Mathematics and Computer Science for Modeling
Unit 6: Differential Equations

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based on materials by Jan Tekülve and Daniel Sabinasz

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# Course Structure

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Lecture Slides/Material

Use the following URL to access the lecture slides:

https://www.ini.rub.de/teaching/courses/preparatory_course_mathematics_and_computer_science_for_modeling_summer_term_2022
Overview

1. Differential Equations
   ➤ Application: Dynamical Systems
   ➤ Solving Differential Equations
   ➤ Qualitative Analysis
Application: Dynamical Systems

- A **dynamical system** is a system of one or more variables that change in time.

- e.g., the location of a falling ball.

- Dynamical systems can often be described with a **differential equation** that describes the rate of change of the system at each point in time, e.g.,

\[ h'(t) = -g \]

- **Solving** this differential equation means finding a function \( h(t) \) that describes the location of the ball at each point in time.

\[ h(t) = h_0 - \frac{1}{2}gt^2 \]
Differential Equations

- Generally, a differential equation describes how the rate of change of a system depends on its current state. For example:

\[ f'(x) = 4f(x) + 5 = g(f(x)) \quad \text{with} \quad g(x) = 4x + 5 \]
Differential Equations

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- A differential equation describes how a system should change in a given state.
Solving Differential Equations

- Given a differential equation of the form \( f'(x) = g(f(x)) \) . . . the original function \( f(x) \) is usually not known.

- Solving a differential equation describes the process of finding an \( f(x) \) that satisfies the differential equation for all \( x \)
  - Example:
    \[
    f'(x) = 4f(x) + 5 \Rightarrow f(x) = \frac{e^{4x} + c}{4} - \frac{5}{4} \text{ and } f'(x) = e^{4x} + c
    \]

- Differential equations entail two equations
  1. The function \( g(f(x)) \) governing the rate of change
  2. The function \( f(x) \) describing the overall behavior
**Derivative vs. Differential equation**

- $f'(x) = cx$

- $f''(x) = cf(x)$
Derivative vs. Differential equation

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  - The rate of change is a scaled version of the function itself:
    $$g(f(x)) = cf(x)$$
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\[ f'(x) = cf(x) \]

- The rate of change is a scaled version of the function itself: \( g(f(x)) = cf(x) \)
- The only function that stays the same when differentiated is the exponential function \( e^x \)
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  - Considering the chain rule the derivative of $e^{cx}$ is exactly $ce^{cx}$ therefore $f(x) = ce^{cx}$
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  - The only function that stays the same when differentiated is the exponential function $e^x$
  - Considering the chain rule the derivative of $e^{cx}$ is exactly $ce^{cx}$ therefore $f(x) = ce^{cx}$
  - Usually a differential equation is not that easily solvable
Dynamical Systems Theory

- Mathematicians want to find solutions to particular differential equations

- **Dynamical Systems Theory** is concerned with analyzing the qualitative behavior of the system
Qualitative Behavior of Differential Equations

\[ f'(x) = y' = \frac{dy}{dx} = -f(x) \]
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Phase Plot
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Phase Plot
Attractors

\[ f'(x) = y' = \frac{dy}{dx} = -f(x) \]
Attractors

\[ f'(x) = y' = \frac{dy}{dx} = -f(x) - 2 \]
Attractors

\[ f'(x) = y' = \frac{dy}{dx} = -f(x) + 1 \]
Repellors

\[ f'(x) = y' = \frac{dy}{dx} = -f(x) \]
Initial Condition Matters

\[ f'(x) = y' = \frac{dy}{dx} = \sin(x) \]
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- **E1**
- **S1**
- **E2**
- **S2**
- **E3**
- **S3**