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Lecture Slides/Material

Use the following URL to access the lecture slides:

https://www.ini.rub.de/teaching/courses/preparatory_course_mathematics_and_computer_science_for_modeling_summer_term_2022
1. Linear Algebra
   ➤ Angles and Trigonometry
   ➤ Vectors
   ➤ Matrices
The number $\pi$
The number π

\[ 75.39 \text{ cm} \]
The number $\pi$
The number $\pi$
The number $\pi$

\[
\frac{75.39}{24} = 3.14159\ldots = \pi \quad \text{and} \quad \frac{56.54}{18} = 3.14159\ldots = \pi
\]
The number $\pi$

$$\frac{75.39}{24} = 3.14159... = \pi \quad \text{and} \quad \frac{56.54}{18} = 3.14159... = \pi$$

Circumference of a circle: $2\pi r$
Measuring Angles

- Defining a full angle as $360^\circ$ is common but actually arbitrary.

\[
\begin{align*}
\text{Rad} \times \text{to Degree:} & \quad x \cdot \frac{180^\circ}{\pi} \\
\text{Degree} \text{d} \text{to Rad:} & \quad d \cdot \frac{\pi}{180^\circ}
\end{align*}
\]
Measuring Angles

- Defining a full angle as $360^\circ$ is common but actually arbitrary.
- Less arbitrary is the use of the actual length of the enclosed arc-segment called the **Radian**.
Measuring Angles

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- Less arbitrary is the use of the actual length of the enclosed arc-segment called the Radian.

- Thus $360^\circ = 2\pi$, $90^\circ = \frac{\pi}{2}$, $180^\circ = \pi$ ...
Measuring Angles

- Defining a full angle as $360^\circ$ is common but actually arbitrary.

- Less arbitrary is the use of the actual length of the enclosed arc-segment called the **Radian**.

- Thus $360^\circ = 2\pi$, $90^\circ = \frac{\pi}{2}$, $180^\circ = \pi$ ... 

- Rad $x$ to Degree: $x \cdot \frac{180^\circ}{\pi}$
Measuring Angles

▶ Defining a full angle as $360^\circ$ is common but actually arbitrary

▶ Less arbitrary is the use of the actual length of the enclosed arc-segment called the **Radian**

▶ Thus $360^\circ = 2\pi$, $90^\circ = \frac{\pi}{2}$, $180^\circ = \pi \ldots$

▶ Rad $x$ to Degree: $x \cdot \frac{180^\circ}{\pi}$

▶ Degree $d$ to Rad: $d \cdot \frac{\pi}{180^\circ}$
Angle Conversion Examples

- Degree to Radians: \( d \cdot \frac{\pi}{180^\circ} \)

\[ \alpha_{\text{deg}} = 34^\circ \]
Angle Conversion Examples

Degree to Radians: \( d \cdot \frac{\pi}{180^\circ} \)

\[
\alpha_{\text{deg}} = 34^\circ \\
= 34^\circ \cdot \frac{\pi}{180^\circ}
\]
Angle Conversion Examples

- **Degree to Radians:** \( d \cdot \frac{\pi}{180^\circ} \)

\[
\alpha_{\text{deg}} = 34^\circ
\]

\[
= 34^\circ \cdot \frac{\pi}{180^\circ}
\]

\[
= \frac{34^\circ \cdot \pi}{180^\circ} = \frac{106.81^\circ}{180^\circ} = 0.593 = \alpha_{\text{rad}}
\]
Angle Conversion Examples

- **Degree to Radians:** $d \cdot \frac{\pi}{180\degree}$
  
  $\alpha_{\text{deg}} = 34\degree$
  
  $= 34\degree \cdot \frac{\pi}{180\degree}$
  
  $= \frac{34\degree \cdot \pi}{180\degree} = \frac{106.81\degree}{180\degree} = 0.593 = \alpha_{\text{rad}}$

- **Radians to Degree:** $x \cdot \frac{180\degree}{\pi}$
**Angle Conversion Examples**

▶ **Degree to Radians:** \( d \cdot \frac{\pi}{180^\circ} \)

\[
\alpha_{\text{deg}} = 34^\circ \\
= 34^\circ \cdot \frac{\pi}{180^\circ} \\
= \frac{34^\circ \cdot \pi}{180^\circ} = \frac{106.81^\circ}{180^\circ} = 0.593 = \alpha_{\text{rad}}
\]

▶ **Radians to Degree:** \( x \cdot \frac{180^\circ}{\pi} \)

\[
\alpha_{\text{rad}} = \frac{3}{4}\pi
\]
Angle Conversion Examples

▶ Degree to Radians: \( d \cdot \frac{\pi}{180^\circ} \)

\[
\alpha_{\text{deg}} = 34^\circ \\
= 34^\circ \cdot \frac{\pi}{180^\circ} \\
= \frac{34^\circ \cdot \pi}{180^\circ} = \frac{106.81^\circ}{180^\circ} = 0.593 = \alpha_{\text{rad}}
\]

▶ Radians to Degree: \( x \cdot \frac{180^\circ}{\pi} \)

\[
\alpha_{\text{rad}} = \frac{3}{4}\pi \\
= \frac{3}{4}\pi \cdot \frac{180^\circ}{\pi} = \frac{180^\circ}{4} = 45^\circ
\]
Angle Conversion Examples

- **Degree to Radians:** \( d \cdot \frac{\pi}{180^\circ} \)

\[
\alpha_{\text{deg}} = 34^\circ \\
= 34^\circ \cdot \frac{\pi}{180^\circ} \\
= \frac{34^\circ \cdot \pi}{180^\circ} = \frac{106.81^\circ}{180^\circ} = 0.593 = \alpha_{\text{rad}}
\]

- **Radians to Degree:** \( x \cdot \frac{180^\circ}{\pi} \)

\[
\alpha_{\text{rad}} = \frac{3}{4} \pi \\
= \frac{3}{4} \cdot \frac{180^\circ}{\pi} \\
= \frac{3}{4} \cdot 180^\circ = 135^\circ = \alpha_{\text{deg}}
\]
Sine and Cosine

- $a^2 + b^2 = c^2$

- $\sin(\alpha) = \frac{b}{c} = \frac{\text{opposite}}{\text{hypotenuse}}$

- $\cos(\alpha) = \frac{a}{c} = \frac{\text{adjacent}}{\text{hypotenuse}}$

- $\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} = \frac{b}{a} = \frac{\text{opposite}}{\text{adjacent}}$
Sine and Cosine

The sine and cosine of an angle can be interpreted as the $x$ and $y$ coordinates of the location on the unit circle at angle $\alpha$.

\[
x = \cos(\alpha) \iff \alpha = \cos^{-1}(x)
\]

\[
y = \sin(\alpha) \iff \alpha = \sin^{-1}(x)
\]

Click here for interactive demo.
Vectors in the Cartesian Coordinate System

- A vector $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ is a geometric object that has length and direction.

- Think of it as an arrow from the origin to the point $(v_x, v_y)$. 

\[ 
\begin{align*}
\text{X} & \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \\
\text{Y} & \quad 0 \quad 0.5 \quad 1 \quad 1.5 \quad 2 \quad 2.5 \quad 3 \quad 3.5 \quad 4 \\
& \quad (2,3) \quad (4,1)
\end{align*}
\]
Vectors in more dimensions

- Vectors can be defined in higher-dimensional coordinate systems as well

- e.g., in 3D: \( \mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \)
Vector Addition

\[
\begin{pmatrix}
  a_x \\
  a_y
\end{pmatrix} + \begin{pmatrix}
  b_x \\
  b_y
\end{pmatrix} = \begin{pmatrix}
  a_x + b_x \\
  a_y + b_y
\end{pmatrix} = \begin{pmatrix}
  c_x \\
  c_y
\end{pmatrix}
\]
Scalar Multiplication

\[ sa = s \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} sa_x \\ sa_y \end{pmatrix} \]

Diagram:
- Vector \( a \) is twice the length of vector \( b \).
- \( a = 2b \)
Exercise 1

1. Compute the circumference of a circle with radius 2 cm
2. Convert an angle of 45° to radians
3. Convert $\frac{3\pi}{2}$ radians to degrees
4. Given a right triangle with $a = 2$, $b = 3$, compute the angle between $a$ and $c$
5. Let $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Compute $2(\mathbf{v} + \mathbf{w})$. 
Length of a vector

- The length of a vector can be calculated using the Pythagorean theorem:

\[ \|v\| = \sqrt{v_x^2 + v_y^2} \]

- Graphical Interpretation:
Scalar Product

The **scalar product** \(< \mathbf{a}, \mathbf{b} >\) or \(\mathbf{a} \cdot \mathbf{b}\) of two vectors is defined as:

\[
< \mathbf{a}, \mathbf{b} > = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ \vdots \end{pmatrix} = a_1 b_1 + a_2 b_2 + \ldots
\]

and results in a **scalar** value.
Scalar Product

- The scalar product is related to the angle between the two vectors:

\[
\langle \mathbf{a}, \mathbf{b} \rangle = |\mathbf{a}| |\mathbf{b}| \cos(\alpha) \quad \iff \quad \frac{\langle \mathbf{a}, \mathbf{b} \rangle}{|\mathbf{a}| |\mathbf{b}|} = \cos(\alpha)
\]

- Graphical Interpretation:
Scalar Product: Special Cases

- If both vectors $a$ and $b$ point in the same direction:
  \[ < a, b > = |a||b|cos(0) = |a||b| \]

- If both vectors $a$ and $b$ are orthogonal to each other:
  \[ < a, b > = |a||b|cos(90^\circ) = 0 \]
**Angle between Vectors**

- The scalar product can be used to calculate the angle between two vectors.

\[
\langle a, b \rangle = |a||b|\cos(\alpha)
\]

\[
\alpha = \cos^{-1}\left(\frac{\langle a, b \rangle}{|a||b|}\right)
\]

\[
\alpha = \cos^{-1}\left(\frac{1 \times 1 + 1 \times 0}{\sqrt{2} \times 1}\right)
\]

\[
\alpha = \frac{\pi}{4} = 45^\circ
\]

October 4, 2022
Exercise 2

1. Let \( \mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \) and \( \mathbf{w} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \). Compute the scalar product \( \langle \mathbf{v}, \mathbf{w} \rangle \) and the vector lengths \( ||\mathbf{v}|| \) and \( ||\mathbf{w}|| \). Next, find the angle between these two vectors.

2. Compute \( 3 \langle 2 \left( \mathbf{w} + \begin{pmatrix} 1 \\ -4 \end{pmatrix} \right), \begin{pmatrix} -2 \\ -2 \end{pmatrix} \rangle \)

3. (optional) Write a python function that can find the angle between two vectors, given as lists. Test the program on the vectors of 1.
Matrices

A matrix is an array or table of numbers arranged in rows and columns:

\[
A = \begin{pmatrix}
1.5 & 2.5 & 4 \\
-1 & 3 & 2 \\
0 & -5 & 2
\end{pmatrix}
\]
Motivation: Linear transformation

- Matrices can specify **linear transformations**

- The $n$-th column of the matrix is a vector that specifies to where the $n$-th dimension of space is mapped (direction and scaling/compression factor)

Rotation by $45^\circ$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Scaling

$$\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$$
Matrix-vector multiplication

Vectors can be multiplied by a matrix, which applies the transformation:

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= x
\begin{pmatrix}
a_{11} \\
a_{21}
\end{pmatrix}
+ y
\begin{pmatrix}
a_{12} \\
a_{22}
\end{pmatrix}
= \begin{pmatrix}
a_{11}x + a_{12}y \\
a_{21}x + a_{21}y
\end{pmatrix}
\]

Rotation by 45°

\[
\begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
1.0 \\
0.4
\end{pmatrix}
\]

Scaling

\[
\begin{pmatrix}
0.5 & 0 \\
0 & 0.5
\end{pmatrix}
\begin{pmatrix}
1.0 \\
0.4
\end{pmatrix}
\]
Matrix-vector multiplication

This works with an arbitrary number of dimensions:

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= x
\begin{pmatrix}
a_{11} \\
a_{21} \\
a_{31}
\end{pmatrix}
+ y
\begin{pmatrix}
a_{12} \\
a_{22} \\
a_{32}
\end{pmatrix}
+ z
\begin{pmatrix}
a_{13} \\
a_{23} \\
a_{33}
\end{pmatrix}
= \begin{pmatrix}
a_{11}x + a_{12}y + a_{13}z \\
a_{21}x + a_{22}y + a_{23}z \\
a_{31}x + a_{32}y + a_{33}z
\end{pmatrix}
\]
Matrix addition

- Matrices can be added:

\[
\begin{pmatrix}
1 & 0 & 2 \\
3 & 2 & 4 \\
1 & 5 & 7
\end{pmatrix}
+ 
\begin{pmatrix}
2 & 1 & 4 \\
6 & 0 & -3 \\
0 & -5 & 2
\end{pmatrix}
= 
\begin{pmatrix}
1+2 & 0+1 & 2+4 \\
3+6 & 2+0 & 4-3 \\
1+0 & 5-5 & 7+2
\end{pmatrix}
= 
\begin{pmatrix}
3 & 1 & 6 \\
9 & 2 & 1 \\
1 & 0 & 9
\end{pmatrix}
Scalar multiplication

Matrices can be multiplied by a scalar:

\[
2 \cdot \begin{pmatrix}
  1 & 0 & 2 \\
  3 & 2 & 4 \\
  1 & 5 & 7 \\
\end{pmatrix} = \begin{pmatrix}
  2 \cdot 1 & 2 \cdot 0 & 2 \cdot 2 \\
  2 \cdot 3 & 2 \cdot 2 & 2 \cdot 4 \\
  2 \cdot 1 & 2 \cdot 5 & 2 \cdot 7 \\
\end{pmatrix} = \begin{pmatrix}
  2 & 0 & 4 \\
  6 & 4 & 8 \\
  2 & 10 & 14 \\
\end{pmatrix}
\]
Matrix multiplication

► Matrices can be multiplied with each other:

\[
\begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{pmatrix}
\begin{pmatrix}
  b_{11} & b_{12} & b_{13} \\
  b_{21} & b_{22} & b_{23} \\
  b_{31} & b_{32} & b_{33}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{pmatrix}
\begin{pmatrix}
  b_{11} \\
  b_{21} \\
  b_{31}
\end{pmatrix},
\begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{pmatrix}
\begin{pmatrix}
  b_{12} \n  b_{22} \n  b_{32}
\end{pmatrix},
\begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{pmatrix}
\begin{pmatrix}
  b_{13} \n  b_{23} \n  b_{33}
\end{pmatrix}
\]

► Note: a vector is also a matrix

► matrix-vector multiplication is a special case of matrix-matrix multiplication
Matrix multiplication

Matrix multiplication is associative: \((AB)C = A(BC)\)

It follows that linear transformations can be composed by multiplication

Example: Rotation followed by scaling

\[
\begin{pmatrix}
0.5 & 0 \\
0 & 0.5
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
= \begin{pmatrix}
\frac{0.5}{\sqrt{2}} & -\frac{0.5}{\sqrt{2}} \\
\frac{0.5}{\sqrt{2}} & \frac{0.5}{\sqrt{2}}
\end{pmatrix}
\]
**Exercise 3**

1. Create a 2x2 matrix that scales a vector by 2 along the first dimension and by 0.5 along the second dimension. Test the matrix by scaling the vector \((5, 10)\).

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 1 & 4 \\
5 & 2 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
2 \\
-1
\end{pmatrix}
\]

2. Compute \[
\begin{pmatrix}
1 & 2 \\
2 & 1
\end{pmatrix} + \begin{pmatrix}
3 & -1 \\
2 & 2
\end{pmatrix}
\]

3. Compute \[
2 \left( \begin{pmatrix}
1 & 2 \\
2 & 1
\end{pmatrix} \right)
\]

4. Create a matrix that rotates a vector by 90° by composing the rotation matrix for 45° rotations two times.

5. (optional) Create a matrix that rotates a vector by 270°. Do not calculate such a matrix by composing it of other matrices. Instead, directly write down the required matrix entries. Start by thinking about what this rotation means geometrically.

6. (optional) Write a python program that can multiply a vector by a matrix. Represent the vector as a list and the matrix as a list of lists, where each inner