Exercise 1: Solution and remarks

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Task 1

1a)

In this task we plot $\dot{x}$ against x. A qualitative plot as shown in Figure 1 is sufficient. Important features of the plot that characterize the dynamics of the system are the linear course of the dynamics, its slope, and the zero crossing at the origin ($\dot{x} = 0, x = 0$).

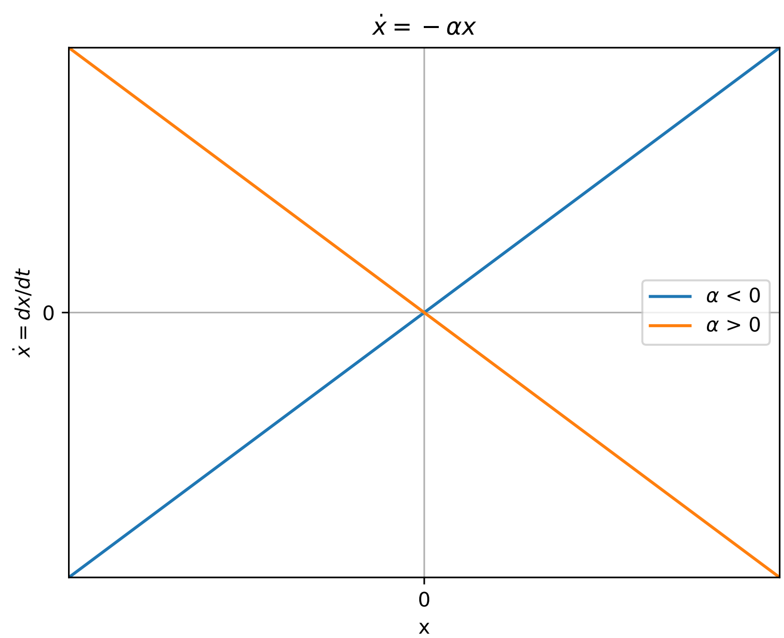

Figure 1: Phase plot of the dynamical system $\dot{x} = -\alpha x$ for $\alpha > 0$ and $\alpha < 0$.

1b)

We integrate the equation $\dot{x} = -\alpha x$ to get the solution:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= -\alpha x \\
\Rightarrow \frac{\mathrm{d}x}{x} &= -\alpha \,\mathrm{d}t \\
\Rightarrow \int \frac{\mathrm{d}x}{x} &= -\alpha \int \mathrm{d}t \\
\Rightarrow \ln(x) &= -\alpha t + C \\
\Rightarrow x(t) &= \exp(-\alpha t + C) \\
\Rightarrow x(t) &= \exp(C) \exp(-\alpha t) \\
\Rightarrow x(t) &= \tilde{C} \exp(-\alpha t) \\
\Rightarrow x(t) &= x_0 \exp(-\alpha t)
\end{aligned}$$

We can write $\exp(C)$ as a new constant $\tilde{C}$ and see that this constant corresponds to the initial condition x_0 by setting t equal to zero.
Next we check if the solution $x(t)$ solves the equation by taking its time-derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = -\alpha x_0 \exp(-\alpha t) = -\alpha x(t).$$

1c)

Now we know the solution and can plot it for two initial conditions x_0 (see Figure 2).

1d)

This is a more advanced question, we may talk about this in the exercise session. Let us first follow the instructions on the exercise sheet and then see what the result actually means!

We want to know the times[1] t_n at which the solution $x(t)$ reaches the value $x_0 / \exp(n)$. With mathematical symbols we can express it this way:

$$x(t_n) = x_0 \exp(-\alpha t_n) \overset{!}{=} x_0 \exp(-n),$$

where the $\overset{!}{=}$ means that we *want* something to be equal, not that we know that it is equal. In fact, we might find out that the expressions on the left-hand side

[1] It is "times" (rather than "time") because you can have any value for n which gives you an infinite number of values for the time. But this does not really have an effect on how you compute it.

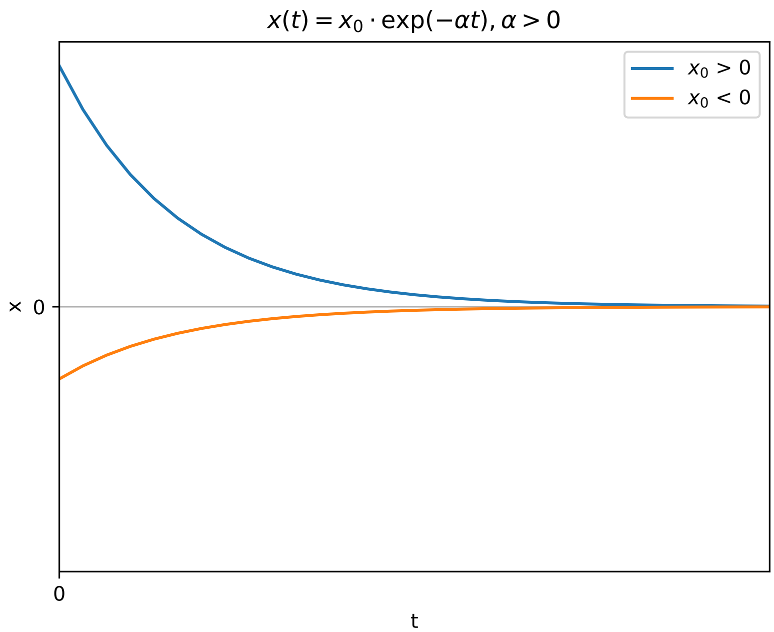

Figure 2: Time course of the solution $x(t) = x_0 \cdot \exp(-\alpha t)$ for a positive initial condition (blue line) and a negative initial condition (orange line).

and right hand-side of $\stackrel{!}{=}$ cannot be equal. However, this is not the case here:

$$\begin{aligned} x_0 \exp(-\alpha t_n) &\stackrel{!}{=} x_0 \exp(-n) \\ \Leftrightarrow \exp -\alpha t_n &= \exp -n \\ \Rightarrow -\alpha t_n &= -n \\ \Leftrightarrow t_n &= \frac{n}{\alpha} \end{aligned}$$

We calculated the times t_n at which the initial condition is reduced to $x_0 / \exp(n)$. In the lectures the *relaxation time* was probably introduced as the "time the initial condition is reduced to its value over e" which equals the time $t_1 = x_0/\mathrm{e}$ with $n = 1$!

Let us now take a look at $t_{n+1} - t_n$, which is the "formal" definition of the relaxation time.

$$t_{n+1} - t_n = \frac{n+1}{\alpha} - \frac{n}{\alpha} = \frac{n-n+1}{\alpha} = \frac{1}{\alpha}$$

We see that the relaxation time is independent of n. This is a fundamental property of exponential decay: It does not matter at which time, t_n, we observe the system, the time after which the solution $x(t_n)$ is reduced to its value over e is always the same. This is why the term “relaxation time” often appears in the context of exponential decay because in other systems it may be more difficult to find such a characteristic time.

1e)

We know from the previous task that the relaxation time depends on α: $t_r = \frac{1}{\alpha}$. For larger values of α the relaxation time decreases, which means that the system relaxes faster to an equilibrium state, here the fixed point at $x = 0$.

In figure 3 we see that the solution with a larger value of α relaxes faster to zero - it reaches the characteristic value $\frac{x(0)}{\mathrm{e}}$ earlier, because its relaxation time is shorter.

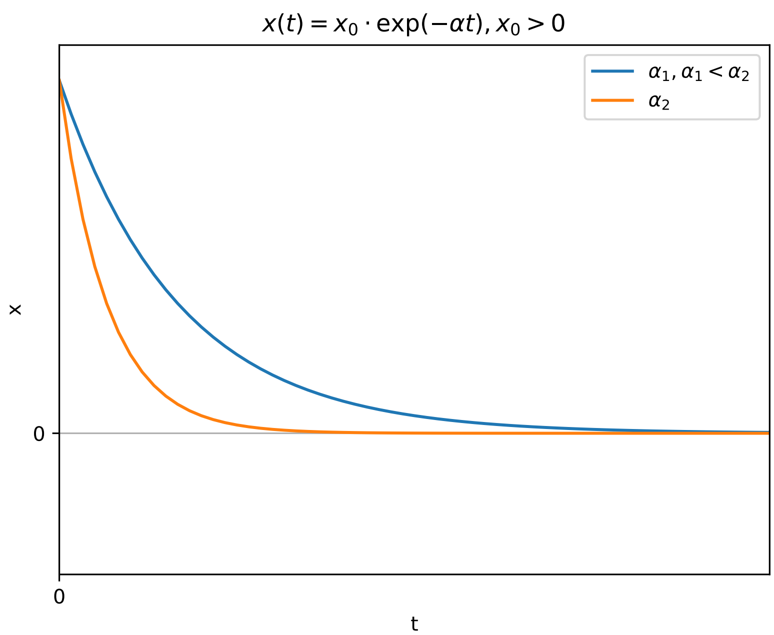

Figure 3: Time course of the solution $x(t) = x_0 \cdot \exp(-\alpha t)$ for two different values of α, where $\alpha_1 < \alpha_2$.

Task 2

2a)

Analogous to the first task we plot the dynamics $\dot{x} = a - x^2$ in Figure 4.

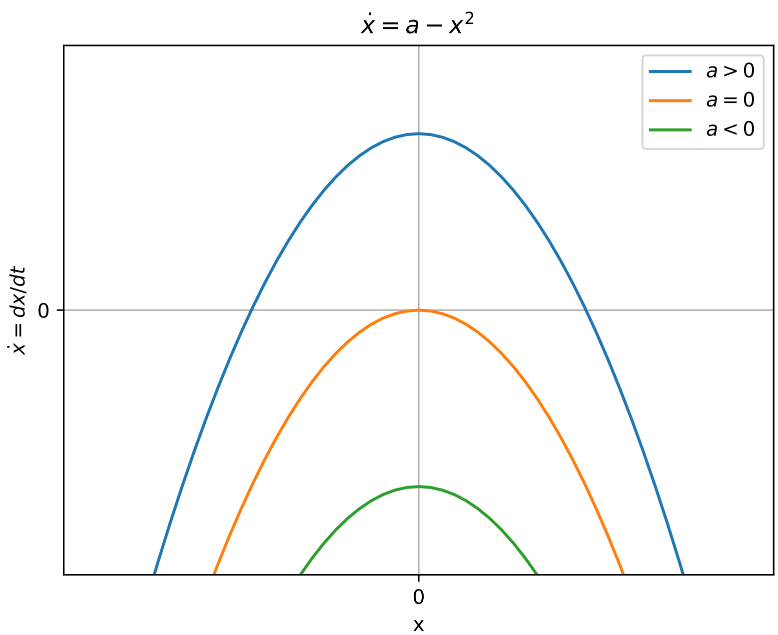

Figure 4: Phase plot of the dynamics $\dot{x} = a - x^2$ for different values of a.

2b)

In this task we calculate the fixed points of the dynamics. They are characterized by a time derivative equal to zero, $\dot{x} = 0$. Since $\dot{x}$ describes how the solution (or the "state" of the dynamical system) changes with time, a value of zero means that the state will not change anymore and is thus *fixed*.

$$\dot{x} = 0 = a - x^2$$
$$\Leftrightarrow x^2 = a$$
$$\Rightarrow x_{1,2} = \pm\sqrt{a}$$

Mind, that when we take the square root of a function, we always obtain two solutions $\pm\sqrt{\ldots}$. We can see this in Figure 4 with $a > 0$: The solution crosses the x-axis twice.

From 1a) we also see that with $a = 0$ there is only one fixed point, as $-\sqrt{0} = +\sqrt{0} = 0$. The last case, $a < 0$, is more interesting: From the first task we see that there cannot be a fixed point since the solution never crosses the x-axis. From our calculation we see that there is no (real) solution to $x^2 = a$ when a is a negative number. Thus, for $a < 0$, there are no fixed points.

2c)

We use Figure 4 and modify it for our "mental simulation" for $a > 0$, we will end up with Figure 5.

First, we mark all the fixed points, that is $x = \pm\sqrt{a}$, because in these points we know that the state will not change over time. Thus, with the initial condition x_0 equal to $-\sqrt{a}$ and $\sqrt{a}$, the asymptotic behavior when time goes to infinity is $x \to -\sqrt{a}$ and $x \to \sqrt{a}$, respectively.

Next we investigate the time derivative $\dot{x}$ for all states x that are smaller than $-\sqrt{a}$: We can see that $\dot{x}$ is negative so that every state $x < -\sqrt{a}$ will be driven toward smaller values. This is indicated by the arrow pointing to "the left" in Figure 5. Since there is no fixed point at values smaller than $-\sqrt{a}$ that could "stop" this behavior the state will go toward minus infinity with time going to infinity.

We already know what happens when we "start" in the fixed point $x = -\sqrt{a}$, thus we focus on the next part, all states between $-\sqrt{a}$ and $\sqrt{a}$. Here the time derivative is positive, so that all states are driven toward larger values, as indicated by the arrows "to the right" in Figure 5. But here the behavior will be "stopped" by the fixed point at $x = \sqrt{a}$. As soon as the state reaches $\sqrt{a}$, its derivative is equal to zero and it will stop its journey toward $+\infty$.

The last part includes all states that are greater than $\sqrt{a}$. Here again, the time derivative is negative and the state is driven toward smaller value. But these states are also "caught" by the fixed point $x = \sqrt{a}$.

From our analysis we can see that the fixed point $x = \sqrt{a}$ *attracts* all states around it; this is why we call it an attractor. It is a stable fixed point because the system relaxes back to the fixed point if it is perturbed out of it. We can also see that this is an attractor because of the negative slope of the dynamics in the fixed point.

The other fixed point $x = -\sqrt{a}$ is a repellor because it *repels* all states around it. It is unstable because if the system is perturbed out of the fixed point $x = -\sqrt{a}$ the state will be pushed away from it. Nevertheless, it is possible that the solution stays exactly in this fixed point if it is "started" there. An example of such an unstable equilibrium state is if you set a pencil or stick upright on the table: It will stay upright, but even a small perturbation will make it fall to the ground and you would never expect it to move back :-)

Figure 5 summarizes our findings for $a > 0$. Considering $a < 0$, we know from the previous tasks that there are no fixed points in which the state would "stay". Because the time derivative is negative for all values of x, the state is driven toward smaller values and in a figure analogous to Figure 5 we would

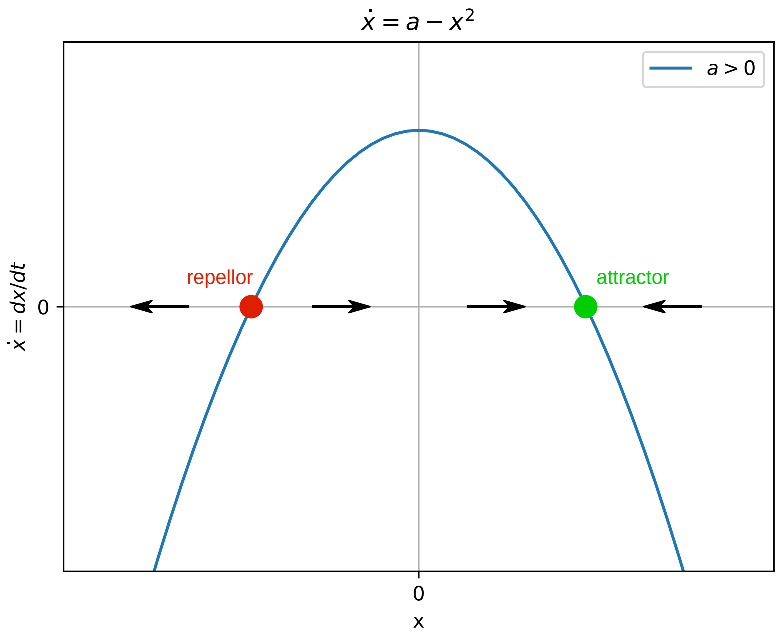

Figure 5: Phase plot for the same dynamics as in Figure 4 for $a > 0$. Arrows denote the asymptotic behavior of the dynamics for different regions of initial conditions. Not depicted is the case that the dynamics will not change if the initial condition is the position of the repellor ($x_0 = -\sqrt{a}$).

draw an arrow pointing "to the left". Thus, with time going to infinity, the state is going to $-\infty$.

2d)

For $a = 0$ we can solve the differential equation analytically to get a better picture of the solution $x(t)$.

In a first step we write $\dot{x}$ as $\frac{\mathrm{d}x}{\mathrm{d}t}$ and then "separate the variables", which means that all terms with an x go to one side of the equation and all those with a t to the other (pretending that $\mathrm{d}x$ and $\mathrm{d}t$ are normal variables). Then we integrate over x on one side and over t on the other:

$$\begin{aligned} \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} &= x^2 \\ \Rightarrow \frac{\mathrm{d}x}{x^2} &= -\mathrm{d}t \\ \Rightarrow \int_{x(0)}^{x(t)} \frac{\mathrm{d}x}{x^2} &= -\int_0^t \mathrm{d}t \\ \Rightarrow \left[-x^{-1}\right]_{x(0)}^{x(t)} &= -\left[t\right]_0^t \end{aligned}$$

We can either integrate with limits 0 to t and $x(0)$ to $x(t)$ or integrate without limits but add a constant. Using limits may be more elegant as we do not have to determine the constant by setting t equal to zero and calculating the initial condition of the solution. In both cases we end up with the solution:

$$\begin{aligned} -x(t)^{-1} + x(0)^{-1} &= -t \\ \Leftrightarrow x(t)^{-1} &= t + x(0)^{-1} \\ \Leftrightarrow x(t) &= \frac{1}{t + x(0)^{-1}} \end{aligned}$$

2e)

With $x(0) > 0$, the denominator of our solution $t + x(0)^{-1}$ is getting larger with time going to infinity. For the solution $x(t)$ this means that it is getting smaller, eventually reaching zero. We can write this mathematically as

$$\lim_{t\to\infty} \frac{1}{t + x(0)^{-1}} = 0.$$

We can also take a look at Figure 4 for $a = 0$. The time derivative is negative, so any state is driven toward smaller values until it reaches the fixed point at $x = 0$, where it stays. The solution is shown in Figure 6 (blue curve), starting at a positive value at $t = 0$ and relaxing to zero.

2f)

It is more complex to understand what happens for $x(0) < 0$. Again we take a look at the denominator $t + x(0)^{-1}$. Now $x(0)$ has a negative value but time t still starts at zero and then takes on positive values (as we are going toward the future, not toward the past!). Thus, we first have some negative values for $x(t)$ that are getting smaller (further toward negative values) as time t approaches the absolute value $|x(0)^{-1}|$. When t reaches $|x(0)^{-1}|$, the denominator $t + x(0)^{-1}$ is equal to zero and the solution *diverges*. This is because the function $x(t)$ is not defined at that point as we divide by zero and the solution reaches -infinity at a

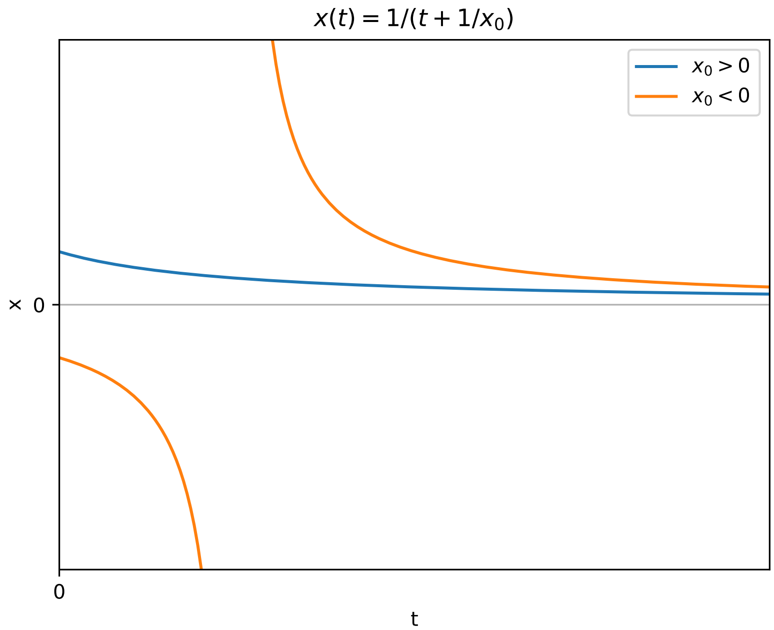

Figure 6: Time course of the solution we computed in 2d) for $a = 0$. For positive initial conditions $x(0) > 0$ the system relaxes to the fixed point at 0 (blue curve). For negative initial conditions $x(0) < 0$, the system moves toward negative infinity at a finite time.

finite time t. This is shown by the orange curve in Figure 6. The system starts at $t = 0$ at some negative value $x(0)$ and then decreases toward $-\infty$. There is no solution for $t = |x(0)^{-1}|$. But there seems to be a solution for $t > |x(0)^{-1}|$, too. However, since our system starts with a negative initial condition it is captured on the branch of the solution diverging to $-\infty$ and the system will not be able to "jump" over from one branch to the other at $t = |x(0)^{-1}|$. In fact, the second branch, which relaxes to zero but comes from $+\infty$, corresponds to the behavior of the system we get with a positive initial condition but shifted on the time axis.

Finally, we can mark the fixed point at $x = 0$ and add the arrows indicating the flow of the dynamical system in Figure 7. We see that the fixed point attracts all states $x > 0$, but repels all states $x < 0$, which is why we call it *marginally stable*. With the parameter a decreasing from positive to negative values we observe a *bifurcation*: the two stable fixed points for $a > 0$ collide, leading to a marginally stable fixed point for $a = 0$, which eventually disappears when a becomes negative.

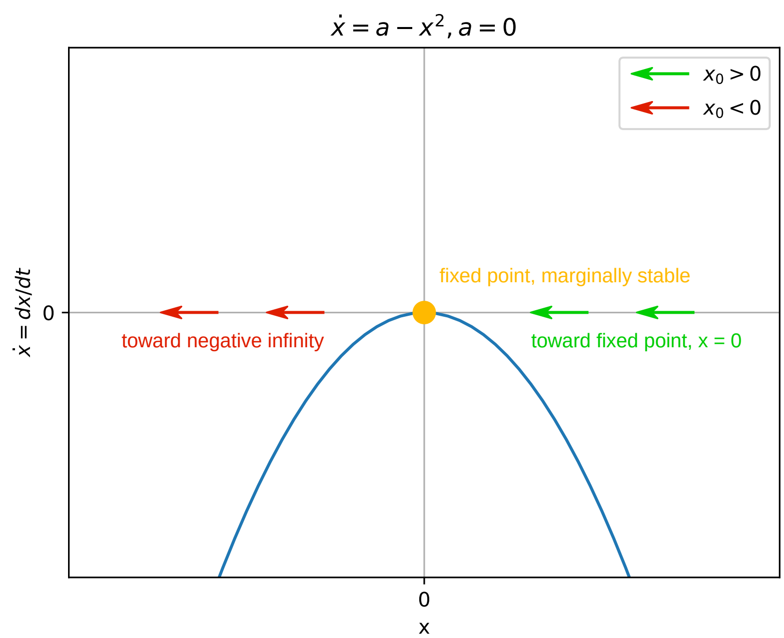

Figure 7: Phase plot for the same dynamics as in Figure 4 for $a = 0$. Green arrows denote the asymptotic behavior of the dynamics for positive initial conditions $x_0 > 0$, red arrows denote the behavior for negative initial conditions $x_0 < 0$.