Mathematics and Computer Science for Modeling
Unit 2: Functions in Math

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Course Structure

Unit	Title	Topics
1	Intro to Programming in Python	*Variables, if Statements, Loops, Functions, Lists*
-	Full-Time Programming Session	*Deepen Programming Skills*
2	Functions in Math	*Function Types and Properties, Plotting Functions, Lists*
3	Linear Algebra	*Vectors, Trigonometry, Matrices*
4	Calculus	*Derivative Definition, Calculating Derivatives*

Course Structure

Unit	Title	Topics
5	Integration	*Geometrical Definition, Calculating Integrals, Numerical Integration*
6	Differential Equations	*Properties of Differential Equations, Euler Approximation, Braitenberg Vehicle*
-	Programming Session & Recap	*Repetition, Questions, Test Topics*
-	07.10.22: Test	

Lecture Slides/Material

Use the following URL to access the lecture slides:

https://www.ini.rub.de/teaching/courses/preparatory_course_mathematics_and_computer_science_for_modeling_summer_term_2022

Sets

- For practical purposes, think of a **set** as a container of objects
- e.g., the set of natural numbers

Sets

- Notation: $\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, \ldots\}$
- Something is either in the set or not in the set
- If something is in the set, we call it an **element** of the set
- e.g., 5 is an element of $\mathbb{N}$, but -3 is not an element of $\mathbb{N}$
- Write $5 \in \mathbb{N}$ and $-3 \notin \mathbb{N}$

Sets

- Instead of listing all the elements, you can describe in natural language what the elements should be
- e.g., $A = \{x \mid x \text{ is an even number}\} = \{0, 2, 4, 6, 8, \ldots\}$

Number Systems

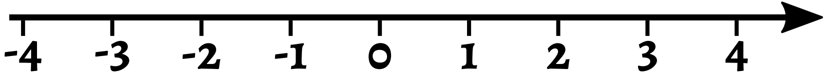
Number Systems

- **Natural Numbers:** $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$
- **Integer Numbers:** $\mathbb{Z} =$
- **Rational Numbers:** $\mathbb{Q}$
- **Real Numbers:** $\mathbb{R}$

Number Systems

Number Systems

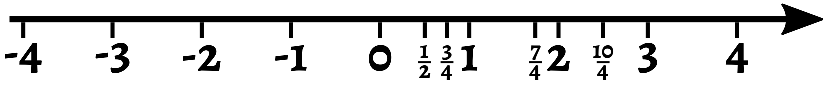
- **Natural Numbers:** $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$
- **Integer Numbers:** $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- **Rational Numbers:** $\mathbb{Q}$
- **Real Numbers:** $\mathbb{R}$



Number Systems

Number Systems

- **Natural Numbers:** $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$
- **Integer Numbers:** $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- **Rational Numbers:** $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}$
- **Real Numbers:** $\mathbb{R}$

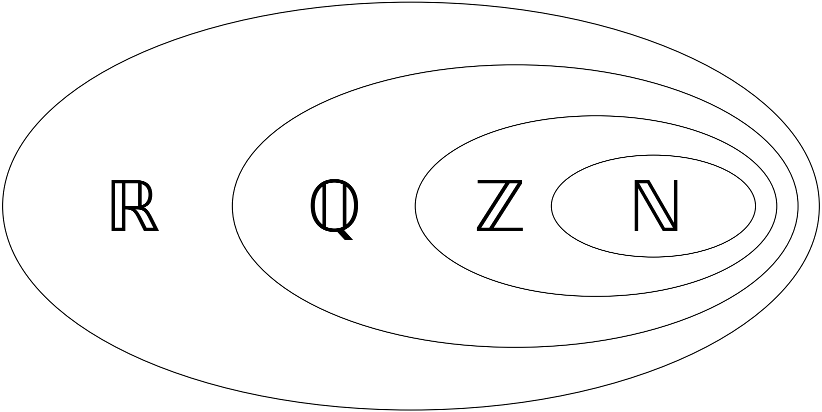


Number Systems

Number Systems

- **Natural Numbers**: $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$
- **Integer Numbers**: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- **Rational Numbers**: $\mathbb{Q} = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$
- **Real Numbers**: $\mathbb{R} = \mathbb{Q} \cup$ irrational numbers

Number Systems



1. Sets and Number Systems

2. Functions in Math

- Definition
- Function Types
- Parametrization
- Multiple Arguments
- Properties

Function Intuition

- Function example: $f(x) = 2x + 3$
- A function, written like this, can be thought of as a formula that can be evaluated to give the value of the function
- e.g.,
  - $f(1) = 2 \cdot 1 + 3 = 5$
  - $f(2) = 2 \cdot 2 + 3 = 6$

Plotting Functions

Tabular Interpretation of: $f(x) = 2x + 3$

x	0	1	2	3	4	5
y						

Plotting Functions

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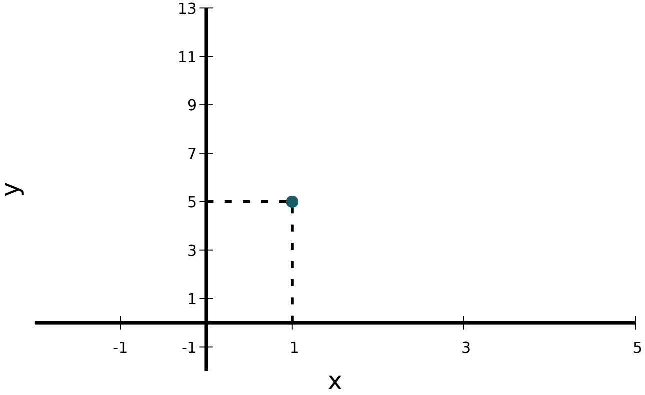
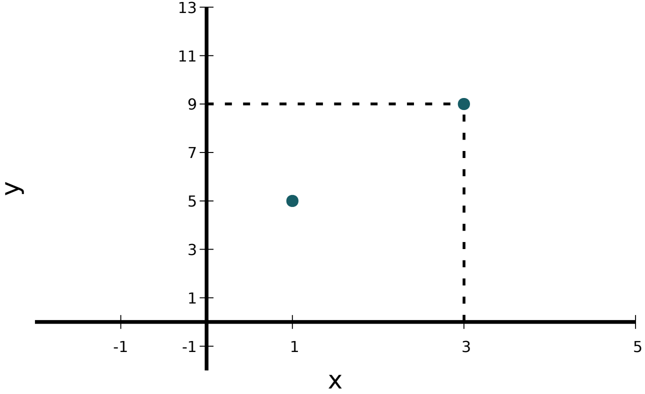


Plotting Functions

Tabular Interpretation of: $f(x) = 2x + 3$

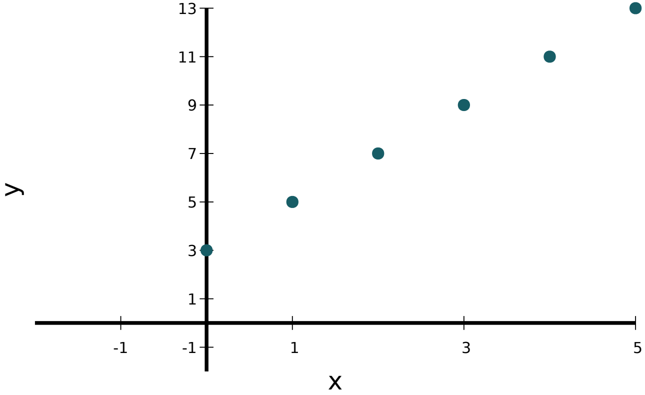
x	0	1	2	3	4	5
y		5		9		



Plotting Functions

Tabular Interpretation of: $f(x) = 2x + 3$

x	0	1	2	3	4	5
y	3	5	7	9	11	13



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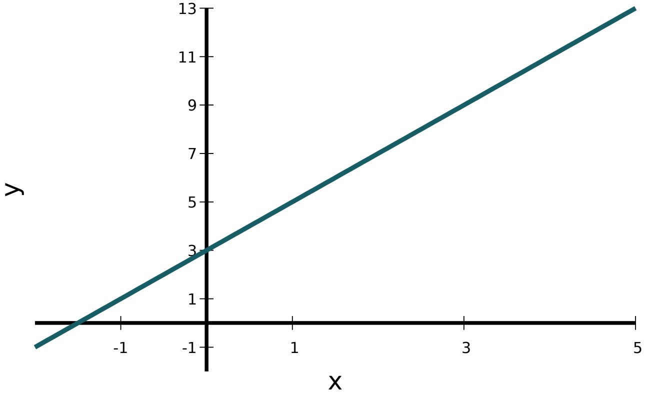
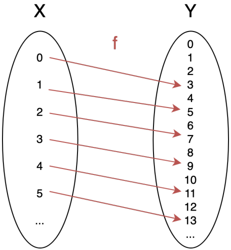


Function Definition

Function

X and Y are two sets.
A **function** $f : X \to Y$ is a mathematical object that assigns each element $x \in X$ exactly one element $y \in Y$.

$$x \to y = f(x)$$



- x is called the **function argument**
- y is called the **function value**
- X is called the **domain**
- Y is called the **codomain**
- The **image** W of $f(x)$ are all values in Y that can be assumed by the function.

Matplotlib

- Matplotlib allows to plot functions:

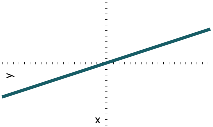
```
import matplotlib.pyplot as plt

numbers = [2*x+3 for x in range(6)]

plt.plot(numbers)
plt.show()
```

Function Types

- **Linear Functions**
  $y = mx + b$



Function Types

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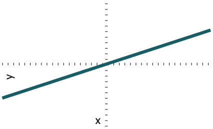
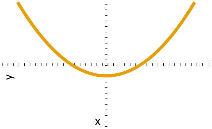


- **Power Functions**
  $y = ax^n$



Function Types

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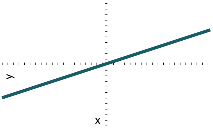

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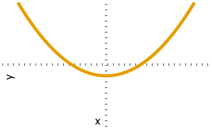
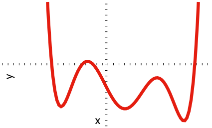


- **Polynomial Functions**
  $y = \sum_{i=0}^{n} a_i x^i$
  $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots a_n x^n$
  describes a polynomial of degree n, where $a_n \neq 0$



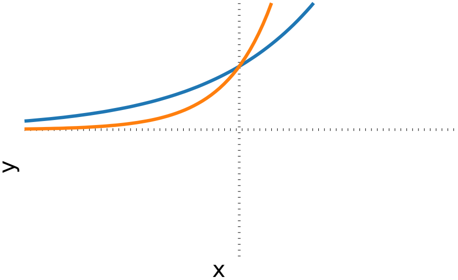
The Summation Symbol

- $\sum_{i=0}^{n} T(i)$ denotes a sum of multiple terms
- The bottom row defines an indexing variable, here i, and specifies an initial value, here 0
- That variable takes on increasing values (0, 1, 2, 3, . . . , n)
- The top row specifies the maximum value for i, here n
- $T(i)$ specifies a term for each i
- $\sum_{i=0}^{n} T(i)$ sums up $T(i)$ for each i
- Thus, $\sum_{i=0}^{n} T(i) = T(0) + T(1) + T(2) + \ldots + T(n)$
- e.g., $\sum_{i=0}^{5} i = 0 + 1 + 2 + 3 + 4 + 5$

Exponentials Functions

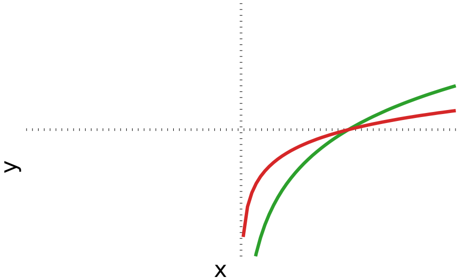
Exponential Functions

$f(x) = e^x$ $g(x) = 10^x$



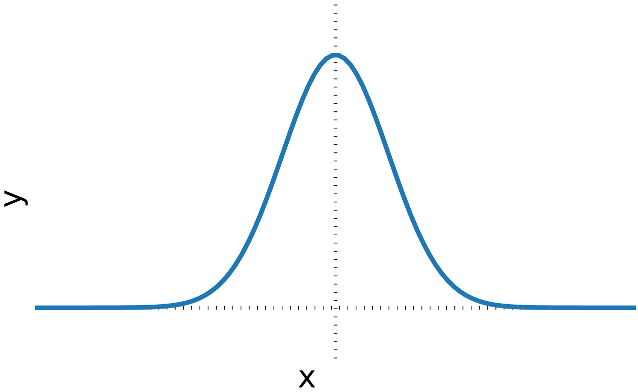
Logarithmic Functions

$h(x) = ln(x)$ $j(x) = log_{10}(x)$



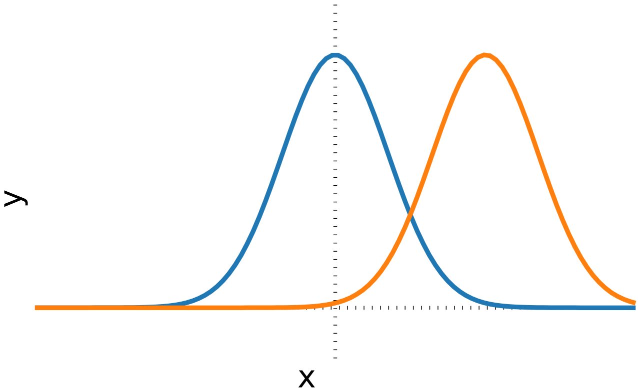
The Gaussian Function

$$f(x) = e^{-x^2}$$



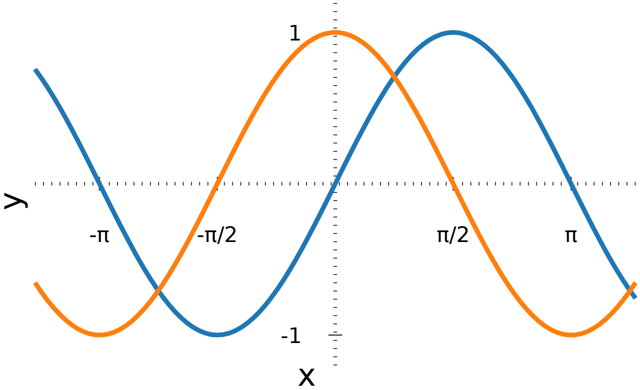
The Gaussian Function

$f(x) = e^{-x^2}$ $g(x) = e^{-(x-2)^2}$



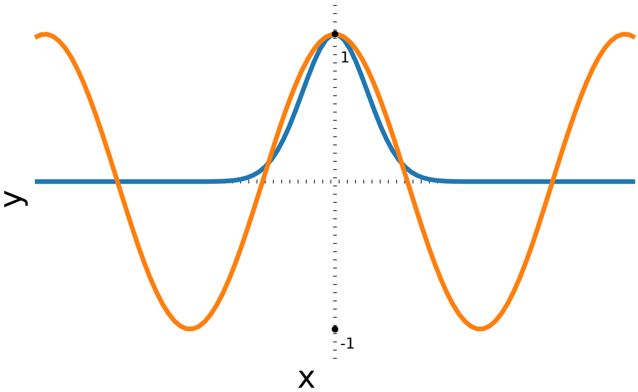
Trigonometric Functions

$f(x) = sin(x)$ $g(x) = cos(x)$



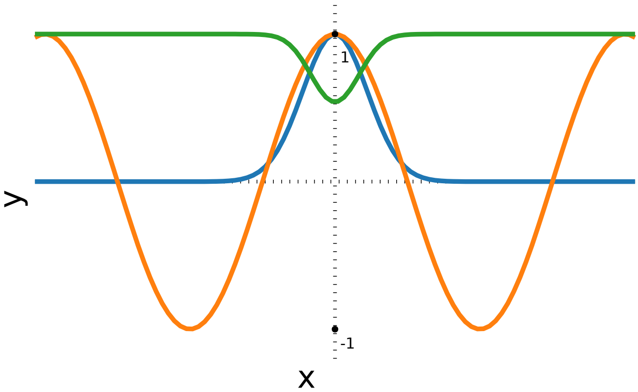
Chaining Functions

$f(x) = e^{-x^2}$ $g(x) = cos(x)$



Chaining Functions

$f(x) = e^{-x^2}$ $\quad g(x) = cos(x)$ $\quad h(x) = g(f(x))$



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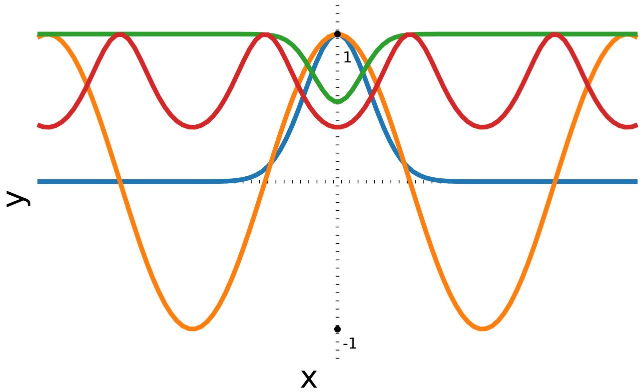
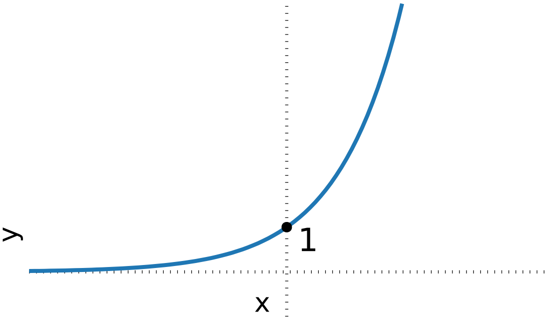
$f(x) = e^{-x^2}$ $g(x) = cos(x)$ $h(x) = g(f(x))$ $j(x) = f(g(x))$



Function Translation

- Translation in y-direction: $\hat{f}(x) = f(x) + b$
- Translation in x-direction: $\hat{f}(x) = f(x - a)$



$f(x) = e^x$

Function Translation

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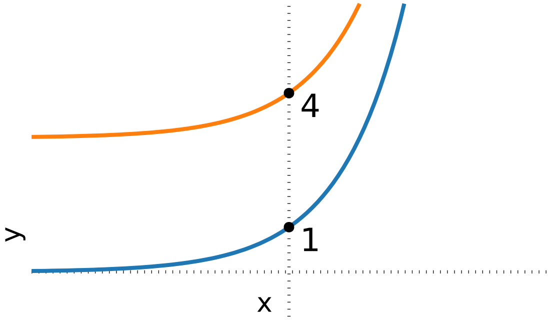

$f(x) = e^x$ $\quad$ $g(x) = e^x + 3$

Function Translation

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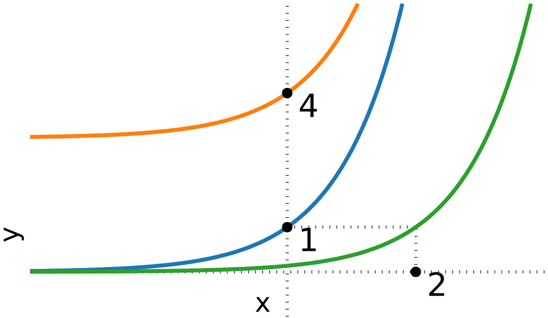
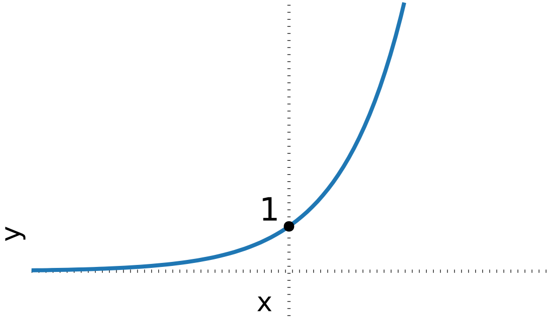


$f(x) = e^x$ $\quad$ $g(x) = e^x + 3$ $\quad$ $h(x) = e^{x-2}$

Function Stretching and Compression

- Stretching/Compression in ***y*-direction**: $\hat{f}(x) = \boldsymbol{d}f(x), d > 0$
- Stretching/Compression in ***x*-direction**: $\hat{f}(x) = f(\boldsymbol{c}x), c > 0$



$f(x) = e^x$

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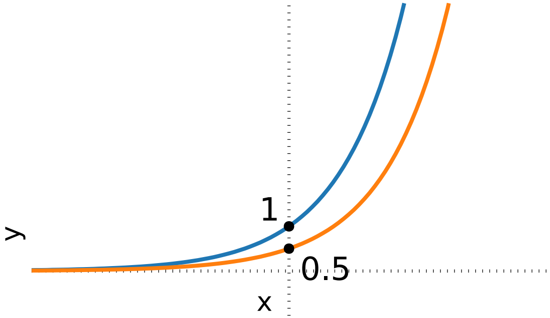

$f(x) = e^x$ $\qquad$ $g(x) = \frac{1}{2}e^x$

Function Stretching and Compression

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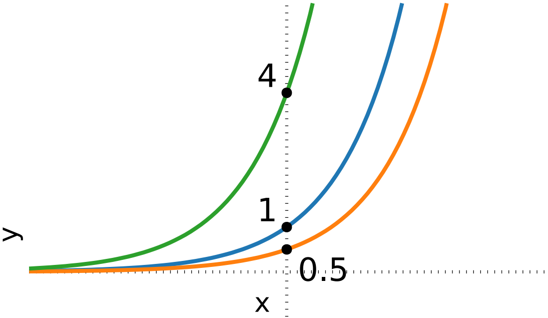

$f(x) = e^x$ $\quad g(x) = \frac{1}{2}e^x$ $\quad h(x) = 4e^x$

Function Stretching and Compression

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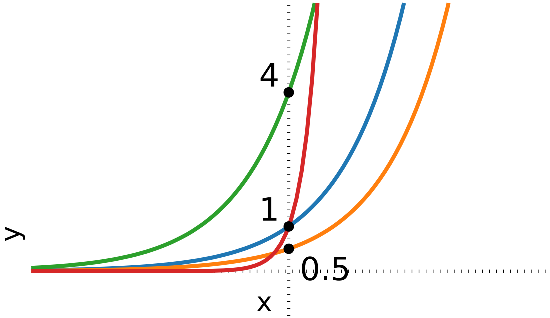
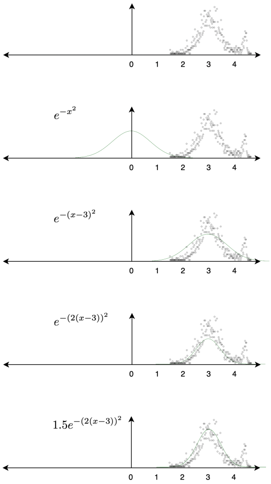


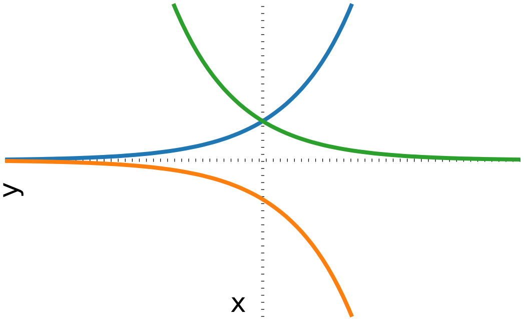
$f(x) = e^x$ $g(x) = \frac{1}{2}e^x$ $h(x) = 4e^x$ $j(x) = e^{4x}$

Example



Function Reflection

- Reflection across the ***y*-axis**: $\hat{f}(x) = f(-x)$
- Reflection across the ***x*-axis**: $\hat{f}(x) = -f(x)$



$f(x) = e^x$ $g(x) = -e^x$ $h(x) = e^{-x}$

Exercise 1

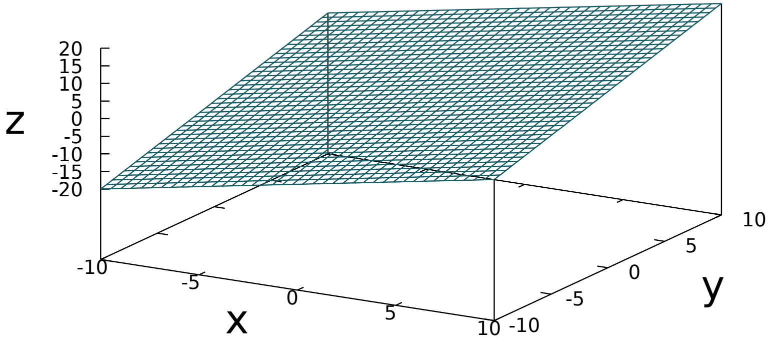
1. Give an example for a natural number, a negative integer, a rational number and an irrational number
2. Which of the following is true? (a) Every real number is rational. (b) Every integer is rational. (c) Every natural number is a real number.
3. Let $f : \mathbb{N} \to \mathbb{R}, x \to 2x + 3$. Identify the function argument, the function value, the domain, the codomain and the image.
4. Create a function $\hat{f}(x)$ by translating $f(x) = e^x$ by -2 in y-direction and by 3 in x-direction.
5. Create a function $\hat{f}(x)$ by stretching $f(x) = e^x$ along the y-axis and compressing it along the x-axis.
6. Create a function $\hat{f}(x)$ by compressing $f(x) = e^x$ along the y-axis and stretching it along the x-axis.

Exercise 2

1. Write a python function that calculates $f(x) = 4x + 3$ and plot it.
2. Define a second function $g(x, a_0, a_1, a_2, a_3)$ that calculates a polynomial of degree 3 with variable coefficients a_0 to a_3 and plot $g(x, 3, 0, 2, 1)$
3. Calculate $f(x)$ or $g(x, 3, 0, 2, 1)$ for x values from 0 to 20. Store the result in a list.
4. (optional) Define a function 'polynomial(a, x)' that receives a list of coefficients 'a' $(a_0, a_1, a_2, \ldots, a_n)$ with a flexible number of items and computes $\sum_{i=0}^{n} a_i x^i$.

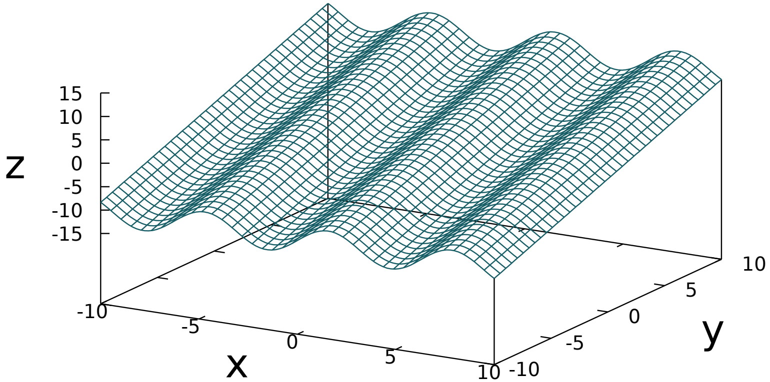
Multiple Arguments

$$f(x, y) = x + y$$



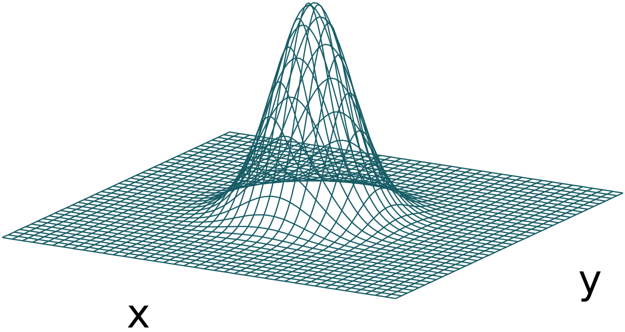
Multiple Arguments

$$f(x,y) = sin(x) + y$$



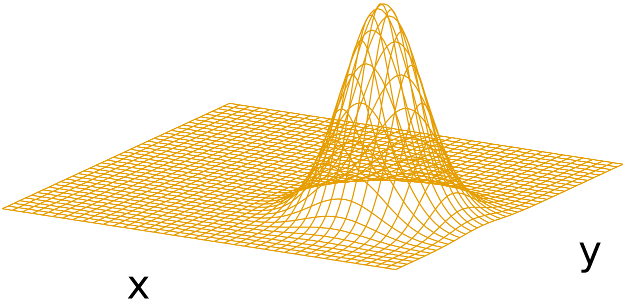
Multiple Arguments

$$f(x,y) = e^{-(x^2+y^2)}$$



Multiple Arguments

$$f(x,y) = e^{-((x-2)^2+(y+1)^2)}$$



Injective, Surjective and Bijective Functions

- An image f is **injective**, if two different elements $x_1 \neq x_2$ are always projected to two different elements $y_1 \neq y_2$
- An image f is **surjective**, if for each element $y \in Y$ one $x \in X$ exists, such that $y = f(x)$
- An image f is **bijective**, if it is injective and surjective

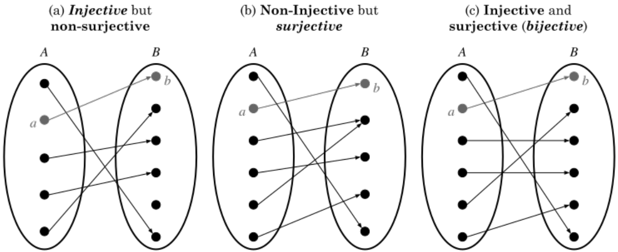


Image source:
`https://commons.wikimedia.org/wiki/File:Injective,_Surjective,_Bijective.svg`

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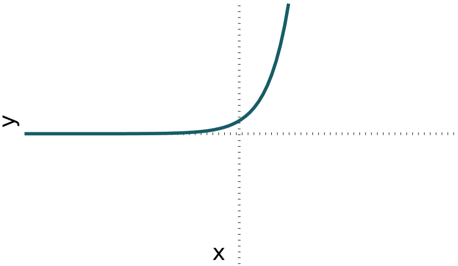
- An image f is **surjective**, if for each element $y \in Y$ one $x \in X$ exists, such that $y = f(x)$

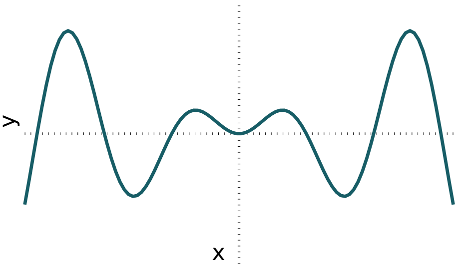
Injective, but not surjective

$f(x) = e^x$



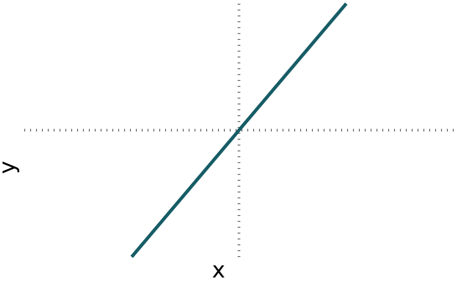
Surjective, but not Injective

$f(x) = x \sin(x)$

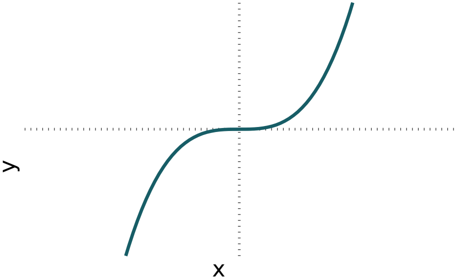


Bijective Function Example

$f(x) = 4x$



$f(x) = x^3$

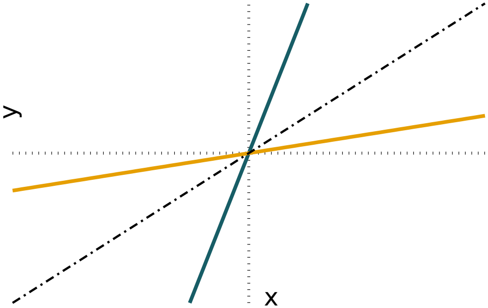


Inverse Function

Definition

Given a bijective **function** $f : X \to Y$, $f^{-1} : Y \to X$ denotes the **inverse function** of f.

It holds that $f^{-1}(f(x)) = x$ for all $x \in X$.



$f(x) = 4x$ $\qquad$ $f^{-1}(x) = \frac{1}{4}x$

Inverse Function

Definition

Given a bijective **function** $f : X \to Y$, $f^{-1} : Y \to X$ denotes the **inverse function** of f.

It holds that $f^{-1}(f(x)) = x$ for all $x \in X$.

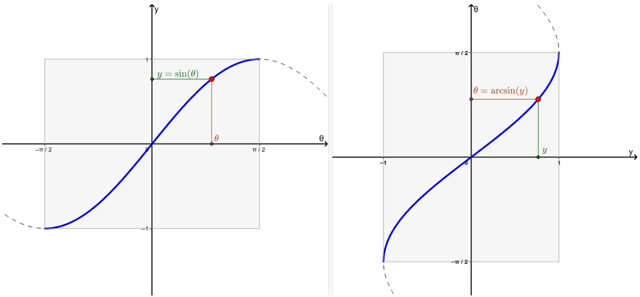


Image source:

`https://www.geogebra.org/m/Efs8QRRF`

Monotonicity

Definition

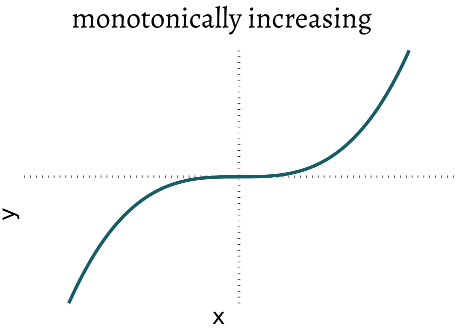
- A function $f : \mathbb{R} \to \mathbb{R}$ is called **monotonically increasing**, if for all x_1, x_2 order is preserved by applying f:

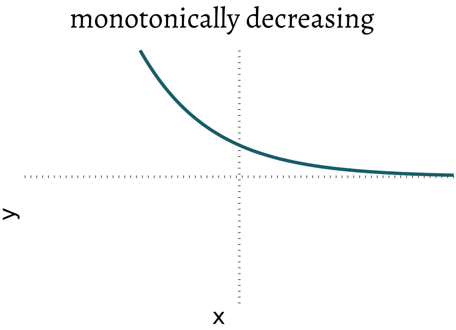
$$x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$$

- A function $f : \mathbb{R} \to \mathbb{R}$ is called **monotonically decreasing**, if for all x_1, x_2 order is reversed by applying f:

$$x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$$

Monoticity Examples





Functions Exercise 3

1. Write a python function that calculates $f(x, y) = 4x^2 + 2(y - 2)^2$ and plot it.
2. Determine the inverse $f^{-1}(x)$ of $f(x) = 2x + 3$
3. For each of the following functions, determine if they are monotonically increasing, monotonically decreasing or neither: $f(x) = x^2$, $f(x) = -x^5$, $f(x) = x^7$