Mathematics and Computer Science for Modeling
Unit 6: Differential Equations

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based on materials by Jan Tekülve and Daniel Sabinasz

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Course Structure

Unit	Title	Topics
1	Intro to Programming in Python	*Variables, if Statements, Loops, Functions, Lists*
-	Full-Time Programming Session	*Deepen Programming Skills*
2	Functions in Math	*Function Types and Properties, Plotting Functions*
3	Linear Algebra	*Vectors, Trigonometry, Matrices*
4	Calculus	*Derivative Definition, Calculating Derivatives*

Course Structure

Unit	Title	Topics
5	Integration	*Geometrical Definition, Calculating Integrals*
6	Differential Equations	*Properties of Differential Equations*
-	07.10.22: Test	

Lecture Slides/Material

Use the following URL to access the lecture slides:

https://www.ini.rub.de/teaching/courses/preparatory_course_mathematics_and_computer_science_for_modeling_summer_term_2022

Overview

1. **Differential Equations**
   - Application: Dynamical Systems
   - Solving Differential Equations
   - Qualitative Analysis

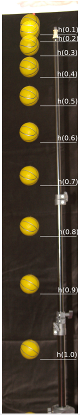
Application: Dynamical Systems

- A **dynamical system** is a system of one or more variables that change in time
- e.g., the location of a falling ball
- Dynamical systems can often be described with a **differential equation** that describes the rate of change of the system at each point in time, e.g.,

$$h'(t) = -g$$

- **Solving** this differential equation means finding a function $h(t)$ that describes the location of the ball at each point in time.

$$h(t) = h_0 - \frac{1}{2}gt^2$$



Differential Equations

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- A differential equation describes how a system should change in a given state.

Solving Differential Equations

- Given a differential equation of the form $f'(x) = g(f(x)) \ldots$ the original function $f(x)$ is usually not known.
- Solving a differential equation describes the process of finding an $f(x)$ that satisfies the differential equation for all x
  - Example:

$$f'(x) = 4f(x) + 5 \Rightarrow f(x) = \frac{e^{4x+c}}{4} - \frac{5}{4} \text{ and } f'(x) = e^{4x+c}$$

- Differential equations entail two equations
  1. The function $g(f(x))$ governing the rate of change
  2. The function $f(x)$ describing the overall behavior

Derivative vs. Differential equation

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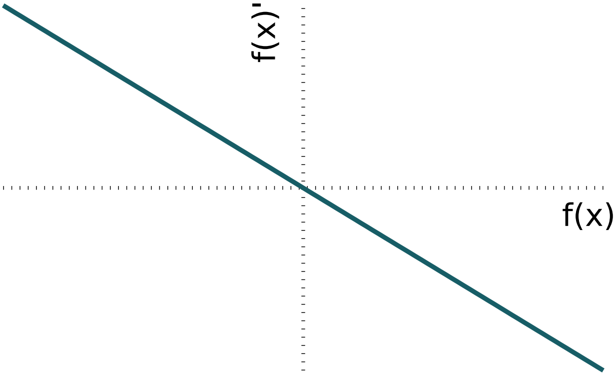
  - Considering the chain rule the derivative of e^{cx} is exactly ce^{cx} therefore $f(x) = ce^{cx}$
  - Usually a differential equation is not that easily solvable

Dynamical Systems Theory

- Mathematicians want to find solutions to particular differential equations
- **Dynamical Systems Theory** is concerned with analyzing the qualitative behavior of the system

Qualitative Behavior of Differential Equations

$$f'(x) = y' = \frac{dy}{dx} = -f(x)$$



Phase Plot

Qualitative Behavior of Differential Equations

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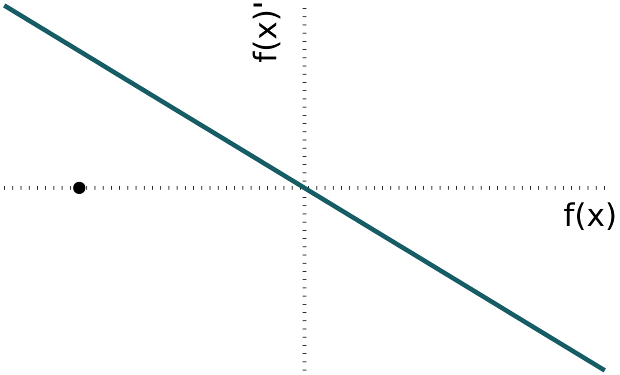

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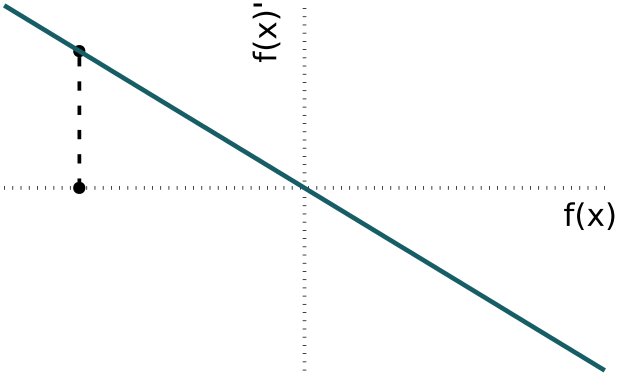

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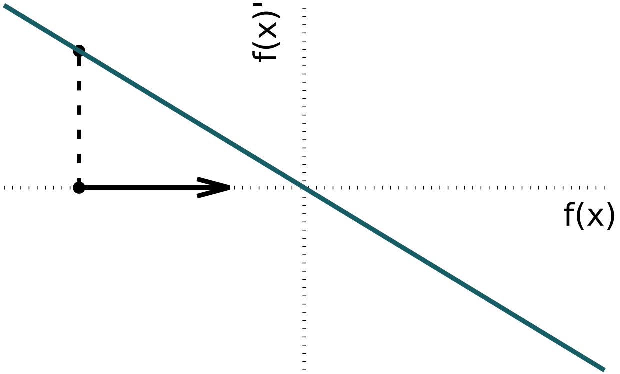

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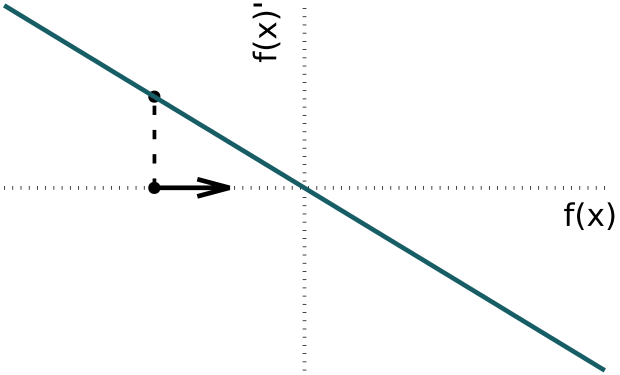

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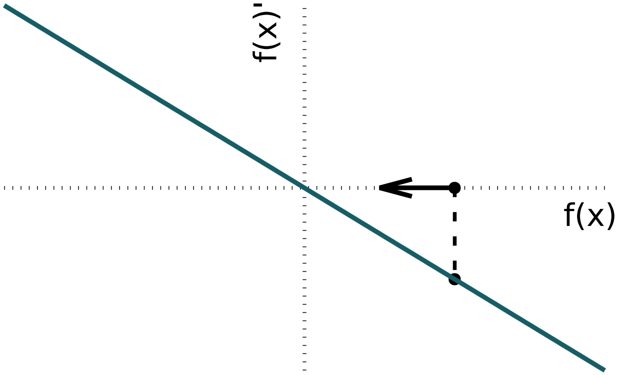

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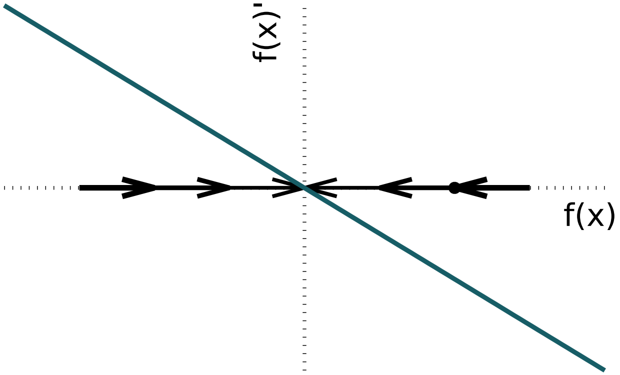
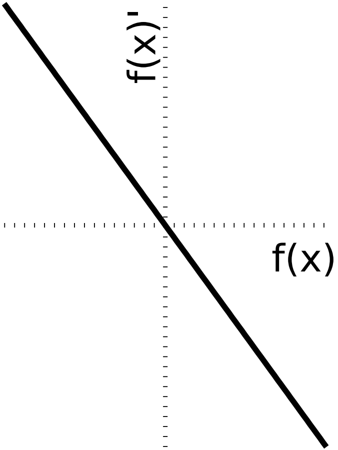


Phase Plot

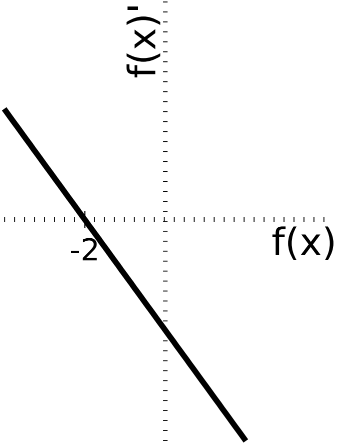
Attractors

$$f'(x) = y' = \frac{dy}{dx} = -f(x)$$



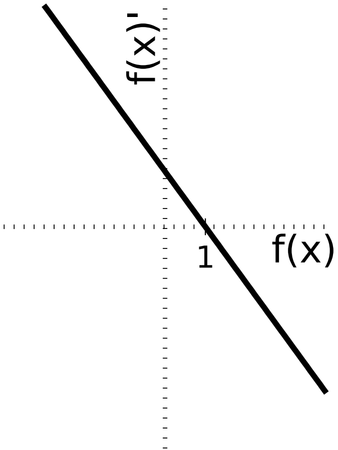
Attractors

$$f'(x) = y' = \frac{dy}{dx} = -f(x) - 2$$



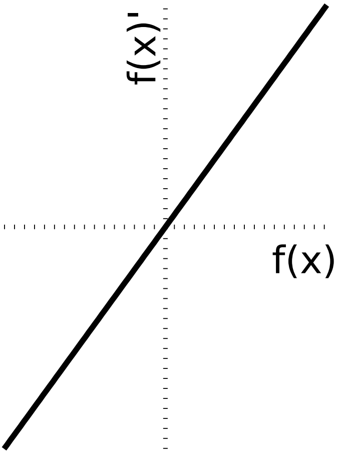
Attractors

$$f'(x) = y' = \frac{dy}{dx} = -f(x) + 1$$



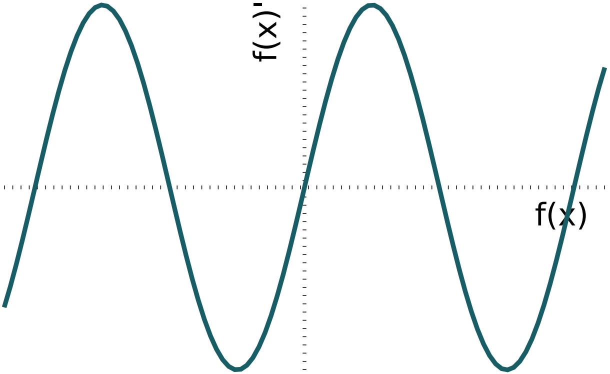
Repellors

$$f'(x) = y' = \frac{dy}{dx} = -f(x)$$



Initial Condition Matters

$$f'(x) = y' = \frac{dy}{dx} = sin(x)$$



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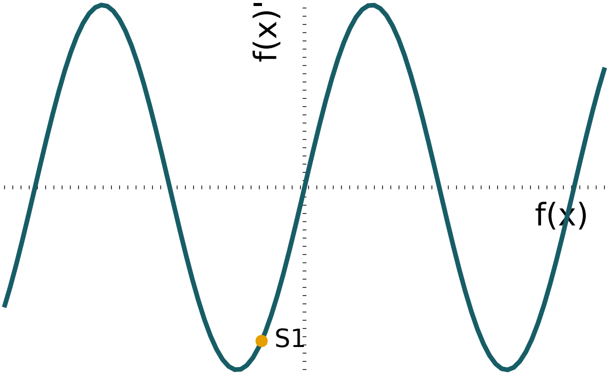

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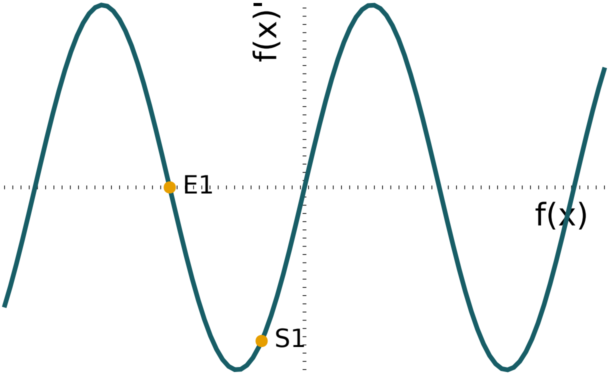

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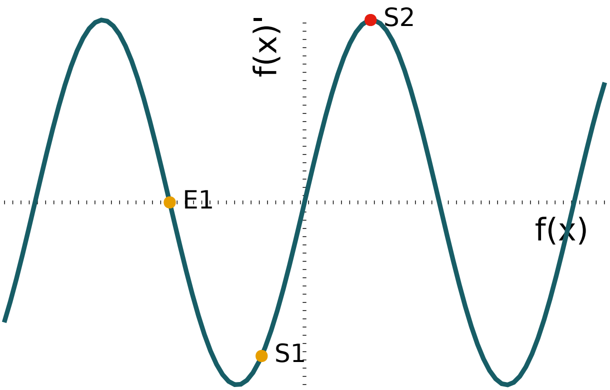

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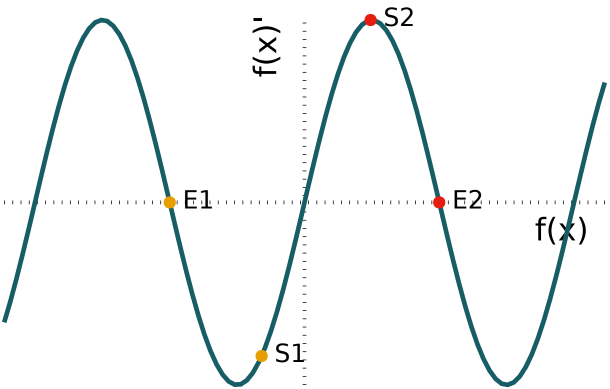

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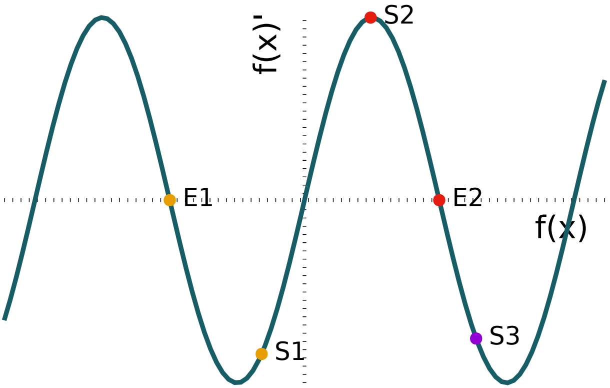

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