Dynamical Systems Tutorial

Sophie Aerdker, Gregor Schöner

Recap Dynamical Systems

- "The present determines the future"
- $dx/\mathrm{d}t = f(x)$ with state x, dynamics $f(x)$, solution $x(t)$ and initial condition $x(0)$
- Phase plot of the dynamics vs. plot of the time course
- Fixed points x_0 : $\mathrm{d}x/\mathrm{d}t = 0 \rightarrow f(x_0) = 0$
- Stability (in linear systems):
  attractor (stable), repellor (unstable),
  marginally stable

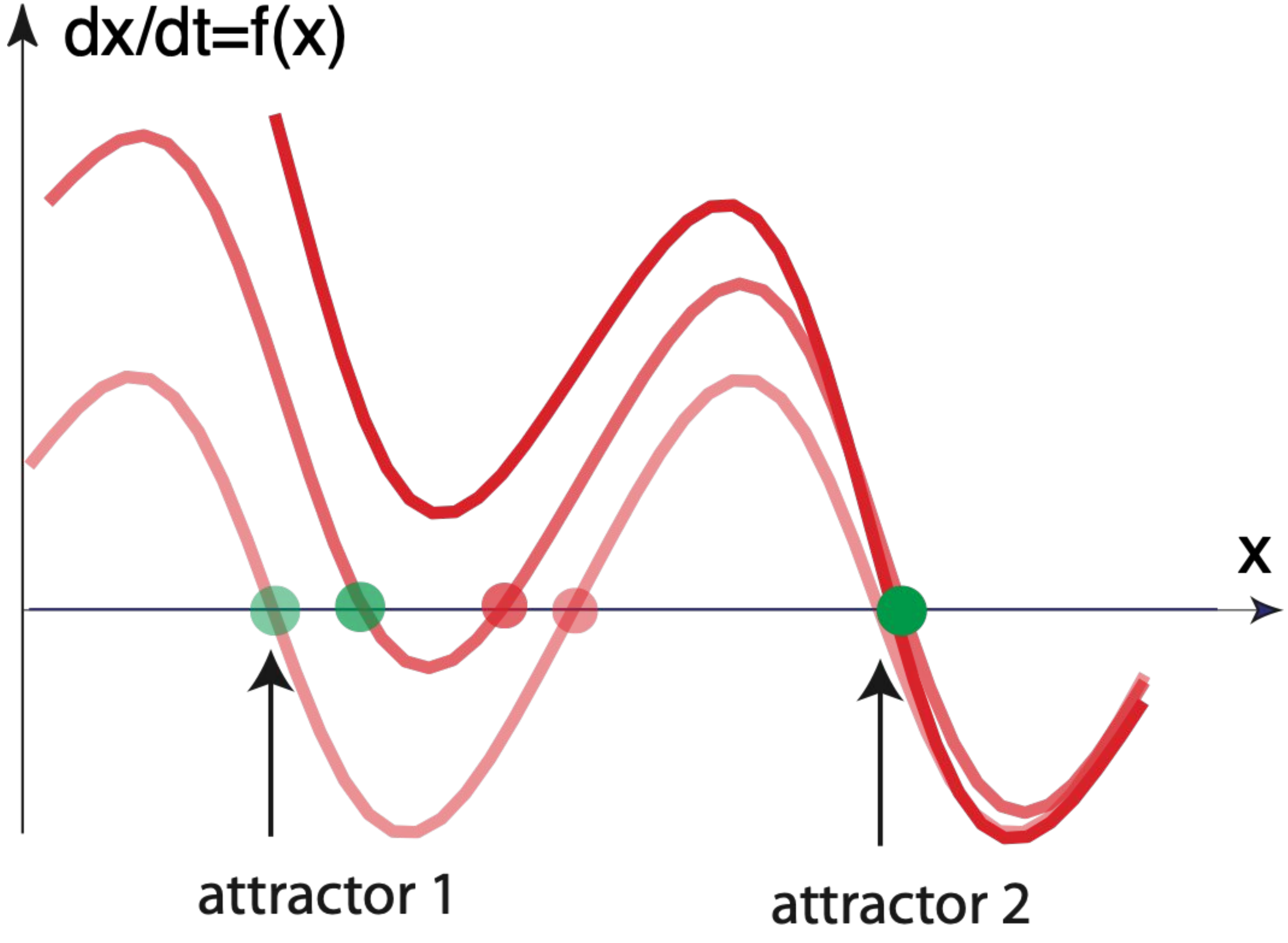
Recap Dynamical Systems

- Analyze slope of *f* or real part of eigenvalues in multiple dimensions
- All real-parts of eigenvalues positive: stable, any real-part of ev positive: unstable, any real-part of ev zero: marginally stable in that direction
- In non-linear dynamical systems, linearize around the fixed point to determine stability (stability as local property of the fixed point)

bifurcations

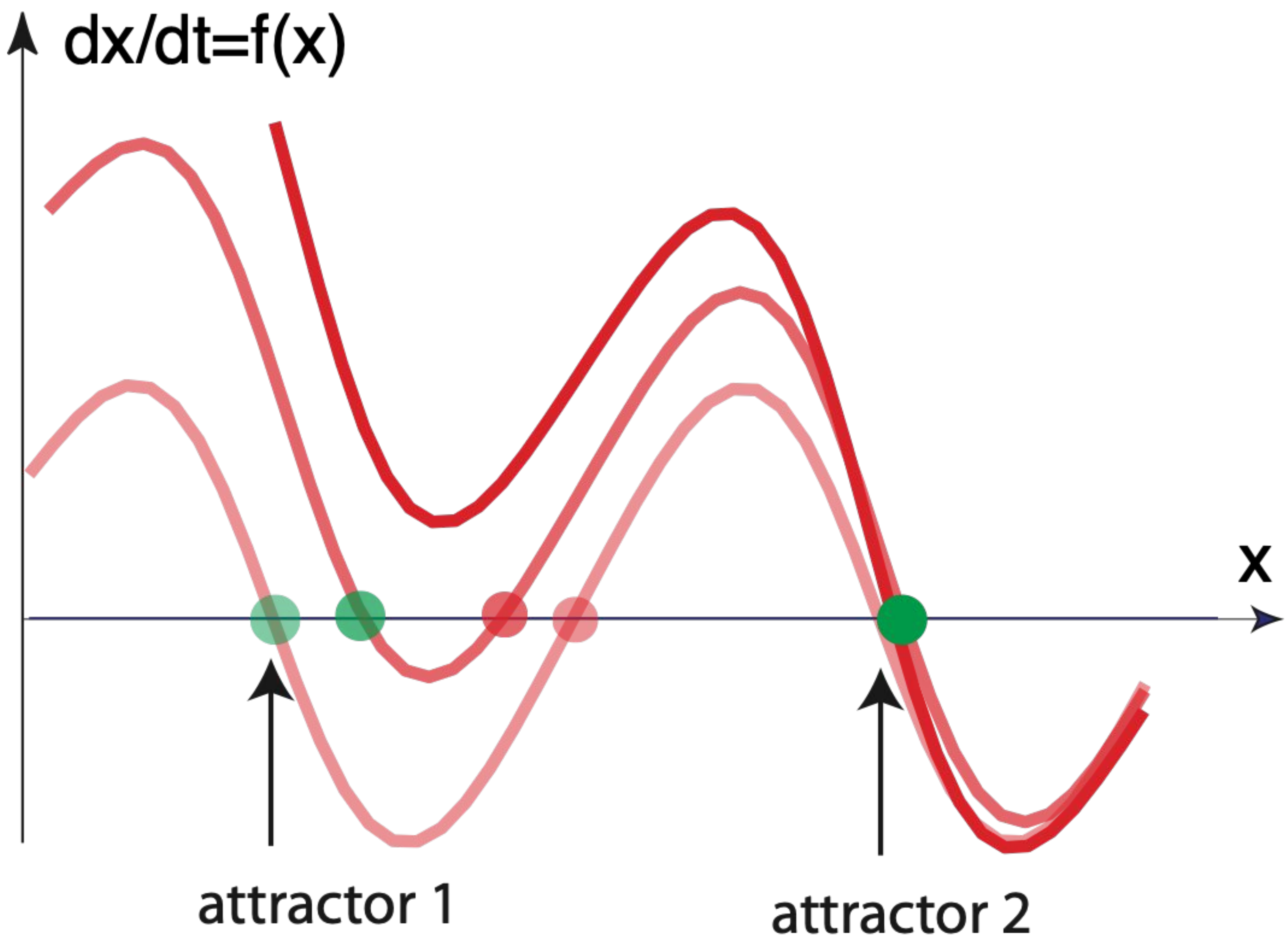
- look now at families of dynamical systems, which depend (smoothly) on parameters
- ask: as the parameters change (smoothly), how do the solutions change (smoothly?)
  - smoothly: topological equivalence of the dynamical systems at neighboring parameter values
  - bifurcation: dynamical systems NOT topological equivalent as parameter changes infinitesimally

bifurcation



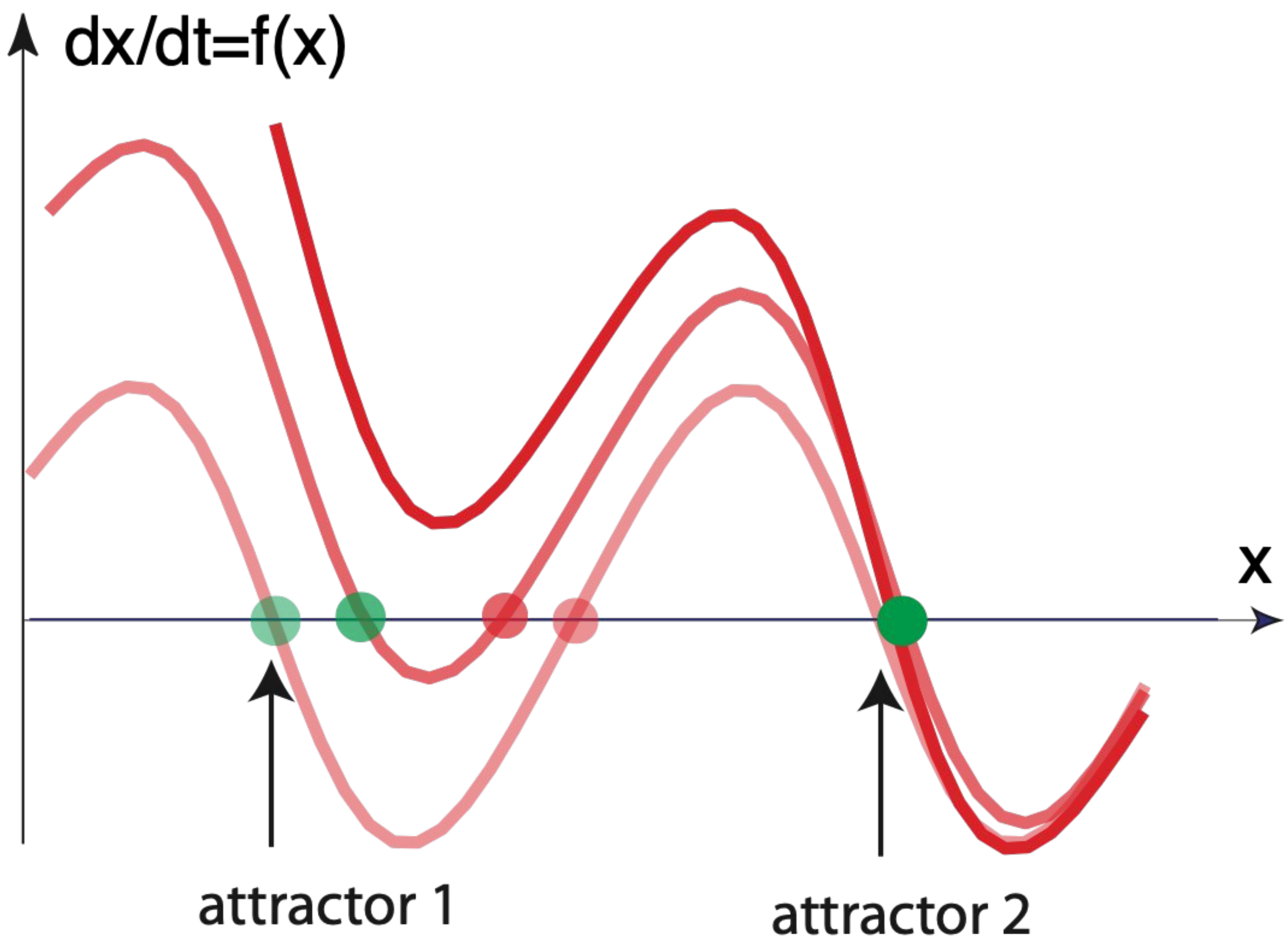
bifurcation

- bifurcation=qualitative change of dynamics (change in number, nature, or stability of fixed points) as the dynamics changes smoothly

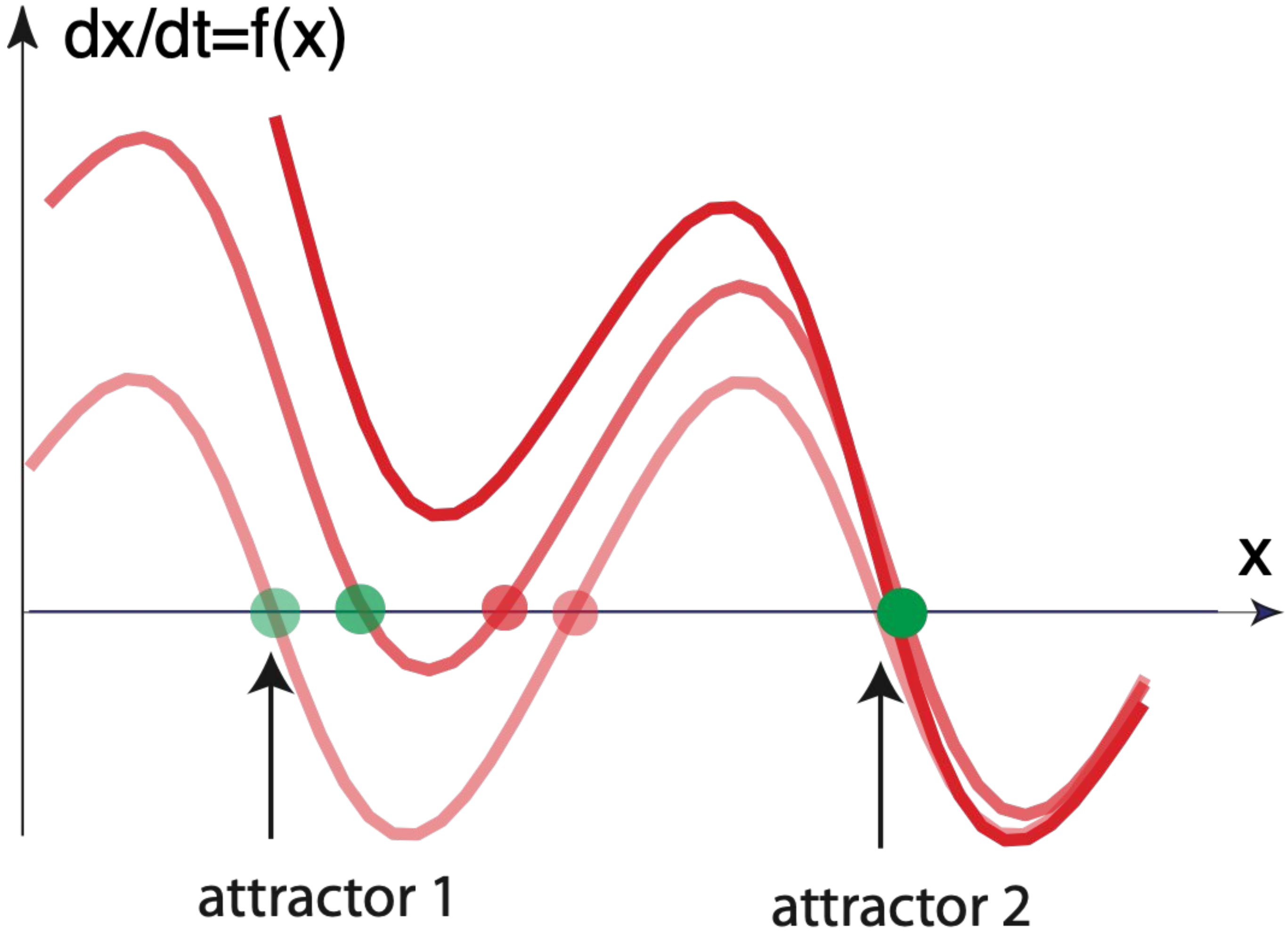


tangent bifurcation

- the simplest bifurcation (co-dimension 0): an attractor collides with a repellor and the two annihilate

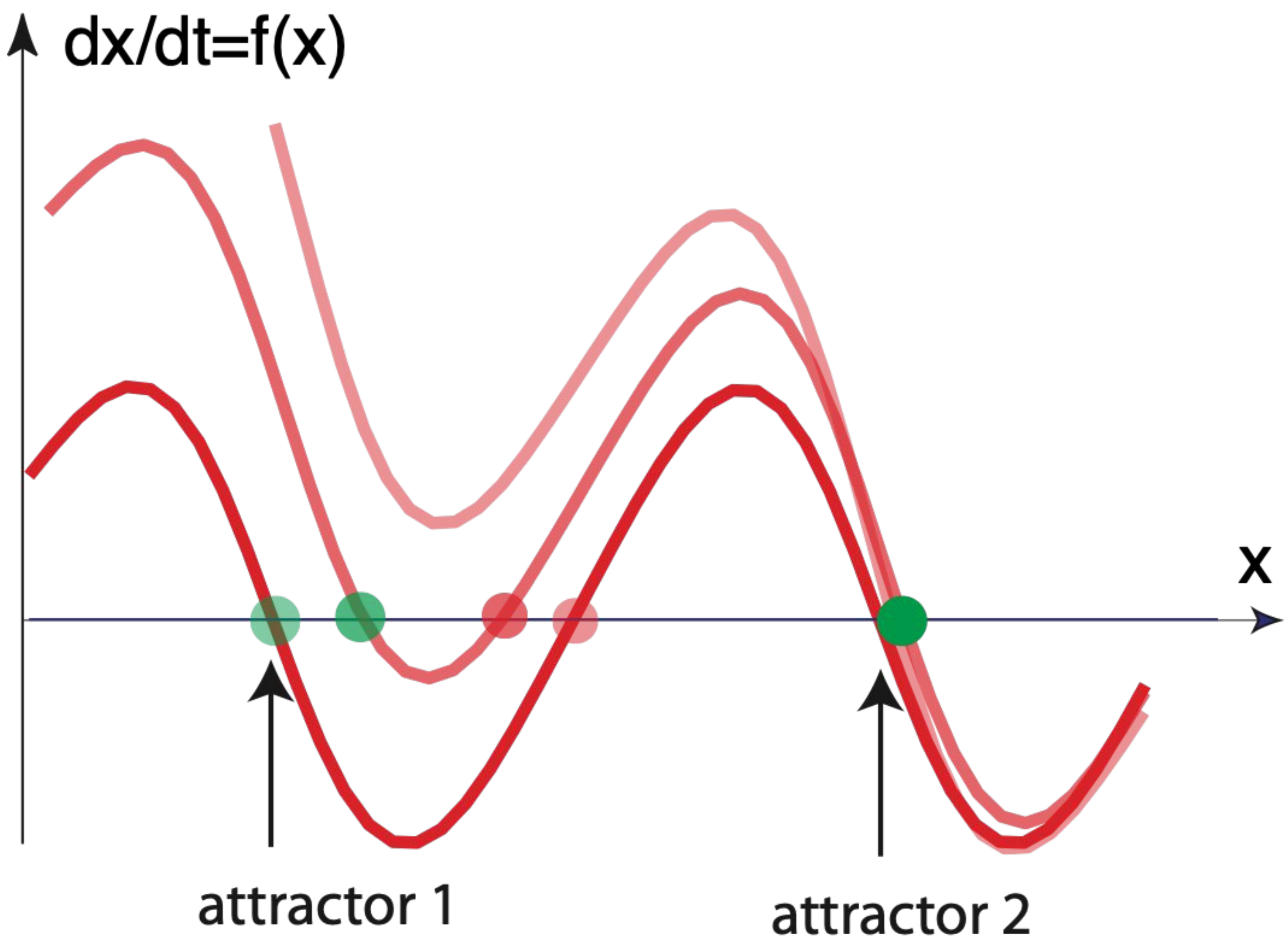


local bifurcation



reverse bifurcation

- changing the dynamics in the opposite direction



bifurcations are instabilities

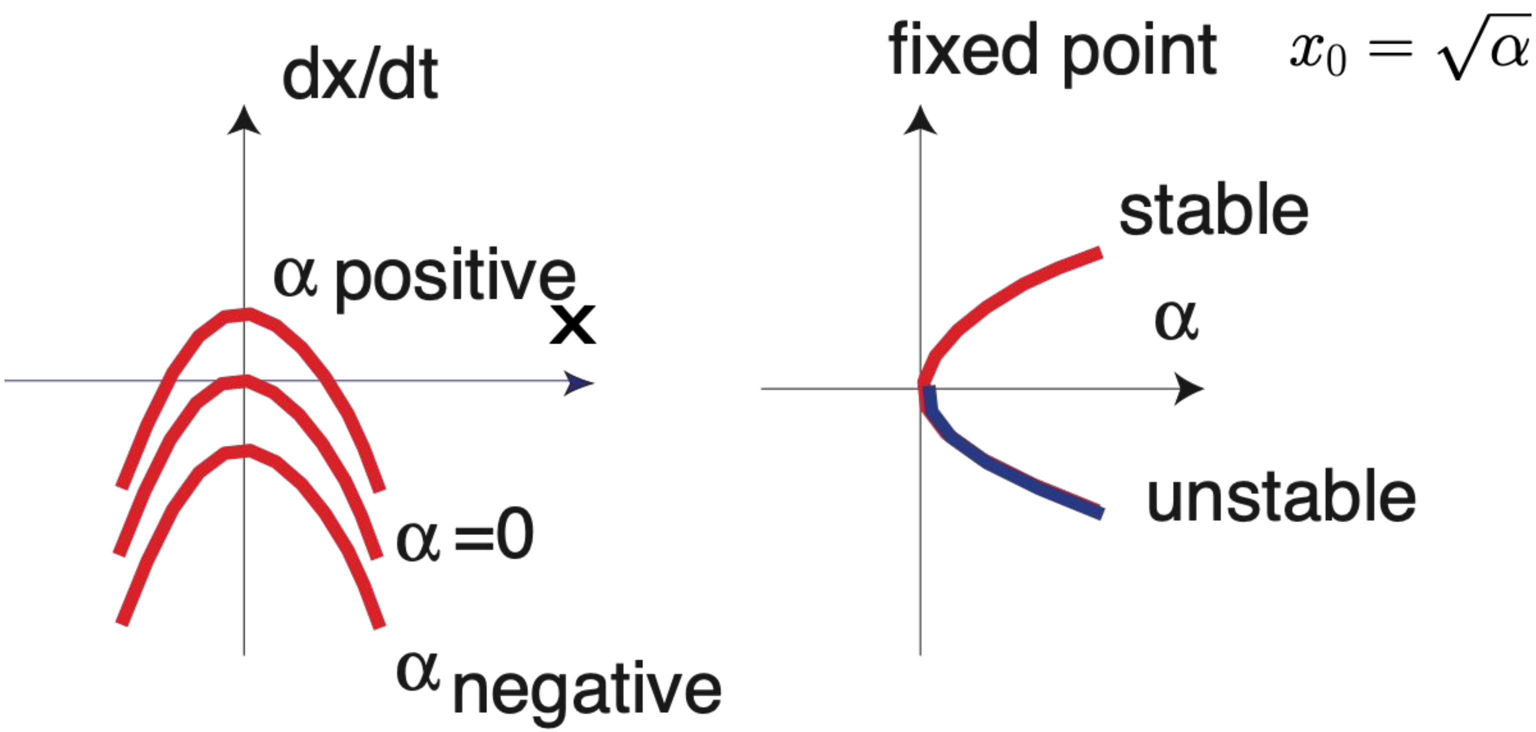
- that is, an attractor becomes unstable before disappearing
- (or the attractor appears with reduced stability)
- formally: a zero-real part is a necessary condition for a bifurcation to occur

tangent bifurcation

- normal form of tangent bifurcation

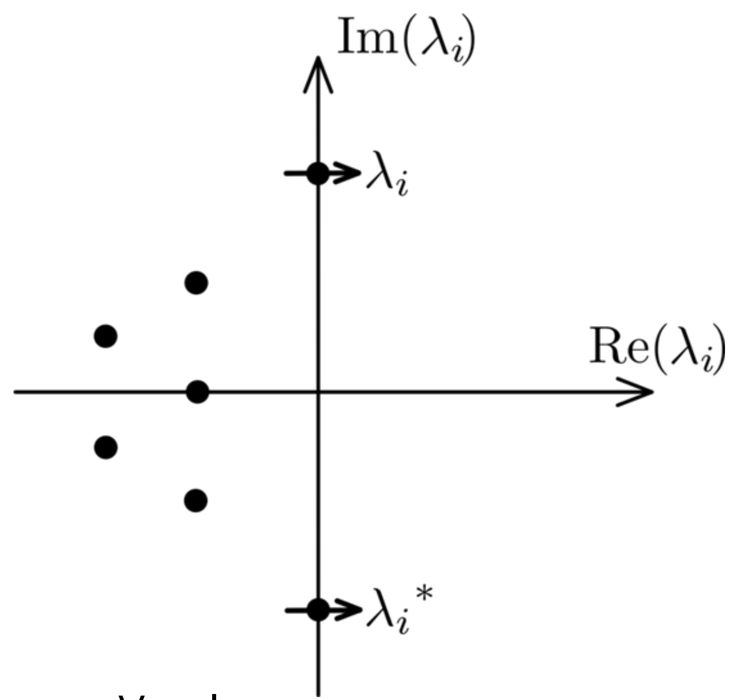
  $\dot{x} = \alpha - x^2$

- (=simplest polynomial equation whose flow is topologically equivalent to the bifurcation)



Hopf theorem

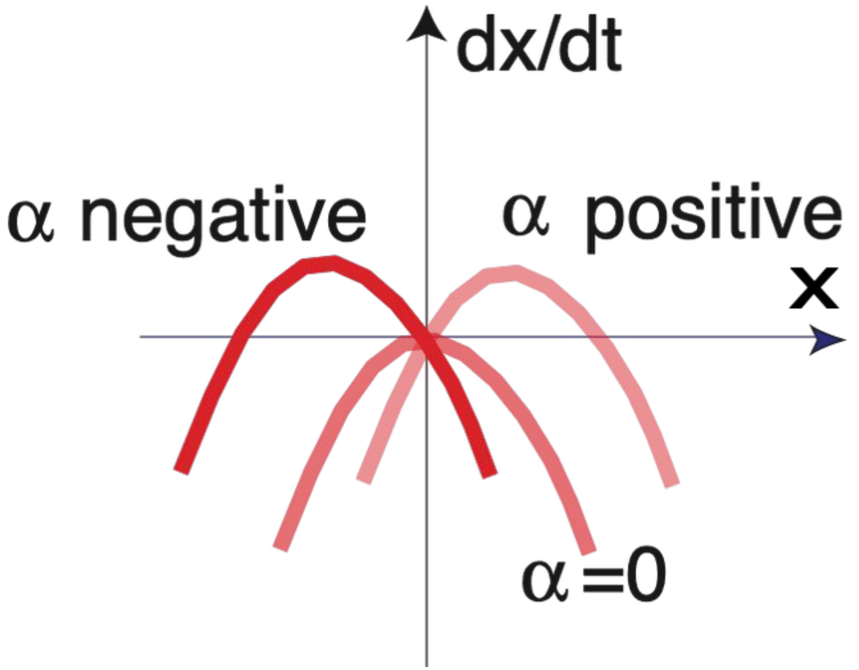
- when a single (or pair of complex conjugate) eigenvalue crosses the imaginary axis, one of four bifurcations occur
  - tangent bifurcation
  - transcritical bifurcation
  - pitchfork bifurcation
  - Hopf bifurcation

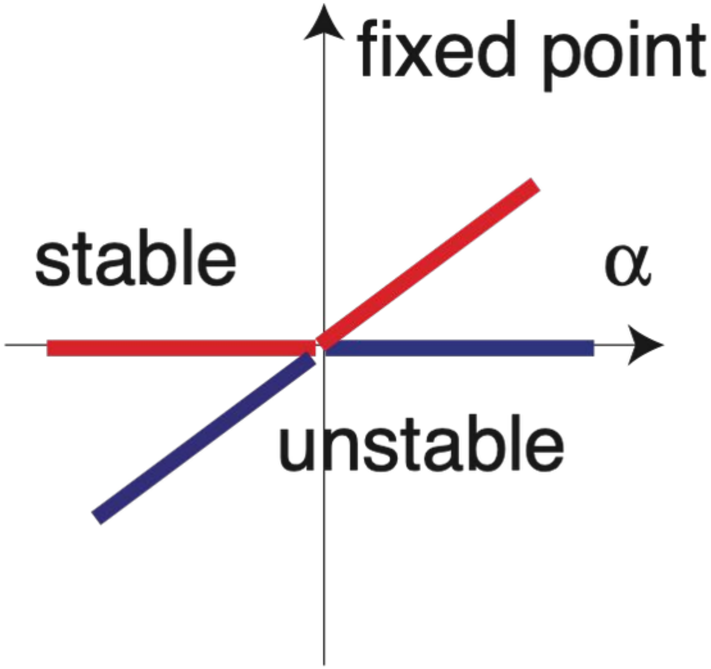


By Hannes Vogel,
https://en.wikipedia.org/wiki/Hopf_bifurcation

transcritical bifurcation

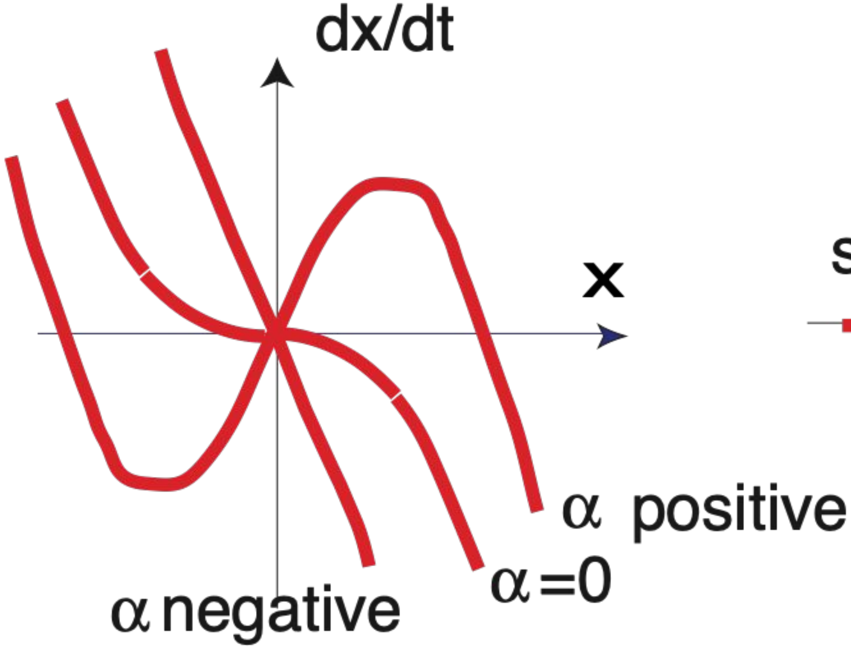
- normal form $\dot{x} = \alpha x - x^2$

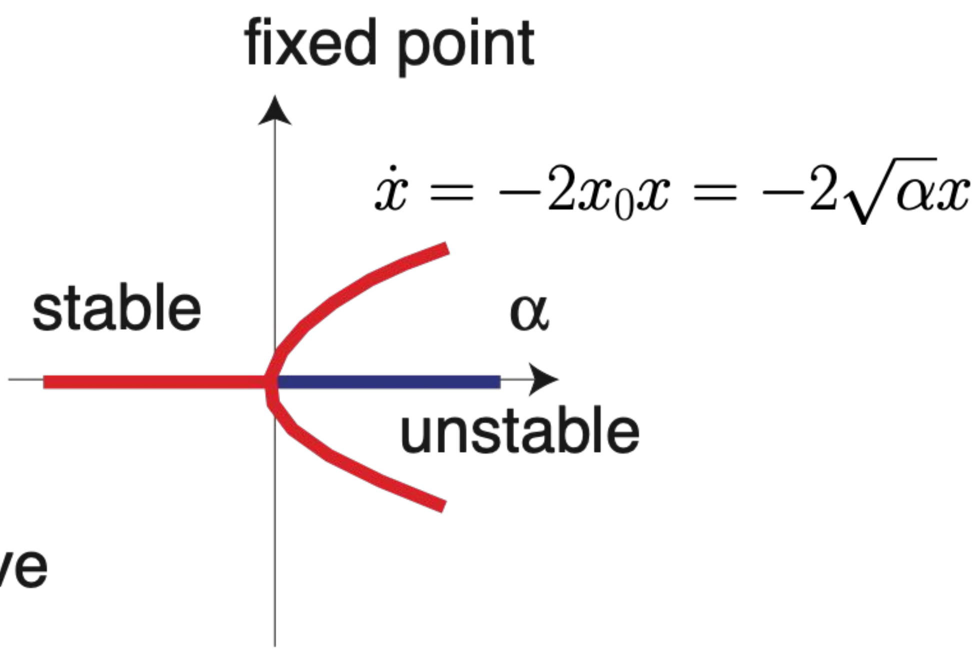




pitchfork bifurcation

- normal form $\dot{x} = \alpha x - x^3$

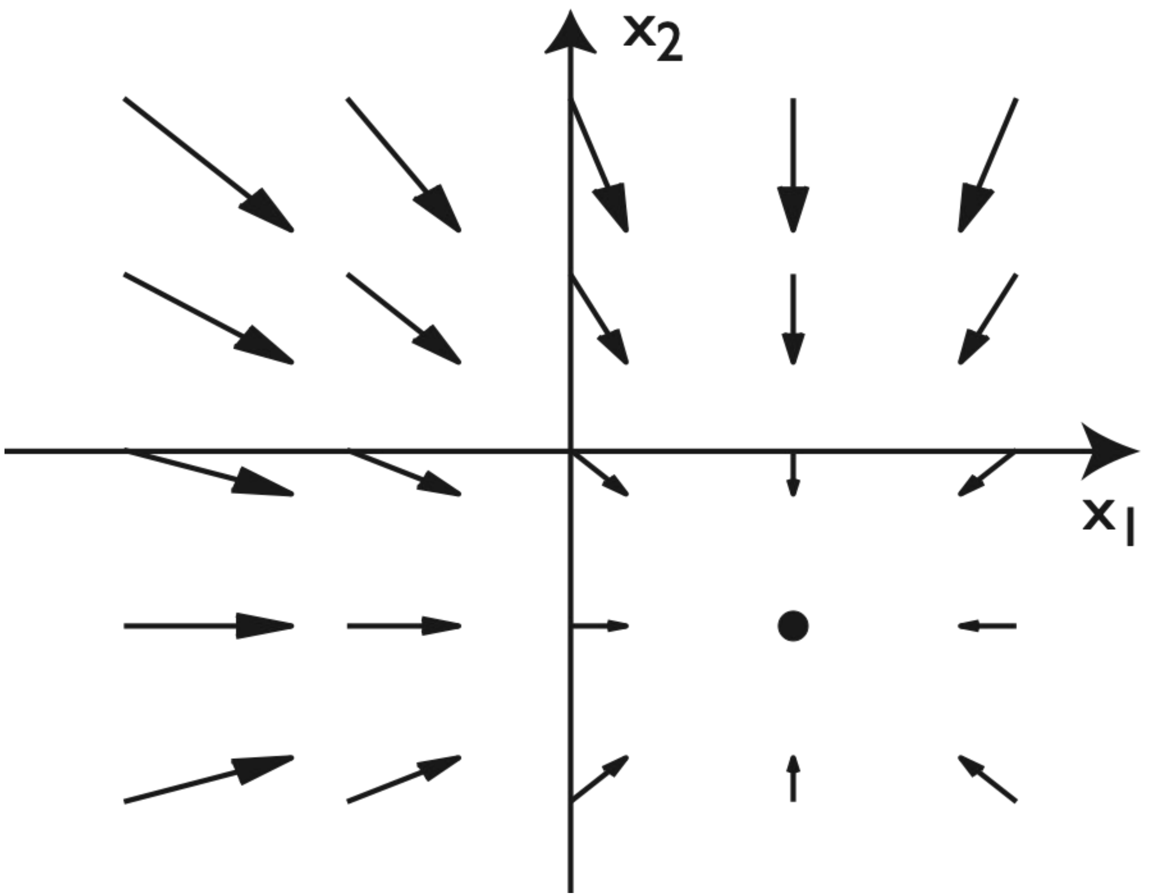




Hopf: need higher dimensions

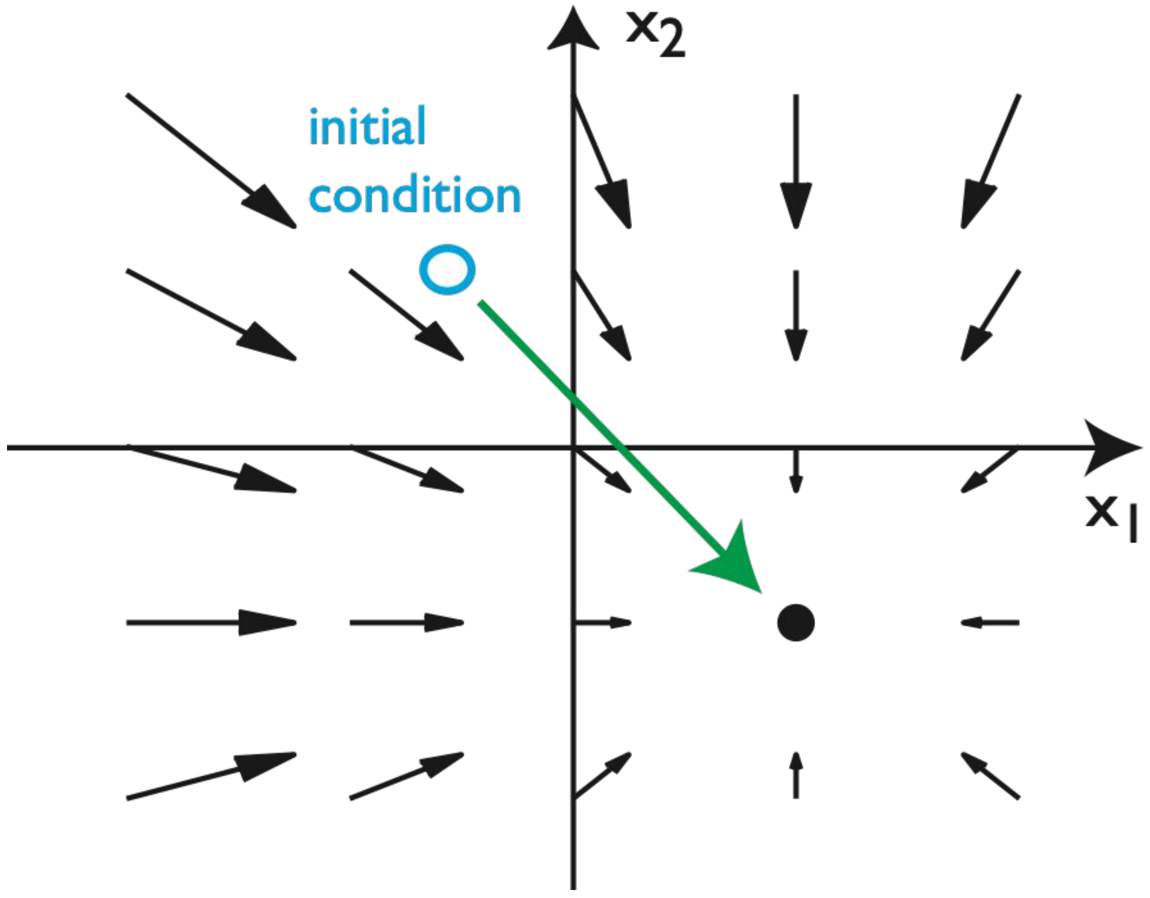
2D dynamical system: vector-field

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$



fixed point, stability, attractor

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

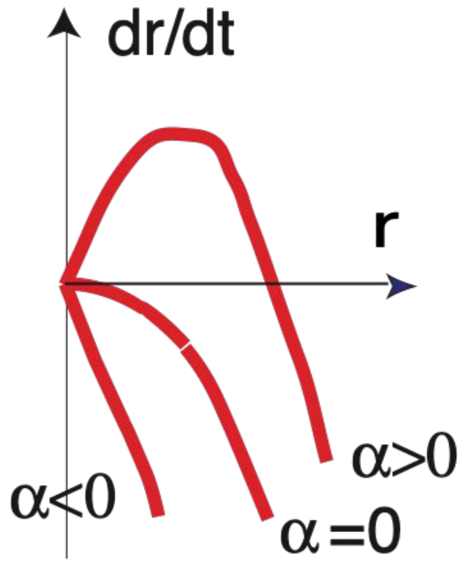


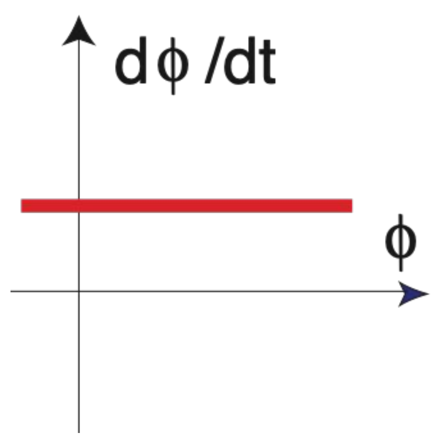
Hopf bifurcation

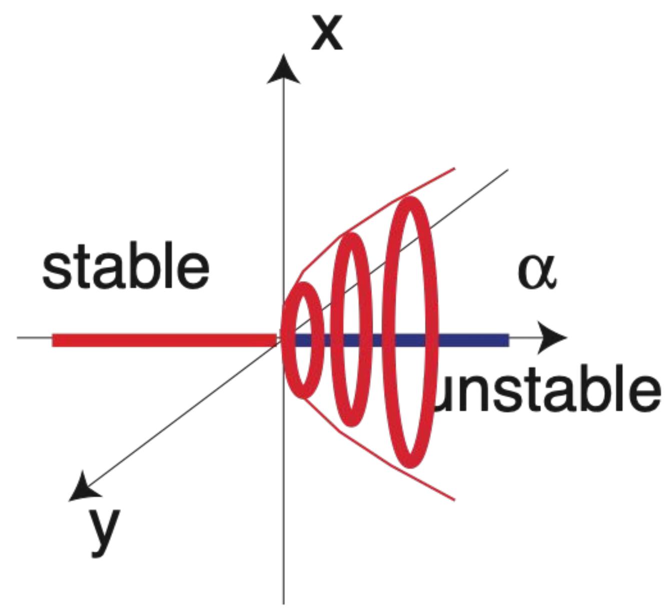
- normal form

$$\dot{r} = \alpha r - r^3$$

$$\dot{\phi} = \omega$$







Hopf bifurcation and limit cycle

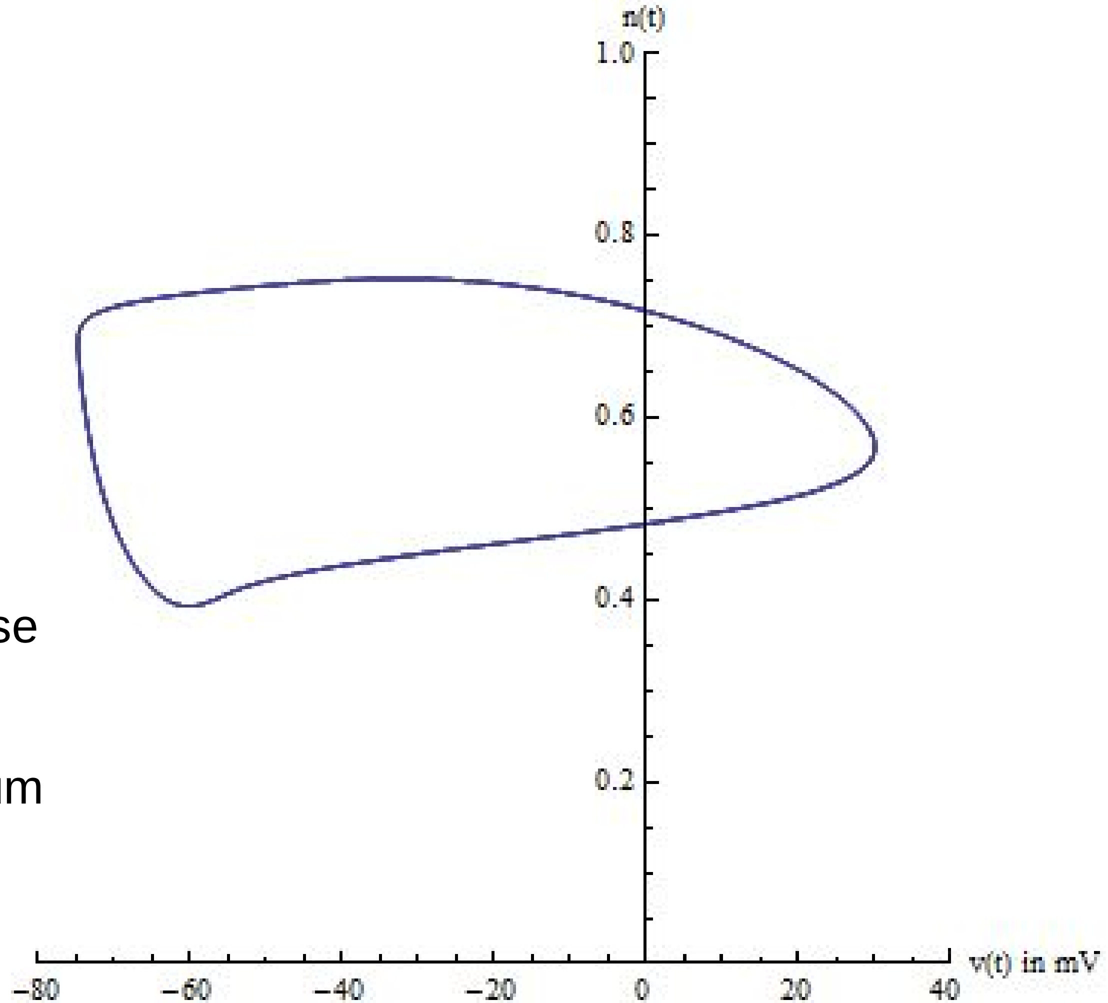
- Stable fixed point → periodic solution when a pair of complex conjugate eigenvalues (of the linearized system) crosses imaginary axis
- Limit cycle = closed trajectory in phase space other trajectories spiral into (for time going to infinity/negative infinity)
- Stable limit cycles lead to oscillations with stable frequency

Example: Hodgkin-Huxley model

- Mathematical model describing the initiation and propagation of action potential in neurons
- 1963 Medicine Nobel Prize
- Non-linear dynamical system with four state variables: membrane potential and flow of ionic currents (sodium, potassium)
- System undergoes a Hopf bifurcation with respect to the injected current

Example: Hodgkin-Huxley model

- Limit Cycle leads to stabilized firing rates
- Firing rate increases with injected current
- Minimal firing rate



Limit cycle with the phase space reduced to two dimensions: membrane potential *V* and potassium gating *n*

By Alexander J. White,
https://en.wikipedia.org/wiki/Hodgkin–Huxley_model

forward dynamics

- given known equation, determined fixed points /limit cycles and their stability
- more generally: determine invariant solutions (stable, unstable and center manifolds)

inverse dynamics

- given solution, find the equation…
- this is the problem faced in design of behavioral dynamics…

inverse dynamics: design

- in the design of behavioral dynamics... you may be given:
- attractor solutions/stable states
- and how they change as a function of parameters/conditions
- => identify the class of dynamical systems using the 4 elementary bifurcations
- and use normal form to provide an exemplary representative of the equivalence class of dynamics