Mathematics and Computer Science for Modeling
Unit 3: Linear Algebra

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based on materials by Jan Tekülve and Daniel Sabinasz

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Course Structure

Unit	Title	Topics
1	Intro to Programming in Python	*Variables, if Statements, Loops, Functions, Lists*
-	Full-Time Programming Session	*Deepen Programming Skills*
2	Functions in Math	*Function Types and Properties, Plotting Functions, Lists*
3	Linear Algebra	*Vectors, Trigonometry, Matrices*
4	Calculus	*Derivative Definition, Calculating Derivatives*

Course Structure

Unit	Title	Topics
5	Integration	*Geometrical Definition, Calculating Integrals, Numerical Integration*
6	Differential Equations	*Properties of Differential Equations, Euler Approximation, Braitenberg Vehicle*
-	Programming Session & Recap	*Repetition, Questions, Test Topics*
-	07.10.22: Test	

Lecture Slides/Material

Use the following URL to access the lecture slides:

https://www.ini.rub.de/teaching/courses/preparatory_course_mathematics_and_computer_science_for_modeling_summer_term_2022

1. Linear Algebra

- Angles and Trigonometry
- Vectors
- Matrices

The number π

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$$\frac{75.39}{24} = 3.14159... = \pi$$

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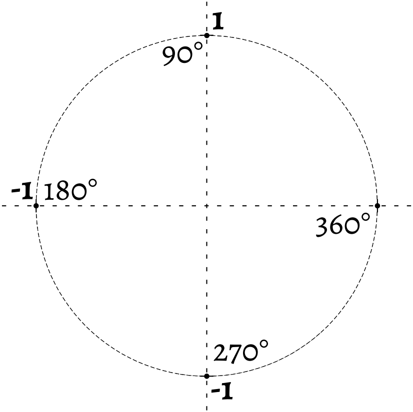
Circumference of a circle: $2\pi r$

Measuring Angles

- Defining a full angle as 360° is common but actually arbitrary



Measuring Angles

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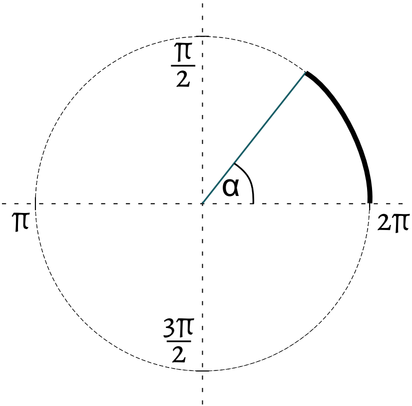
- Less arbitrary is the use of the actual length of the enclosed arc-segment called the **Radian**



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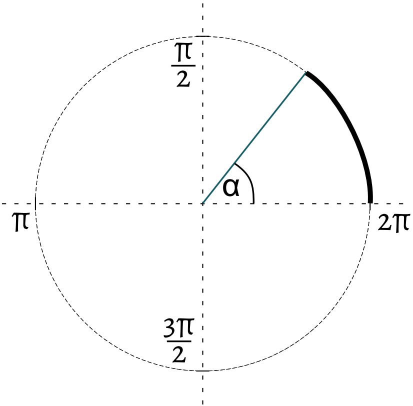
- Thus $360^\circ = 2\pi$, $90^\circ = \frac{\pi}{2}$, $180^\circ = \pi \ldots$



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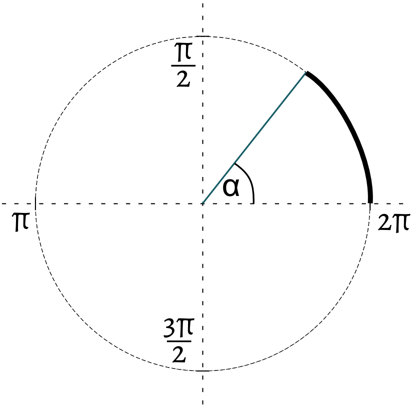
- Rad x to Degree: $x \cdot \frac{180^\circ}{\pi}$



Measuring Angles

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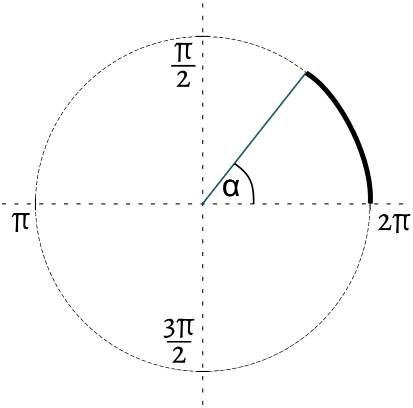
- Degree d to Rad: $d \cdot \frac{\pi}{180°}$



Angle Conversion Examples

- Degree to Radians: $d \cdot \frac{\pi}{180^\circ}$

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- Radians to Degree: $x \cdot \frac{180^\circ}{\pi}$

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$$= \frac{3}{4}\pi \cdot \frac{180^\circ}{\pi}$$

Angle Conversion Examples

- Degree to Radians: $d \cdot \frac{\pi}{180^\circ}$

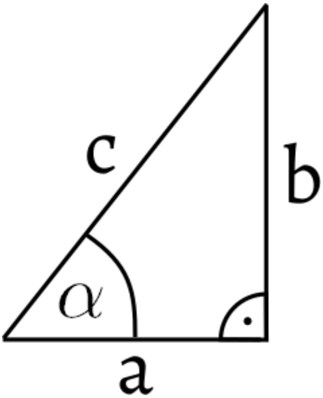
$$\begin{aligned} \alpha_{\text{deg}} &= 34^\circ \\ &= 34^\circ \cdot \frac{\pi}{180^\circ} \\ &= \frac{34^\circ \cdot \pi}{180^\circ} = \frac{106.81^\circ}{180^\circ} = 0.593 = \alpha_{\text{rad}} \end{aligned}$$

- Radians to Degree: $x \cdot \frac{180^\circ}{\pi}$

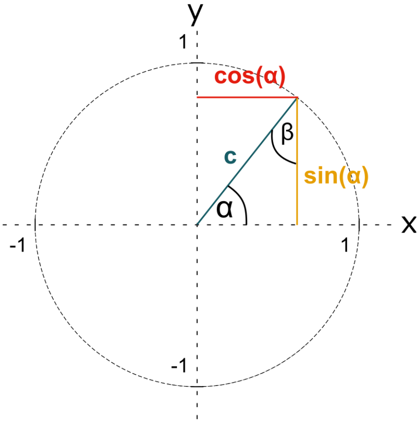
$$\begin{aligned} \alpha_{\text{rad}} &= \frac{3}{4}\pi \\ &= \frac{3}{4}\pi \cdot \frac{180^\circ}{\pi} \\ &= \frac{3}{4} \cdot 180^\circ = 135^\circ = \alpha_{\text{deg}} \end{aligned}$$

Sine and Cosine

- $a^2 + b^2 = c^2$
- $sin(\alpha) = \frac{b}{c} = \frac{\text{opposite}}{\text{hypothenuse}}$
- $cos(\alpha) = \frac{a}{c} = \frac{\text{adjacent}}{\text{hypothenuse}}$
- $tan(\alpha) = \frac{sin(\alpha)}{cos(\alpha)} = \frac{b}{a} = \frac{\text{opposite}}{\text{adjacent}}$



Sine and Cosine



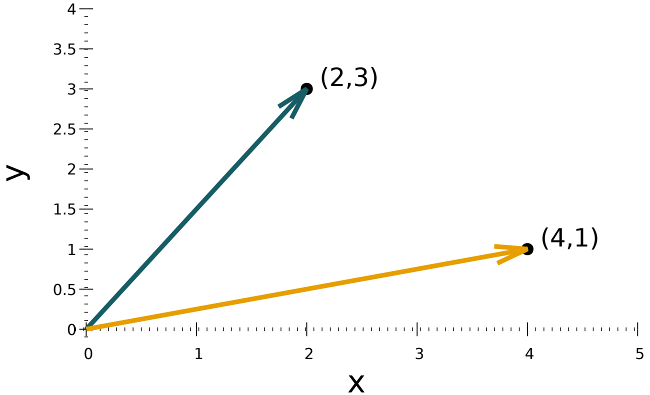
- The sine and cosine of an angle can be interpreted as the x and y coordinates of the location on the unit circle at angle α.

$$x = cos(\alpha) \iff \alpha = cos^{-1}(x)$$
$$y = sin(\alpha) \iff \alpha = sin^{-1}(x)$$

- Click here for interactive demo.

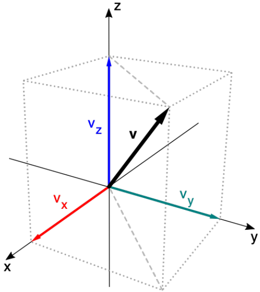
Vectors in the Cartesian Coordinate System

- A vector $\boldsymbol{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ is a geometric object that has **length** and **direction**
- Think of it as an arrow from the origin to the point (v_x, v_y)



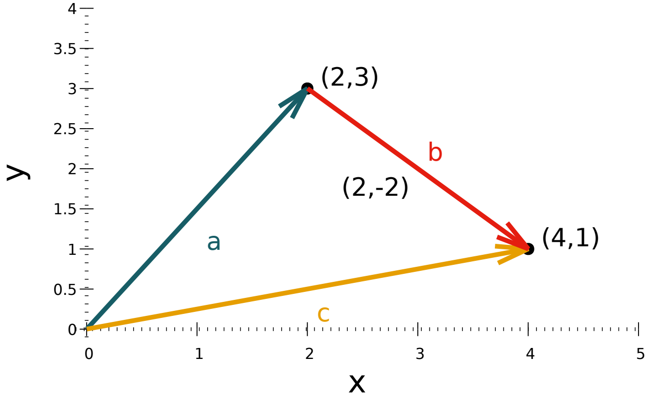
Vectors in more dimensions

- Vectors can be defined in higher-dimensional coordinate systems as well
- e.g., in 3D: $\boldsymbol{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$



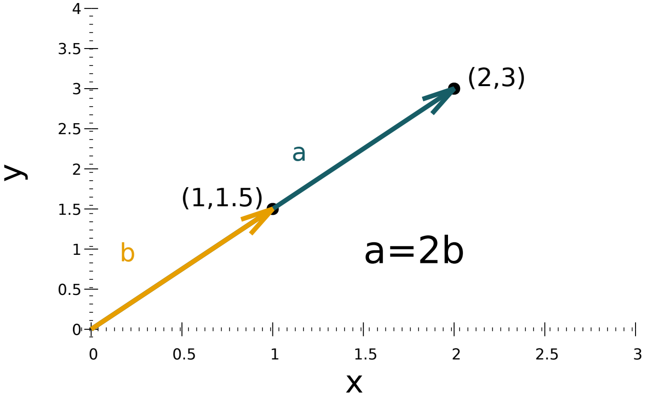
Vector Addition

$$\begin{pmatrix} a_x \\ a_y \end{pmatrix} + \begin{pmatrix} b_x \\ b_y \end{pmatrix} = \begin{pmatrix} a_x + b_x \\ a_y + b_y \end{pmatrix} = \begin{pmatrix} c_x \\ c_y \end{pmatrix}$$



Scalar Multiplication

$$s\mathbf{a} = s\begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} sa_x \\ sa_y \end{pmatrix}$$



Exercise 1

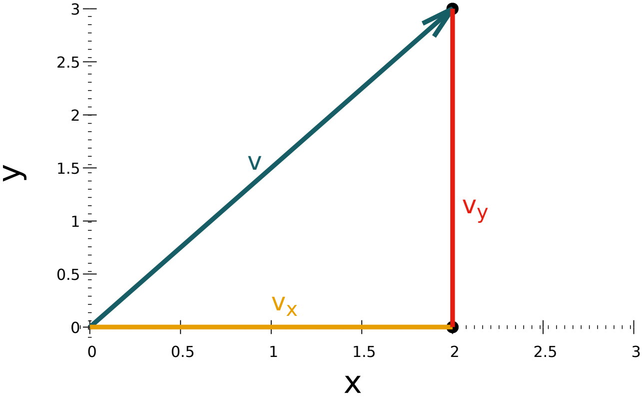
1. Compute the circumference of a circle with radius 2 cm
2. Convert an angle of 45° to radians
3. Convert $\frac{3\pi}{2}$ radians to degrees
4. Given a right triangle with $a = 2$, $b = 3$, compute the angle between a and c
5. Let $\boldsymbol{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Compute $2(\boldsymbol{v} + \boldsymbol{w})$.

Length of a vector

- The length of a vector can be calculated using the Pythagorean theorem:

$$||v|| = \sqrt{v_x^2 + v_y^2}$$

- Graphical Interpretation:



Scalar Product

- The **scalar product** $< \boldsymbol{a}, \boldsymbol{b} >$ or $\boldsymbol{a} \cdot \boldsymbol{b}$ of two vectors is defined as:

$$< \boldsymbol{a}, \boldsymbol{b} > = < \begin{pmatrix} a_1 \\ a_2 \\ \dots \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ \dots \end{pmatrix} > = a_1 b_1 + a_2 b_2 + \dots$$

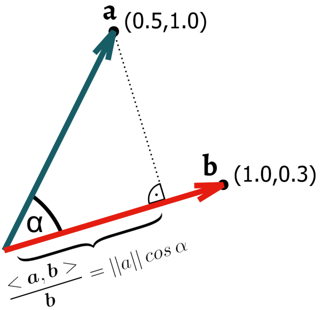
  and results in a **scalar** value.

Scalar Product

- The scalar product is related to the angle between the two vectors:

$$< \boldsymbol{a}, \boldsymbol{b} >= |\boldsymbol{a}||\boldsymbol{b}|cos(\alpha) \iff \frac{< \boldsymbol{a}, \boldsymbol{b} >}{|\boldsymbol{a}||\boldsymbol{b}|} = cos(\alpha)$$

- Graphical Interpretation:



Scalar Product: Special Cases

- If both vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ point in the same direction:

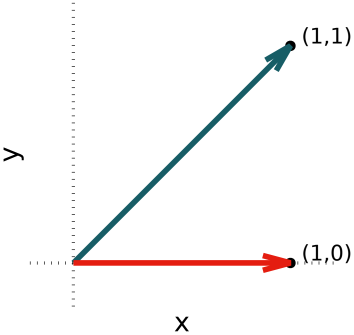
$$< \boldsymbol{a}, \boldsymbol{b} >= |\boldsymbol{a}||\boldsymbol{b}|cos(0) = |\boldsymbol{a}||\boldsymbol{b}|$$

- If both vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal to each other:

$$< \boldsymbol{a}, \boldsymbol{b} >= |\boldsymbol{a}||\boldsymbol{b}|cos(90^\circ) = 0$$

Angle between Vectors

- The scalar product can be used to calculate the angle between two vectors



$$< \boldsymbol{a}, \boldsymbol{b} > = |\boldsymbol{a}||\boldsymbol{b}| cos(\alpha)$$

$$\alpha = cos^{-1}\left(\frac{< \boldsymbol{a}, \boldsymbol{b} >}{|\boldsymbol{a}||\boldsymbol{b}|}\right)$$

$$\alpha = cos^{-1}\left(\frac{1 * 1 + 1 * 0}{\sqrt{2} * 1}\right)$$

$$\alpha = cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\alpha = \frac{\pi}{4} = 45^\circ$$

Exercise 2

1. Let $\boldsymbol{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Compute the scalar product $< v, w >$ and the vector lengths $||v||$ and $||w||$. Next, find the angle between these two vectors.
2. Compute $3 < 2\left(w + \begin{pmatrix} 1 \\ -4 \end{pmatrix}\right), \begin{pmatrix} -2 \\ -2 \end{pmatrix} >$
3. (optional) Write a python function that can find the angle between two vectors, given as lists. Test the program on the vectors of 1.

Matrices

- A **matrix** is an array or table of numbers arranged in rows and columns:

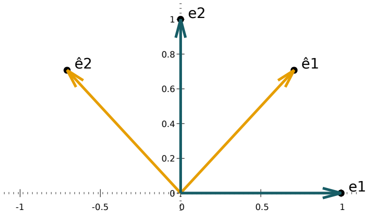
$$\boldsymbol{A} = \begin{pmatrix} 1.5 & 2.5 & 4 \\ -1 & 3 & 2 \\ 0 & -5 & 2 \end{pmatrix}$$

Motivation: Linear transformation

- Matrices can specify **linear transformations**
- The n-th column of the matrix is a vector that specifies to where the n-th dimension of space is mapped (direction and scaling/compression factor)

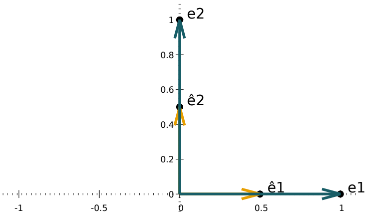
Rotation by 45°

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$



Scaling

$$\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$$



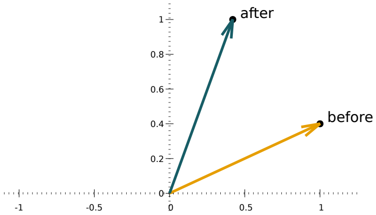
Matrix-vector multiplication

- Vectors can be multiplied by a matrix, which applies the transformation:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + y \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{21}y \end{pmatrix}$$

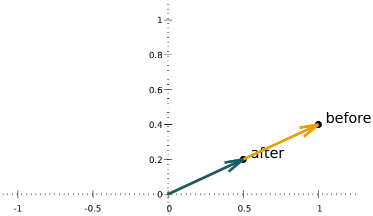
Rotation by 45°

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1.0 \\ 0.4 \end{pmatrix}$$



Scaling

$$\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} 1.0 \\ 0.4 \end{pmatrix}$$



Matrix-vector multiplication

- This works with an arbitrary number of dimensions:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} + y \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} + z \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{pmatrix}$$

Matrix addition

- Matrices can be added:

$$\begin{pmatrix} 1 & 0 & 2 \\ 3 & 2 & 4 \\ 1 & 5 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 4 \\ 6 & 0 & -3 \\ 0 & -5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+2 & 0+1 & 2+4 \\ 3+6 & 2+0 & 4-3 \\ 1+0 & 5-5 & 7+2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & 6 \\ 9 & 2 & 1 \\ 1 & 0 & 9 \end{pmatrix}$$

Scalar multiplication

- Matrices can be multiplied by a scalar:

$$2 \cdot \begin{pmatrix} 1 & 0 & 2 \\ 3 & 2 & 4 \\ 1 & 5 & 7 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 & 2 \cdot 0 & 2 \cdot 2 \\ 2 \cdot 3 & 2 \cdot 2 & 2 \cdot 4 \\ 2 \cdot 1 & 2 \cdot 5 & 2 \cdot 7 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 4 \\ 6 & 4 & 8 \\ 2 & 10 & 14 \end{pmatrix}$$

Matrix multiplication

- Matrices can be multiplied with each other:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{11} & b_{32} & b_{33} \end{pmatrix}$$

$$= \left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix}, \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix}, \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} \right)$$

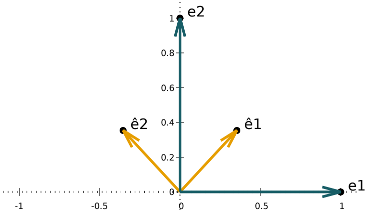
- Note: a vector is also a matrix
- matrix-vector multiplication is a special case of matrix-matrix multiplication

Matrix multiplication

- Matrix multiplication is associative: $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$
- It follows that linear transformations can be composed by multiplication

Example: Rotation followed by scaling

$$\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{0.5}{\sqrt{2}} & -\frac{0.5}{\sqrt{2}} \\ \frac{0.5}{\sqrt{2}} & \frac{0.5}{\sqrt{2}} \end{pmatrix}$$



Exercise 3

1. Create a 2x2 matrix that scales a vector by 2 along the first dimension and by 0.5 along the second dimension. Test the matrix by scaling the vector $\begin{pmatrix} 5 \\ 10 \end{pmatrix}$.
2. Compute $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 5 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$
3. Compute $2\left(\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}\right)$
4. Create a matrix that rotates a vector by 90° by composing the rotation matrix for 45° rotations two times.
5. (optional) Create a matrix that rotates a vector by 270°. Do not calculate such a matrix by composing it of other matrices. Instead, directly write down the required matrix entries. Start by thinking about what this rotation means geometrically.
6. (optional) Write a python program that can multiply a vector by a matrix. Represent the vector as a list and the matrix as a list of lists, where each inner