Mathematics and Computer Science for Modeling
Unit 5: Integration

Daniel Sabinasz
based on materials by Jan Tekülve and Daniel Sabinasz

Institut für Neuroinformatik, Ruhr-Universität Bochum

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# Course Structure

<table>
<thead>
<tr>
<th>Unit</th>
<th>Title</th>
<th>Topics</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Intro to Programming in Python</td>
<td>Variables, if Statements, Loops, Functions, Lists</td>
</tr>
<tr>
<td></td>
<td>Full-Time Programming Session</td>
<td>Deepen Programming Skills</td>
</tr>
<tr>
<td>2</td>
<td>Functions in Math</td>
<td>Function Types and Properties, Plotting Functions</td>
</tr>
<tr>
<td>3</td>
<td>Linear Algebra</td>
<td>Vectors, Trigonometry, Matrices</td>
</tr>
<tr>
<td>4</td>
<td>Calculus</td>
<td>Derivative Definition, Calculating Derivatives</td>
</tr>
</tbody>
</table>
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<table>
<thead>
<tr>
<th>Unit</th>
<th>Title</th>
<th>Topics</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>Integration</td>
<td>Geometrical Definition, Calculating Integrals</td>
</tr>
<tr>
<td>6</td>
<td>Differential Equations</td>
<td>Properties of Differential Equations</td>
</tr>
<tr>
<td>-</td>
<td>07.10.22: Test</td>
<td></td>
</tr>
</tbody>
</table>
Lecture Slides/Material

Use the following URL to access the lecture slides:

https://www.ini.rub.de/teaching/courses/preparatory_course_mathematics_and_computer_science_for_modeling_summer_term_2022
Overview

1. Motivation

2. Mathematics
   ➤ Approximating the Area under a Curve
   ➤ Calculating the Area under a curve
   ➤ Improper Integrals

3. Exercise
From Velocity to Position

You drove 30 km/h for 6 hours. How far did you drive?
From Velocity to Position

You drove 30 km/h for 6 hours. How far did you drive?

\[
30 \frac{\text{km}}{\text{h}} \times 6 \text{h} = 180 \text{km}
\]
**From Velocity to Position**

You drove 30 km/h for 6 hours. How far did you drive?

\[ 30 \frac{\text{km}}{\text{h}} \times 6 \text{h} = 180 \text{km} \]
From Velocity to Position

Let’s say you slowed down for the last 3 hours. How far did you get?
From Velocity to Position

Let’s say you slowed down for the last 3 hours. How far did you get?

$$30 \frac{\text{km}}{\text{h}} \times 3\text{h} + 20 \frac{\text{km}}{\text{h}} \times 3\text{h} = 150\text{km}$$
From Velocity to Position

What if you mixed it up to not get bored?
From Velocity to Position

What if you mixed it up to not get bored?

$$30 \frac{\text{km}}{\text{h}} \times 1\text{h} + 40 \frac{\text{km}}{\text{h}} \times 2\text{h} + 10 \frac{\text{km}}{\text{h}} \times 1\text{h} + 20 \frac{\text{km}}{\text{h}} \times 2\text{h} = 160\text{km}$$
From Velocity to Position

But how about something realistic?
Approximation

- Not all areas can be calculated with rectangles
Approximation

- Not all areas can be calculated with rectangles
- One can however approximate them
Approximation

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- One can however approximate them
- The more rectangles the better the approximation becomes
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Midpoint Riemann Sum

Calculating Midpoints

The **Midpoint Riemann Sum** is a way of approximating an integral with finite sums. The area under the curve in a given interval \([x_i, x_{i+1}]\) can be approximated as the area of a rectangle with width \(\Delta x = x_{i+1} - x_i\) and height \(f\left(\frac{x_i + x_{i+1}}{2}\right)\):

\[
f\left(\frac{x_i + x_{i+1}}{2}\right) \Delta x
\]

The sum over all intervals yields an estimation of the area under the curve:

\[
I_M = \sum_{i}^{n} f\left(\frac{x_i + x_{i+1}}{2}\right) \Delta x
\]
Midpoint Sums
Midpoint Sums
Midpoint Sums
From Sums to Integrals

**Midpoint Sum:** $f\left(\frac{x_i+x_{i+1}}{2}\right) \Delta x$

The larger the number $n$ of intervals, the smaller $\Delta x$ and the better our approximation.

Definite Integral

The definite integral of a function $f(x)$ between the lower boundary $a$ and the upper boundary $b$

$$\int_{a}^{b} f(x) \, dx$$

is defined as the size of the area between $f$ and the $x$-axis inside the boundaries. Areas above the $x$-axis are considered positively and areas below negatively.
From Sums to Integrals

**Midpoint Sum:** \( f\left(\frac{x_i + x_{i+1}}{2}\right) \Delta x \)

The larger the number \( n \) of intervals, the smaller \( \Delta x \) and the better our approximation.

What if \( n \) becomes infinitely large and \( \Delta x \) becomes infinitely small?
From Sums to Integrals

Midpoint Sum: \( f\left(\frac{x_i + x_{i+1}}{2}\right) \Delta x \)

The larger the number \( n \) of intervals, the smaller \( \Delta x \) and the better our approximation.

What if \( n \) becomes infinitely large and \( \Delta x \) becomes infinitely small?

Definite Integral

The **definite integral** of a function \( f(x) \) between the **lower boundary** \( a \) and the **upper boundary** \( b \)

\[
\int_{a}^{b} f(x) \, dx
\]

is defined as the size of the area between \( f \) and the \( x \)-axis inside the boundaries. Areas above the \( x \)-axis are considered positively and areas below negatively.
Definite Integral

\[ f(x) = \cos(x) \quad \int_{0}^{2\pi} \cos(x) \, dx \]
Indefinite Integral

\[ f(x) = \cos(x) \quad \int_0^x \cos(x') \, dx' \]
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**Indefinite Integral**

\[ f(x) = \cos(x) \quad \int_0^x \cos(x') \, dx' = \sin(x) \]
Indefinite Integral

\[ f(x) = \cos(x) \quad \int_0^x \cos(x') \, dx' = \int \cos(x') \, dx' \]
The Antiderivative

Definition

If \( f \) is a function with domain \([a, b] \rightarrow \mathbb{R}\) and there is a function \( F \), which is differentiable in the interval \([a, b]\) with the property that

\[
F'(x) = f(x),
\]

then \( F \) is considered an \textit{antiderivative} of \( f \).
The Antiderivative

Definition

If $f$ is a function with domain $[a, b] \rightarrow \mathbb{R}$ and there is a function $F$, which is differentiable in the interval $[a, b]$ with the property that

$$F'(x) = f(x),$$

then $F$ is considered an antiderivative of $f$.

Properties of an antiderivative

- Differentiation removes constants, therefore $F(x) + c$ for any constant $c$ is also an antiderivative.

- Unlike with differentiation there are no fixed rules to compute an antiderivative from a given $f$. 
A function and its antiderivative

\[ f(x) = x \quad F(x) = \frac{1}{2}x^2 \]
The Fundamental Theorem of Calculus

First Fundamental Theorem of Calculus

One of the antiderivatives of a function can be obtained as the indefinite integral:

\[ \int f(x')\,dx' = F(x) \]

- Intuition: The rate of change of the area under \( f(x) \) is \( f(x) \)
The Fundamental Theorem of Calculus

Second Fundamental Theorem of Calculus

If $f$ is integrable and continuous in $[a, b]$, then the following holds for each antiderivative $F$ of $f$

$$\int_{a}^{b} f(x) \, dx = [F(x)]_{a}^{b} = F(b) - F(a)$$

Example:

- Area under $f(x) = x$ between values 1 and 2

$$\int_{1}^{2} x \, dx = \left[ \frac{1}{2} x^2 \right]_{1}^{2} = \frac{1}{2} 2^2 - \frac{1}{2} 1^2 = 1.5$$
Definite Integral Example

\[ f(x) = x \quad F(x) = \frac{1}{2}x^2 \quad \int_1^2 f(x)\,dx = F(2) - F(1) \]
The Integral is a Linear Operator

Integration Rules

- **Summation**

\[
\int_a^b f(x) + g(x) = \int_a^b f(x) + \int_a^b g(x)
\]
The Integral is a Linear Operator

Integration Rules

- **Summation**

\[ \int_a^b f(x) + g(x) = \int_a^b f(x) + \int_a^b g(x) \]

- **Scalar Multiplication**

\[ \int_a^b cf(x) = c \int_a^b f(x) \]
The Integral is a Linear Operator

Integration Rules

- **Summation**

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\int_a^b f(x) + g(x) = \int_a^b f(x) + \int_a^b g(x)
\]

- **Scalar Multiplication**

\[
\int_a^b cf(x) = c \int_a^b f(x)
\]

- **Boundary Transformations**

\[
\int_a^b f(x) + \int_b^c f(x) = \int_a^c f(x) \quad \int_a^b f(x) = -\int_b^a f(x)
\]
Improper Integrals

Infinite Intervals

It is possible to calculate the area in infinitely large intervals. Intervals with an infinite boundary are called **Improper Integrals**

\[
\int_{a}^{\infty} f(x) \, dx = \lim_{b \to \infty} \int_{a}^{b} f(x) \, dx
\]

**Example:**

- Convergent improper integral

\[
\int_{1}^{\infty} x^{-2} \, dx = \lim_{b \to \infty} \int_{1}^{b} x^{-2} \, dx = \lim_{b \to \infty} \left[ -x^{-1} \right]_{1}^{b} = \lim_{b \to \infty} (-b^{-1} + 1) = 1
\]
Exercise

Answer the following tasks using a piece of paper and a pocket calculator.

1. Given the Antiderivative \( F(x) = 12x^2 + 5x \) of the function \( f(x) \), calculate the area between \( f(x) \) and the x-axis in the interval of \([-3, 5]\).

2. Calculate \( \int_{0}^{\pi} \cos(x) \, dx \). Before applying the formula, look at a plot of \( \cos(x) \). What kind of result would you expect?
Exercise Solutions

1. The antiderivative is already given, therefore you only need to plug-in the beginning and the end of the interval.

\[
F(x) \bigg|_a^b = F(b) - F(a) = F(5) - F(3) = 12 \times 5^2 + 5 \times 5 - 12 \times (-3)^2 + 5 \times (-3) = 325 - 93 = 232
\]

2. Looking at the plot of \( \cos(x) \) you can see that exactly the same area is enclosed above the x-axis as below the x-axis, therefore the total area has to be zero.

To verify this analytically, you need to figure out the antiderivative of \( \cos(x) \) first. From the lecture you know that \( F(x) = \sin(x) \).

\[
F(x) \bigg|_a^b = F(b) - F(a) = \sin(\pi) - \sin(0) = 0 - 0 = 0
\]
Exercise Solutions

1. The antiderivative is already given, therefore you only need to plug-in the beginning and the end of the interval.

\[
[F(x)]_a^b = F(b) - F(a) = F(5) - F(3) \\
= 12 \times 5^2 + 5 \times 5 - (12 \times (-3)^2 + 5 \times (-3)) = 325 - 93 = 232
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[F(x)]_a^b = F(b) - F(a) = F(\pi) - F(0) = \sin(\pi) - \sin(0) = 0 - 0 = 0
\]