]

attractor 1

attractor 2
local bifurcation

\[ \frac{dx}{dt} = f(x) \]

attractor 1

attractor 2
reverse bifurcation

changing the dynamics in the opposite direction

\[ \frac{dx}{dt} = f(x) \]

attractor 1

attractor 2
bifurcations are instabilities

that is, an attractor becomes unstable before disappearing

(or the attractor appears with reduced stability)

formally: a zero-real part is a necessary condition for a bifurcation to occur
tangent bifurcation

- Normal form of tangent bifurcation
  \[ \dot{x} = \alpha - x^2 \]

- (=simplest polynomial equation whose flow is topologically equivalent to the bifurcation)

\[ x_0 = \sqrt{\alpha} \]
Hopf theorem

When a single (or pair of complex conjugate) eigenvalue crosses the imaginary axis, one of four bifurcations occur:

- tangent bifurcation
- transcritical bifurcation
- pitchfork bifurcation
- Hopf bifurcation
transcritical bifurcation

\[ \dot{x} = \alpha x - x^2 \]

**normal form**

- \( \alpha \) positive
- \( \alpha \) negative
- \( \alpha = 0 \)
pitchfork bifurcation

**normal form**

\[ \dot{x} = \alpha x - x^3 \]

\[ \dot{x} = -2x_0x = -2\sqrt{\alpha}x \]

- **Fixed point**
- **Stable**
- **Unstable**

\( \alpha \) positive

\( \alpha \) negative

\( \alpha = 0 \)
Hopf: need higher dimensions
2D dynamical system: vector-field

\[ \dot{x}_1 = f_1(x_1, x_2) \]
\[ \dot{x}_2 = f_2(x_1, x_2) \]
\[ \dot{x}_1 = f_1(x_1, x_2) \]
\[ \dot{x}_2 = f_2(x_1, x_2) \]
fixed point, stability, attractor

\[ \dot{x}_1 = f_1(x_1, x_2) \]
\[ \dot{x}_2 = f_2(x_1, x_2) \]
Hopf bifurcation

\[ \dot{r} = \alpha r - r^3 \]
\[ \dot{\phi} = \omega \]

normal form

\[ \frac{dr}{dt} \]
\[ \frac{d\phi}{dt} \]

\[ \alpha < 0 \]
\[ \alpha = 0 \]
\[ \alpha > 0 \]
forward dynamics

given known equation, determined fixed points / limit cycles and their stability

more generally: determine invariant solutions (stable, unstable and center manifolds)
inverse dynamics

given solution, find the equation…

this is the problem faced in design of behavioral dynamics…
inverse dynamics: design

- in the design of behavioral dynamics… you may be given:
  - attractor solutions/stable states
  - and how they change as a function of parameters/conditions
  - => identify the class of dynamical systems using the 4 elementary bifurcations
  - and use normal form to provide an exemplary representative of the equivalence class of dynamics