Dynamical Systems Tutorial

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Dynamical Systems Tutorial

- What makes a system “dynamic”?
- Basic math for describing dynamical systems
- Conceptual understanding
- Stability
- Bifurcation
- Review (?)

Dynamical System

- “A dynamical system is a function with an attitude” – E. Scheinerman, Invitation to Dynamical systems
- Variable(s) or *state* that change over time, *x(t)*
- Rule or *function* describing how the state evolves with time, *f(x(t))*
- Universal language of science
  - physics, engineering, chemistry, biology, economics, ...

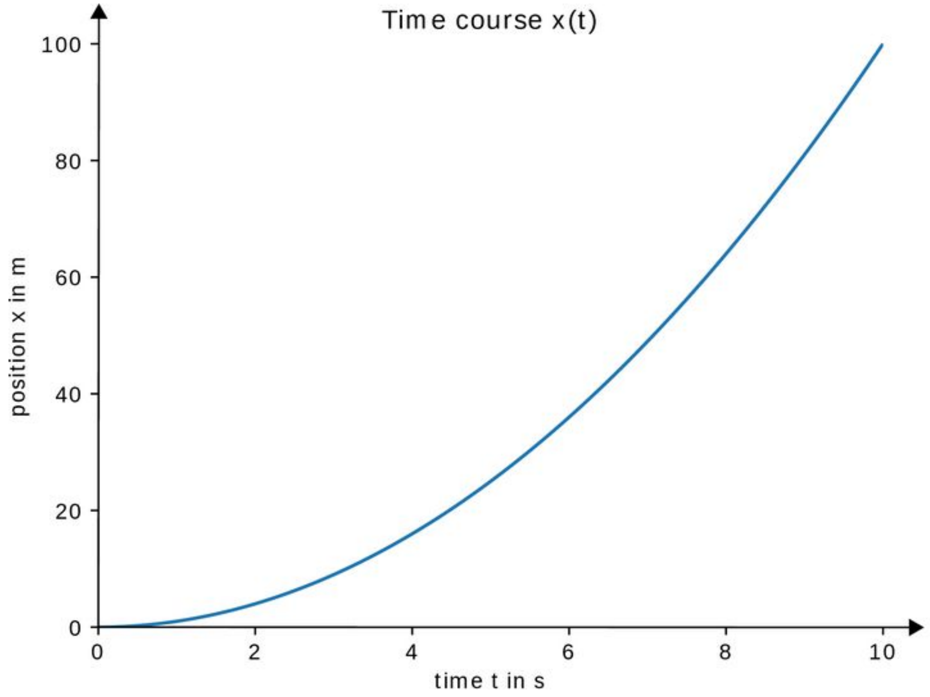
Recap: time-variation and rate of change

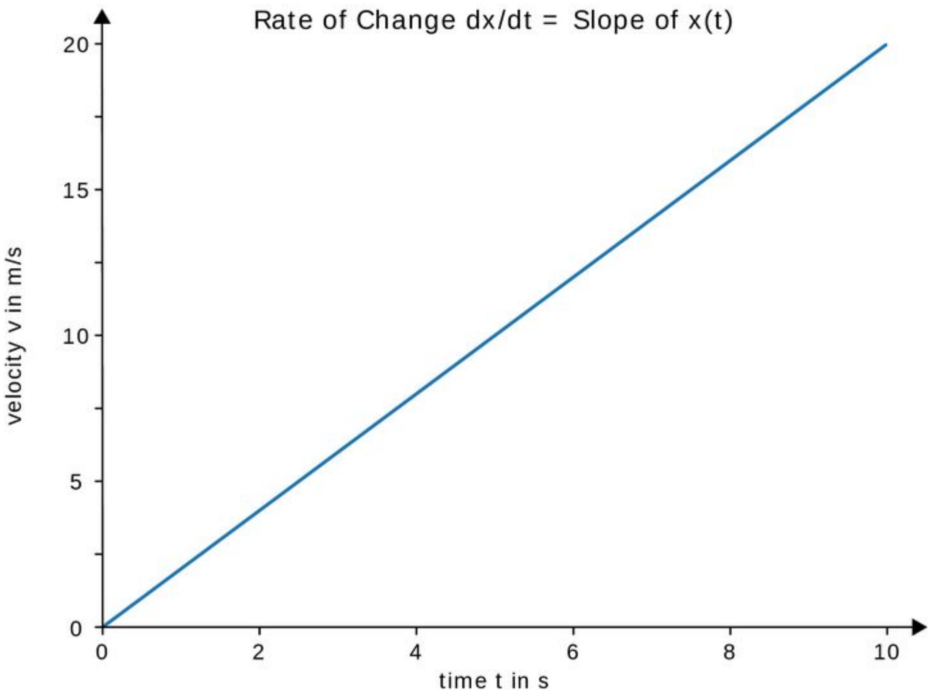
- Variable or state $x(t)$
- rate of change $\mathrm{d}x/\mathrm{d}t$

Recap: time-variation and rate of change

- example:
  - variable x(t) = position
  - rate of change dx/dt = velocity
- example:
  - variable v(t) = velocity
  - rate of change ?

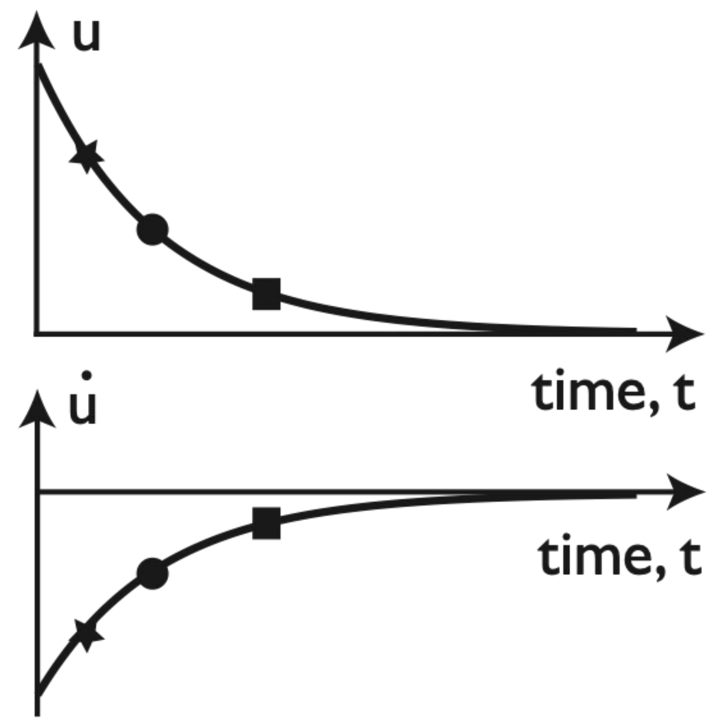
Recap: time-variation and rate of change



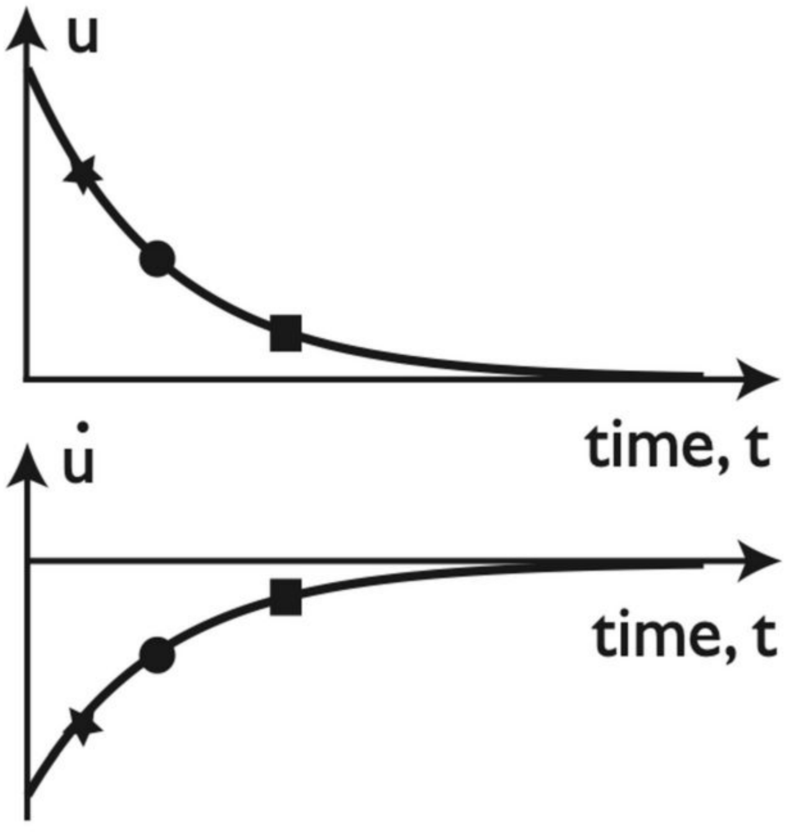


Recap: time-variation and rate of change

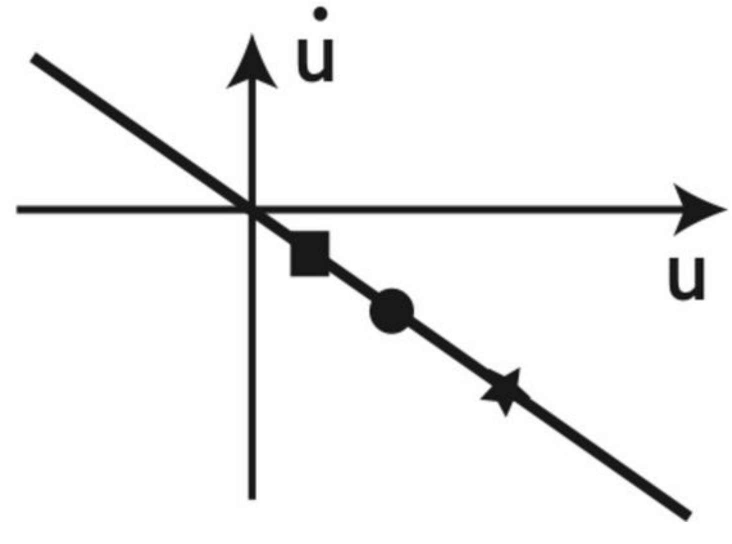
- trajectory: time course of a dynamical variable
- rate of change: slope of the trajectory



dynamical system: relationship between a variable and its rate of change, du/dt = f(u)



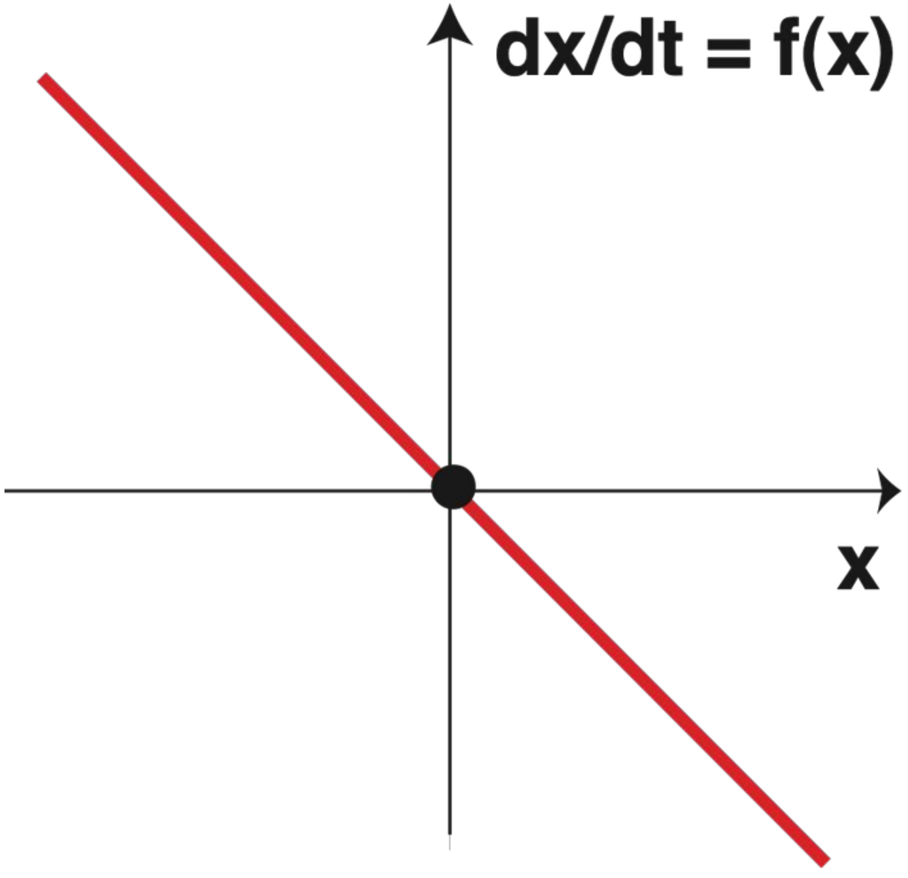
Time course of u and du/dt



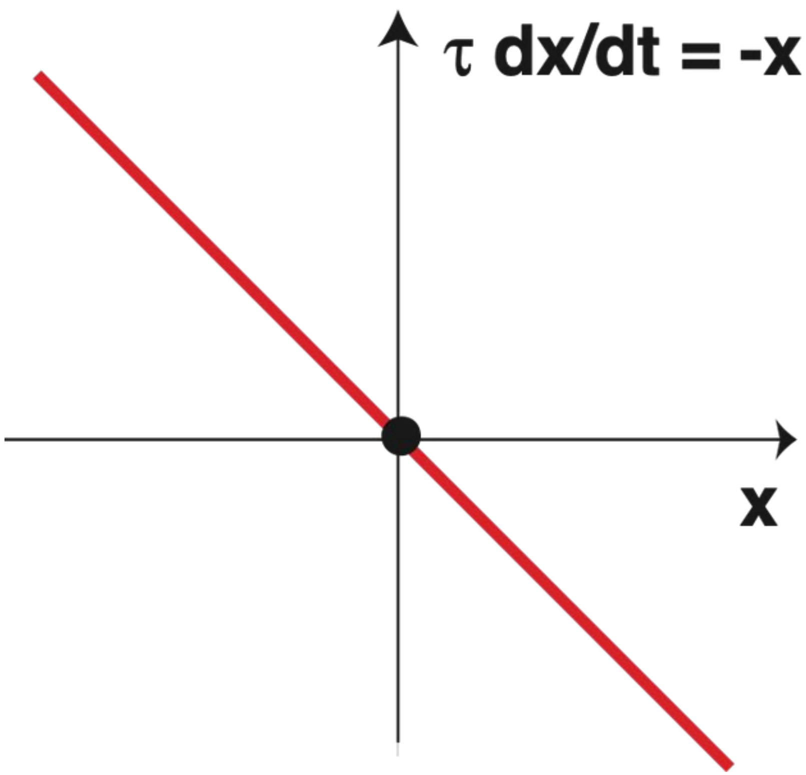
Phase plot of the dynamics

linear dynamical system

$$dx/dt = f(x) = ax+b$$



solution of linear dynamical systems



1. Guess:
A function whose derivative is the function itself but with some factor?

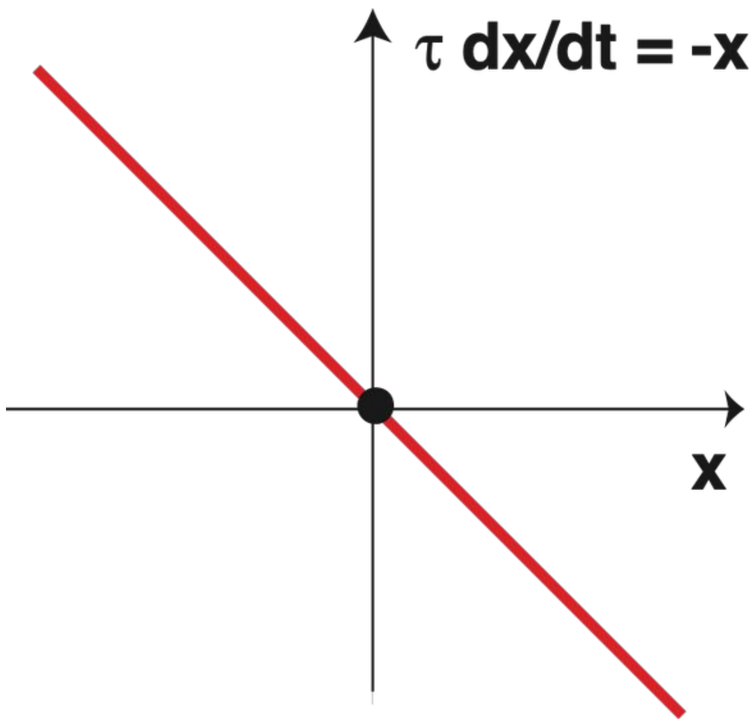
2. Calculating:
Exercise sheet!

$$\mathrm{d}x/\mathrm{d}t = -\tau^{-1}x$$

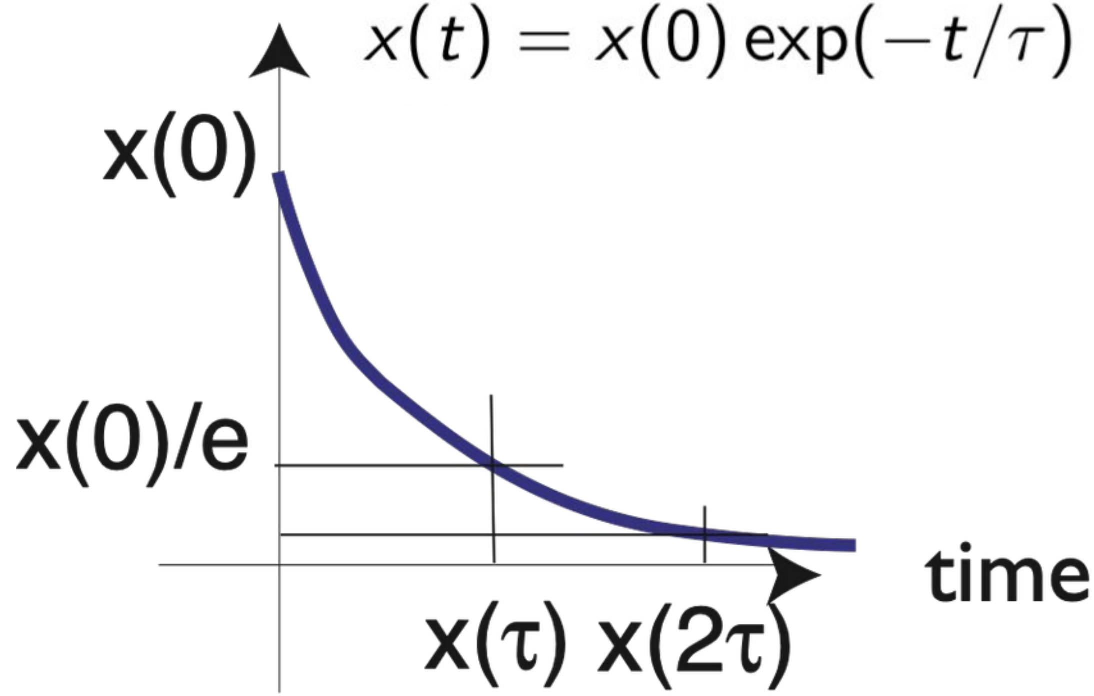
$$x(t) = x(0)\exp(-t/\tau)$$

Verify that x(t) is a solution of the linear differential equation!

solution of linear dynamical systems

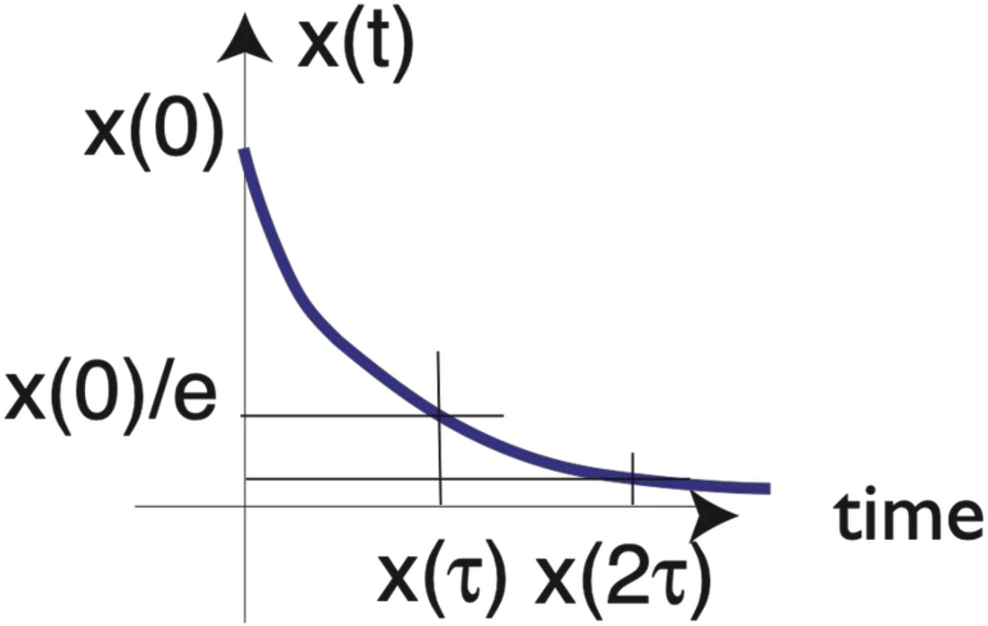


Phase plot of dynamical system



Time course of the solution x(t)

exponential relaxation to *attractors* with time scale τ



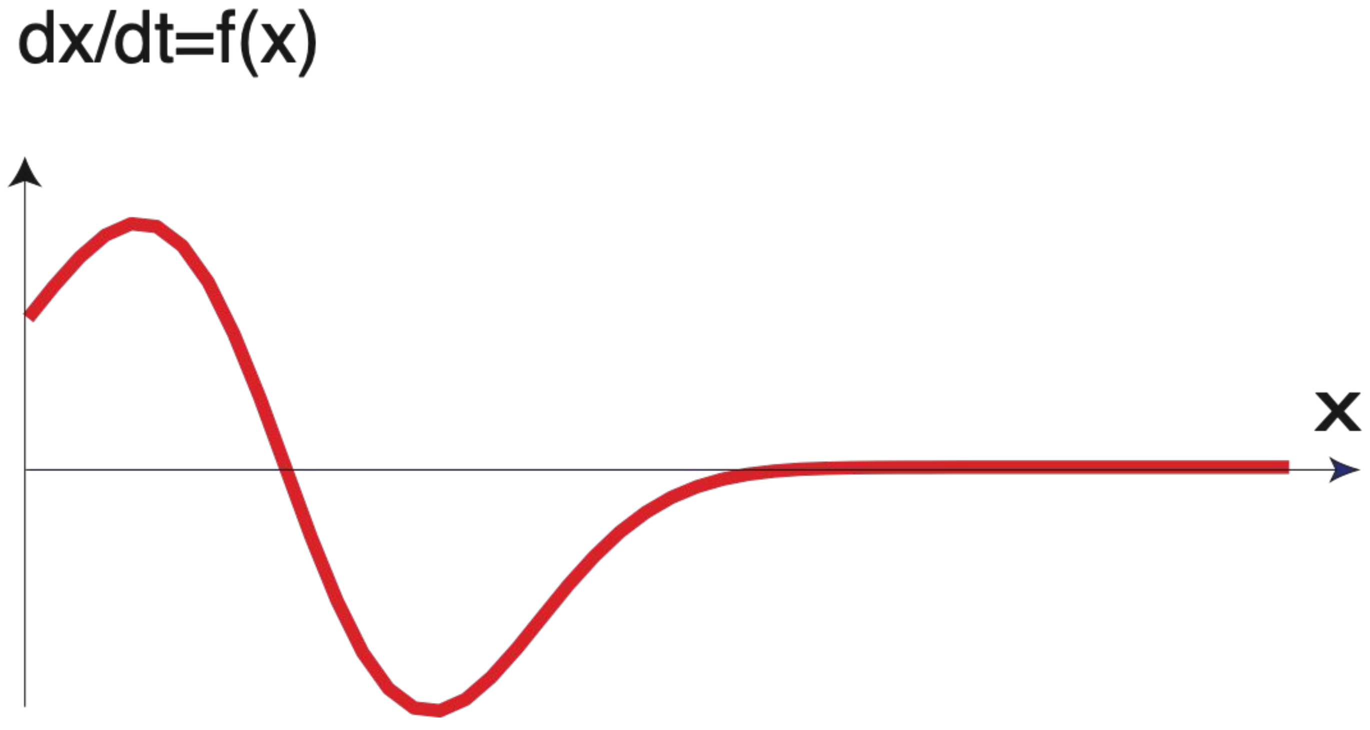
Time course of the solution x(t)

τ is the time at which the initial condition x(0) is reduced to x(0)/e (or at any other time point t)

→ Exercise sheet!

Dynamical system

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$$

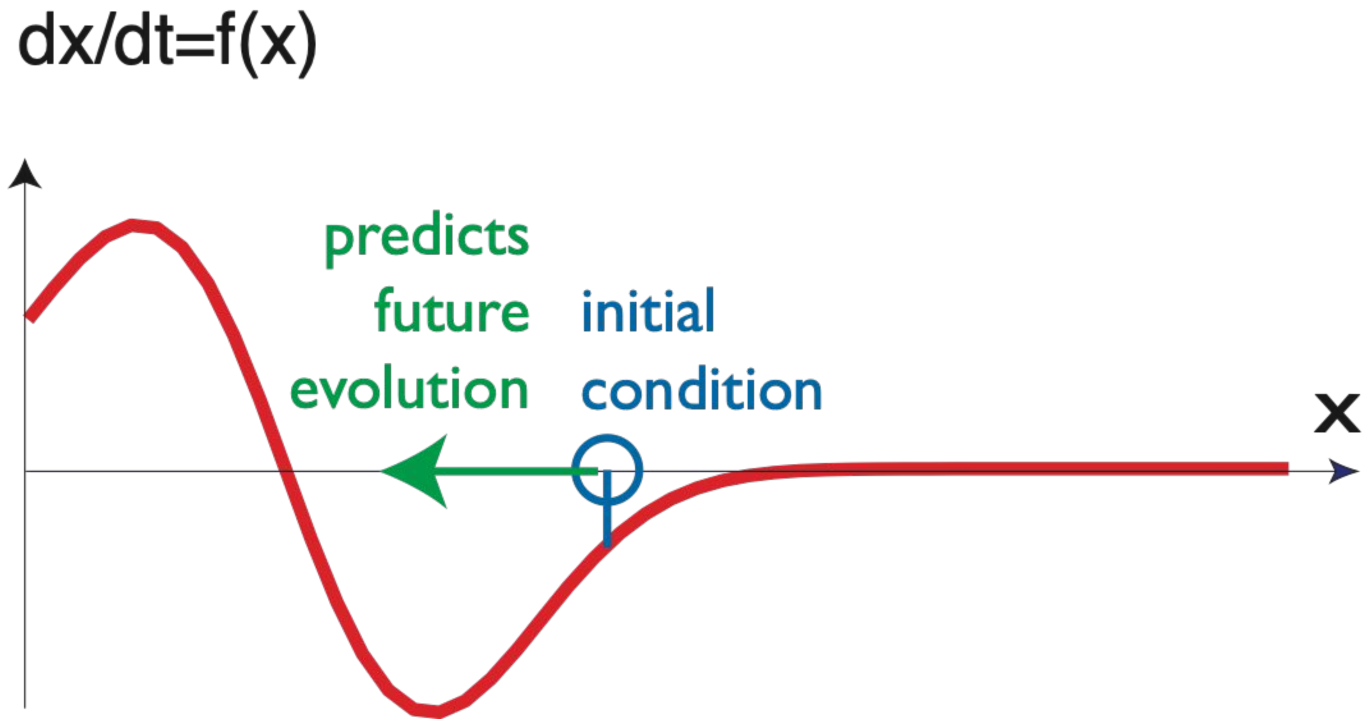


Phase plot of the dynamical system

Dynamical system

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$$

- the present determines the future



Dynamical system

$$\dot{x} = \frac{dx}{dt} = f(x)$$

- *x* spans the state space (can be vector-valued or even function valued)
- *f(x)* is the "dynamics" of *x* (or vector-field)
- *x(t)* is a solution of the dynamical systems with initial condition x_0 when the rate of change of *x(t)* obeys $dx/dt=f(x)$ and $x(0)=x_0$

Different forms of dynamical systems

- one-dimensional (ordinary) differential equation: initial value determines the future

$$\dot{x} = f(x)$$

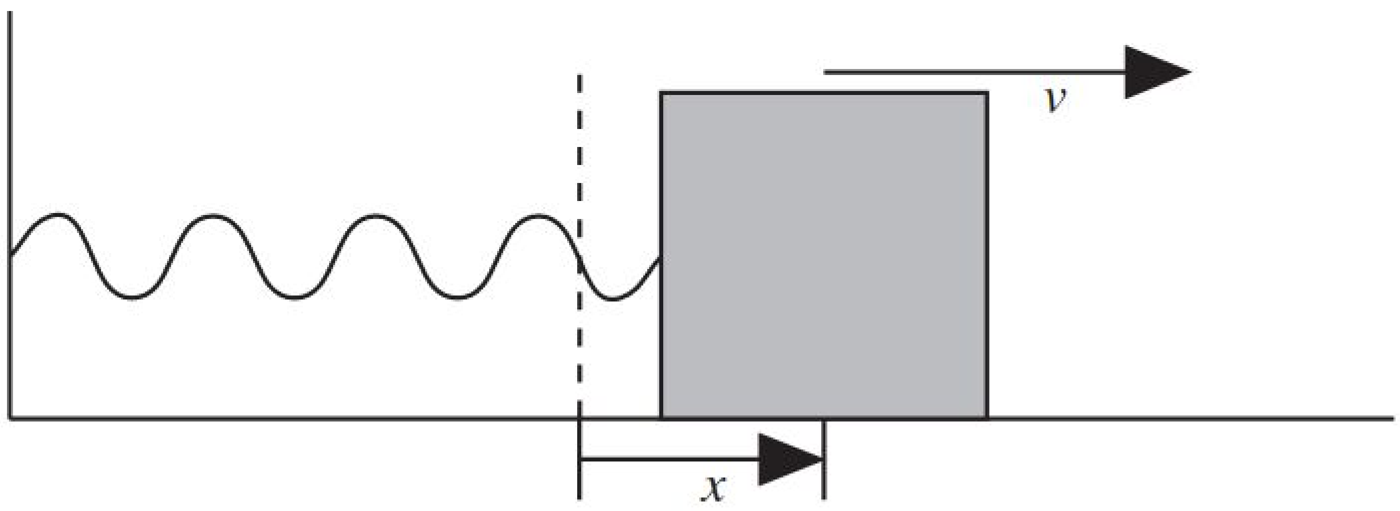
→ E.g. the linear dynamical system we analyzed

Different forms of dynamical systems

- vector-valued (ordinary) differential equation
- a vector of initial states determines the future, systems of differential equations:

$\dot{\mathbf{x}} = f(\mathbf{x})$ where $\mathbf{x} = (x_1, x_2, \ldots x_{\mathrm{n}})$

Example of vector-valued differential equations: Ideal spring



From “Invitation to Dynamical Systems by E. Scheinerman

Example of vector-valued differential equations: Ideal spring

Hooke's law: $\dot{v} = -k/m \cdot x$

We know: $\dot{x} = v$

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k/m & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$$

Example of vector-valued differential equations: Ideal spring

With $x(0) = 1$, verify that a solution is:

$$\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{k/m}\, t) \\ -\sqrt{k/m}\sin(\sqrt{k/m}\, t) \end{pmatrix}$$

Position and velocity of the mass bounce back and forth (without friction)!

Different forms of dynamical systems

- discrete time
- At each state x_k, the state at the following time point is given by:

- $x_{k+1} = f(x_k)$ and an initial condition x_0

Example of discrete time dynamical system: Bank account

With x being the money in our bank account and r the interest rate which is payed out annually, the state of your bank account in year k can be described by: $x_{k+1} = (1 + r)x_k$ with the initial state x_0

Let $r = 4\%$ and $x_0 = \$100$:

$$x_1 = 1.04 \cdot \$100 = \$104$$

$$x_2 = 1.04 \cdot \$104 = \$108.16$$

$$x_3 = 1.04 \cdot \$108.16 = \$112.49$$

$$\ldots$$

Other forms of dynamical systems

- partial differential equations
- integro-differential equations
- delay differential equations
- Functional differential equations

Numerical solutions

- Use the discrete to approximate the continuous:

  compute solution, $x(t_i)$, by iterating through time, $t_i = i\,\Delta t,\ i=0,1,\ldots N$

- for example: (forward Euler)

$$x_{i+1} = x_i + \Delta t\ f(x_i)$$

- => code / simulation

Fixed point

- is a constant solution of the dynamical system
- that is a state with zero rate of change:

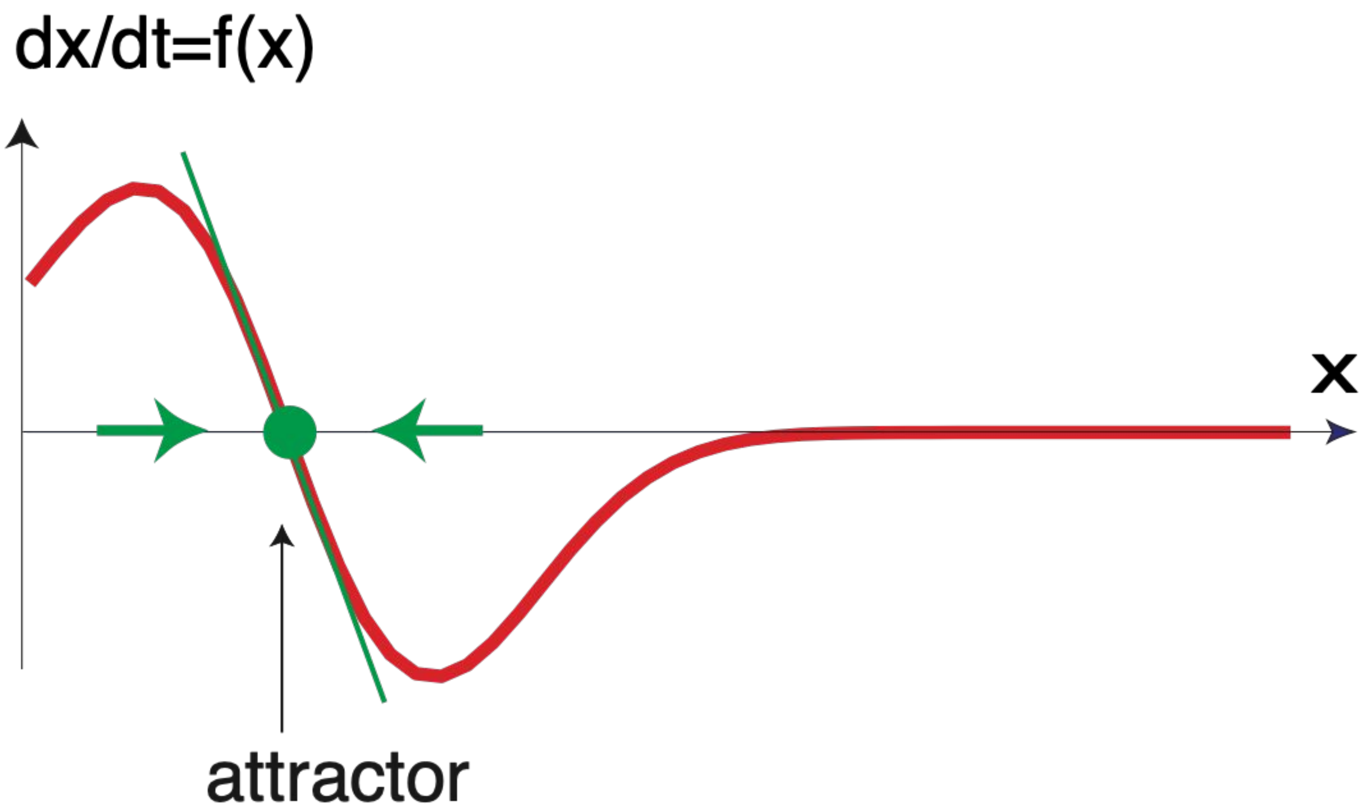
$$\dot{x} = f(x)$$

$$\dot{x} = 0 \Rightarrow f(x_0) = 0$$

Zero crossings in the phase plot!

attractor

- fixed point, to which neighboring initial conditions converge = attractor



stability

- mathematically really: asymptotic stability
- defined: a fixed point is asymptotically stable, when solutions of the dynamical system that start nearby converge in time to the fixed point

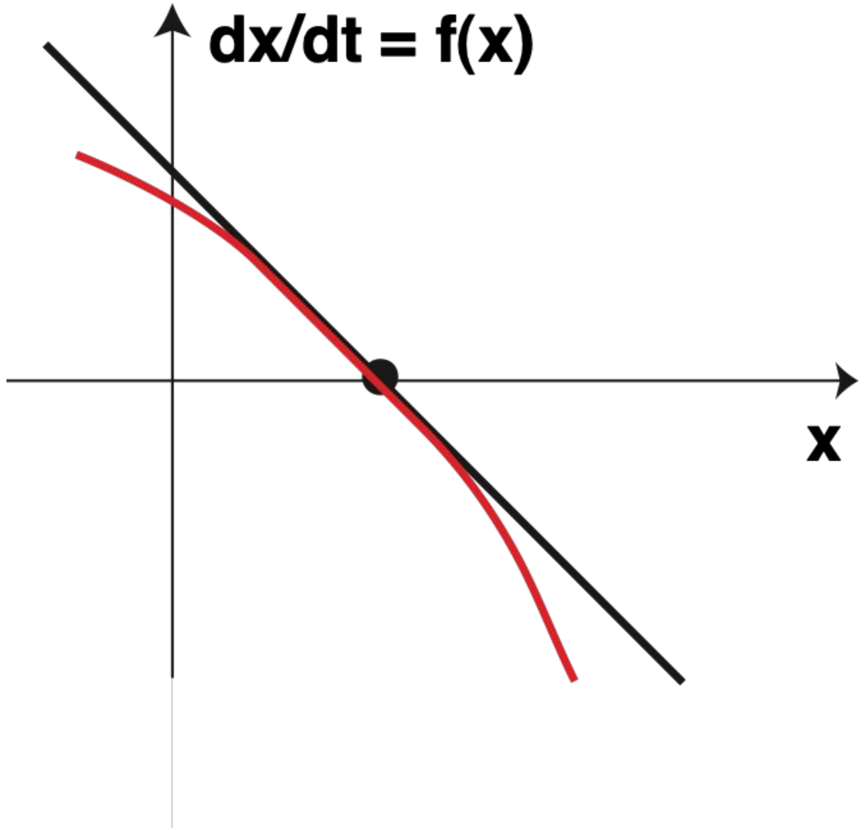
stability

- the mathematical concept of stability (which we do not use) requires only that nearby solutions stay nearby
- Definition: a fixed point is unstable if it is not stable in that more general sense,
  - that is: if nearby solutions do not necessarily stay nearby (may diverge)

How to tell whether a fixed point is stable or unstable?

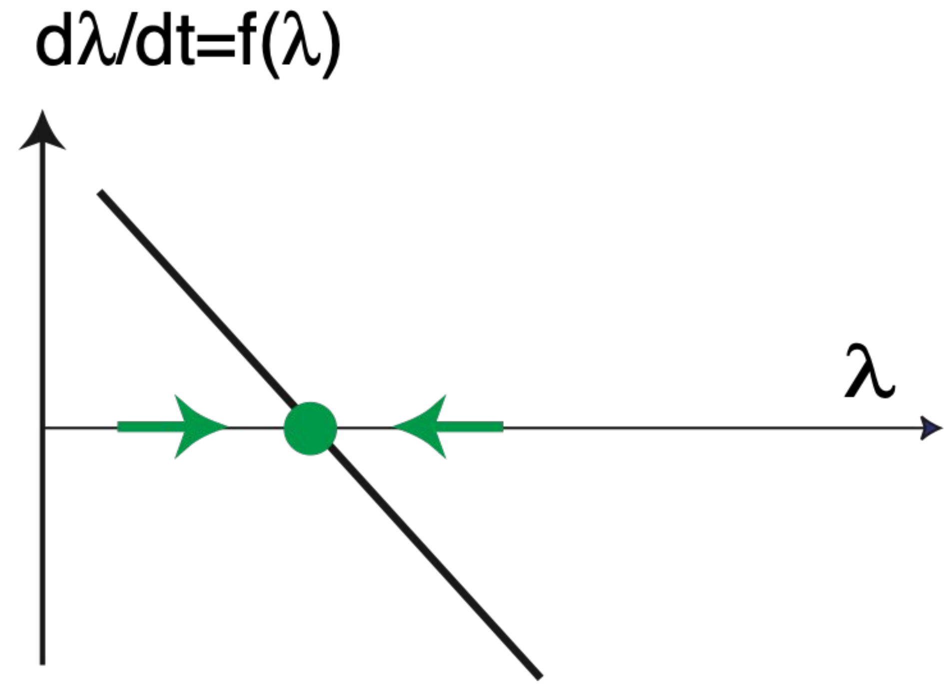
linear approximation near fixed point

- non-linearity as a small perturbation/deformation of linear system
- Only need to analyze the linearized system!



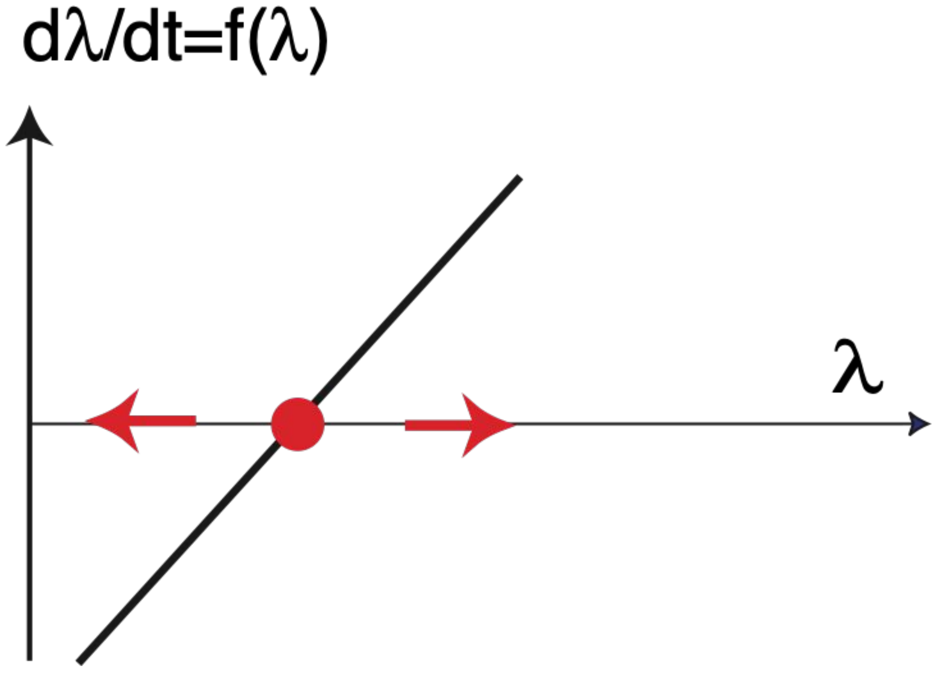
stability in a (one-dimensional) linear system

- if the slope of the linear system is negative, the fixed point is (asymptotically stable)



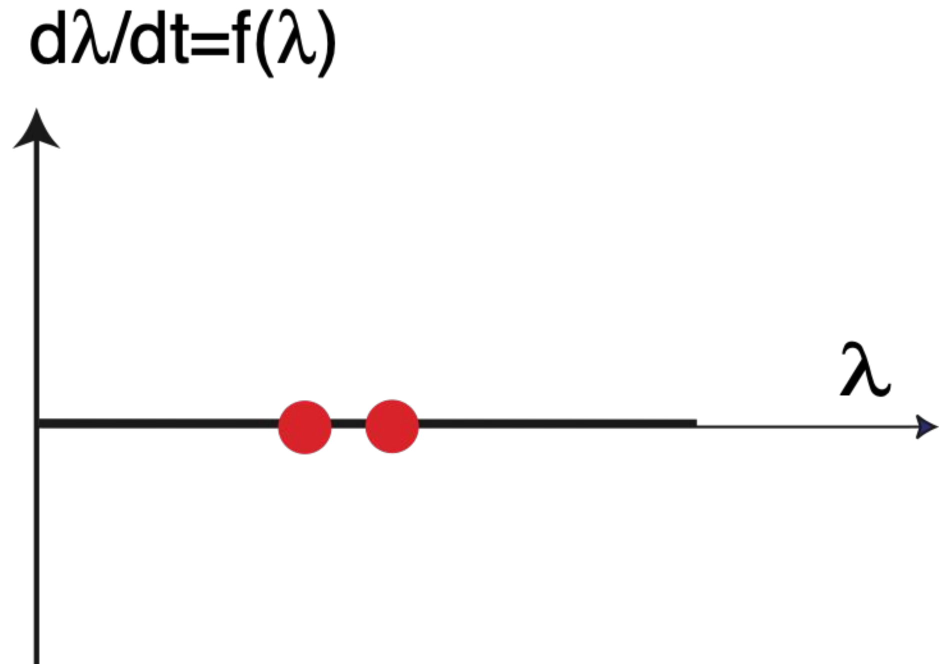
stability in a linear system

- if the slope of the linear system is positive, then the fixed point is unstable



stability in a linear system

- if the slope of the linear system is zero, then the system is indifferent (marginally stable: stable but not asymptotically stable)



stability in linear systems

- generalization to multiple dimensions:
  - if the real-parts of all Eigenvalues are negative: stable
  - if the real-part of any Eigenvalue is positive: unstable
  - if the real-part of any Eigenvalue is zero: marginally stable in that direction (stability depends on other eigenvalues)

Eigenvectors and Eigenvalues

The eigenvectors $\mathbf{v}$ of a matrix A are:

$$A\mathbf{v} = \lambda\mathbf{v}$$

with the scalar λ, which is the eigenvalue associated with the eigenvector $\mathbf{v}$

The eigenvalue λ denotes the factor by which the eigenvector $\mathbf{v}$ is scaled

Solution of the linear system

$$\dot{\mathbf{x}} = A\mathbf{x}$$

$$\rightarrow \mathbf{x}(t) = \mathbf{x}(0)\exp(At)$$

e raised to a matrix?

Solution of the linear system

$\dot{\mathbf{x}} = A\mathbf{x}$

$\rightarrow \mathbf{x}(t) = \mathbf{x}(0) \exp(At)$

e raised to a matrix? Yes, if A is *diagonal*!

With $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ follows: $\exp(At) = \begin{pmatrix} \exp(\lambda_1 t) & 0 \\ 0 & \exp(\lambda_2 t) \end{pmatrix}$

We can use the eigenvalues to *diagonalize* the matrix A

Why is only the real part important?

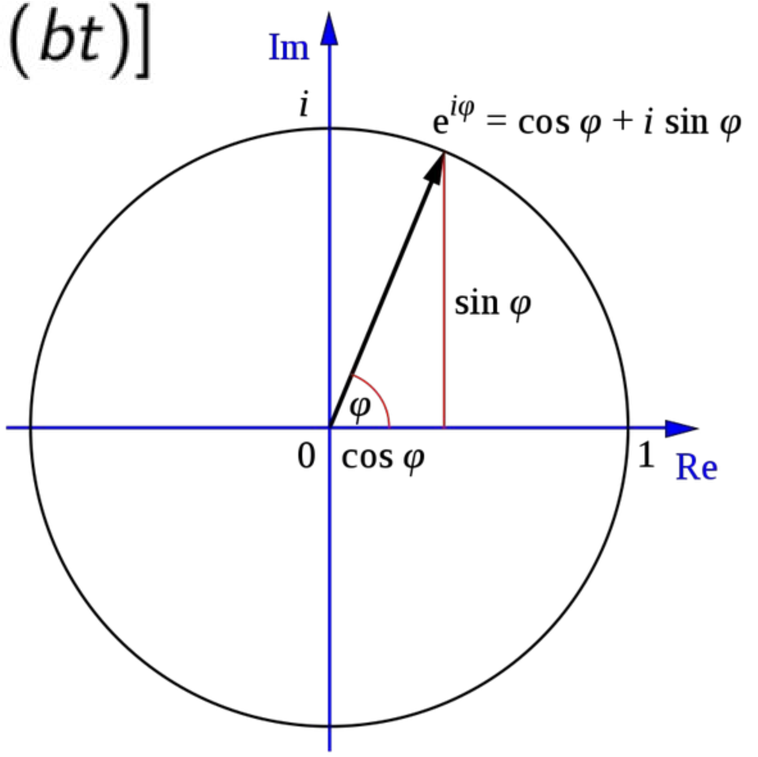
Eigenvalues may be complex, $\lambda = a + \mathrm{i}\, b$

With Euler's formula:

$$\exp((a + \mathrm{i}b)t) = \exp(at)\,[\cos(bt) + \mathrm{i}\ \sin(bt)]$$

So that the amplitude is defined by the

real part of the eigenvalue!



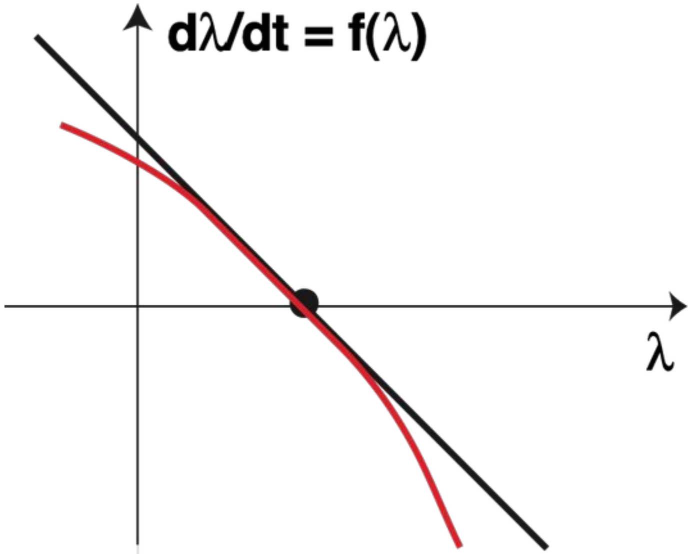
https://en.wikipedia.org/wiki/Euler's_formula

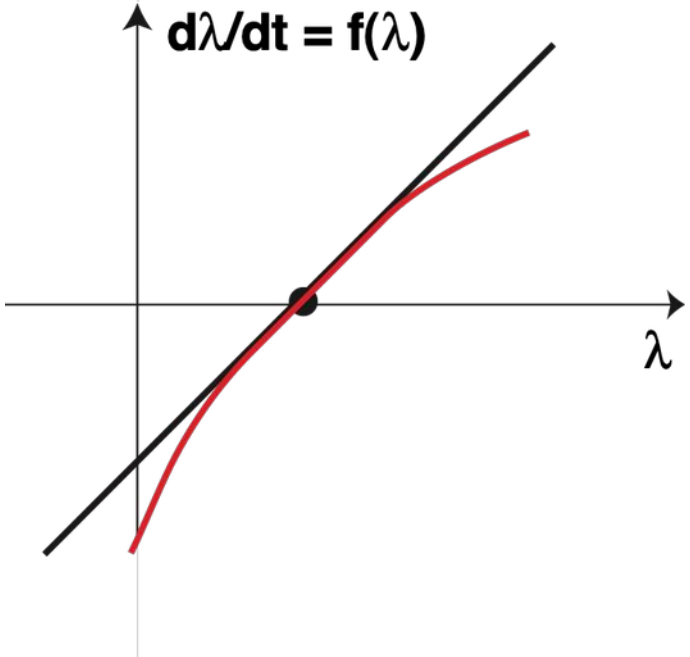
stability in nonlinear systems

- stability is a local property of the fixed point
- => linear stability theory
  - the eigenvalues of the linearization around the fixed point determine stability
  - all real-parts negative: stable
  - any real-part positive: unstable
  - any real-part zero: undecided: now nonlinearity decides (non-hyberpolic fixed point)

stability in nonlinear systems

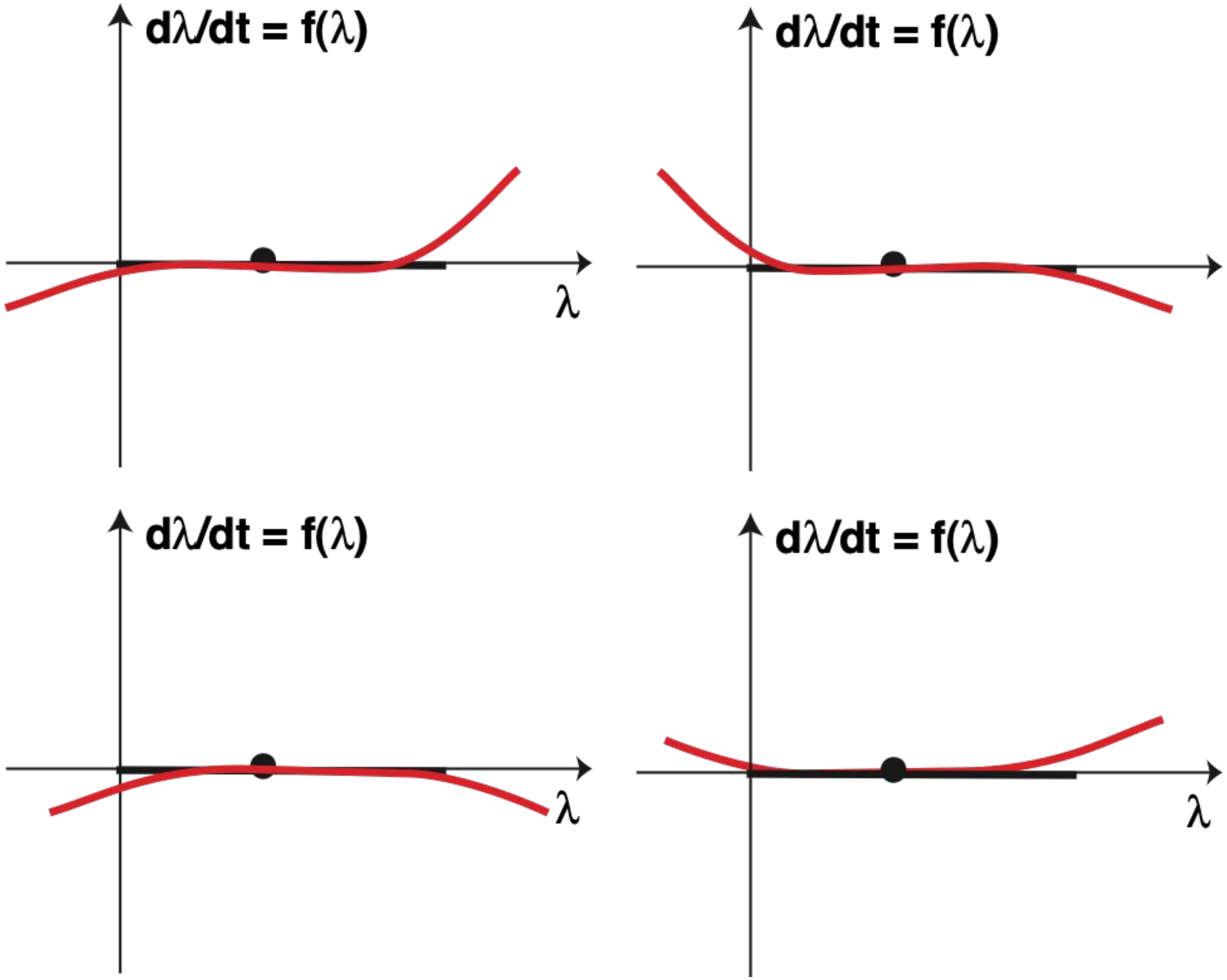
- all real-parts negative: stable
- any real-part positive: unstable





stability in nonlinear systems

- any real-part zero: undecided: now nonlinearity decides (non-hyberpolic fixed point)



Example: Pendulum

Nonlinear dynamical system:

$\dot{\theta} = \omega$

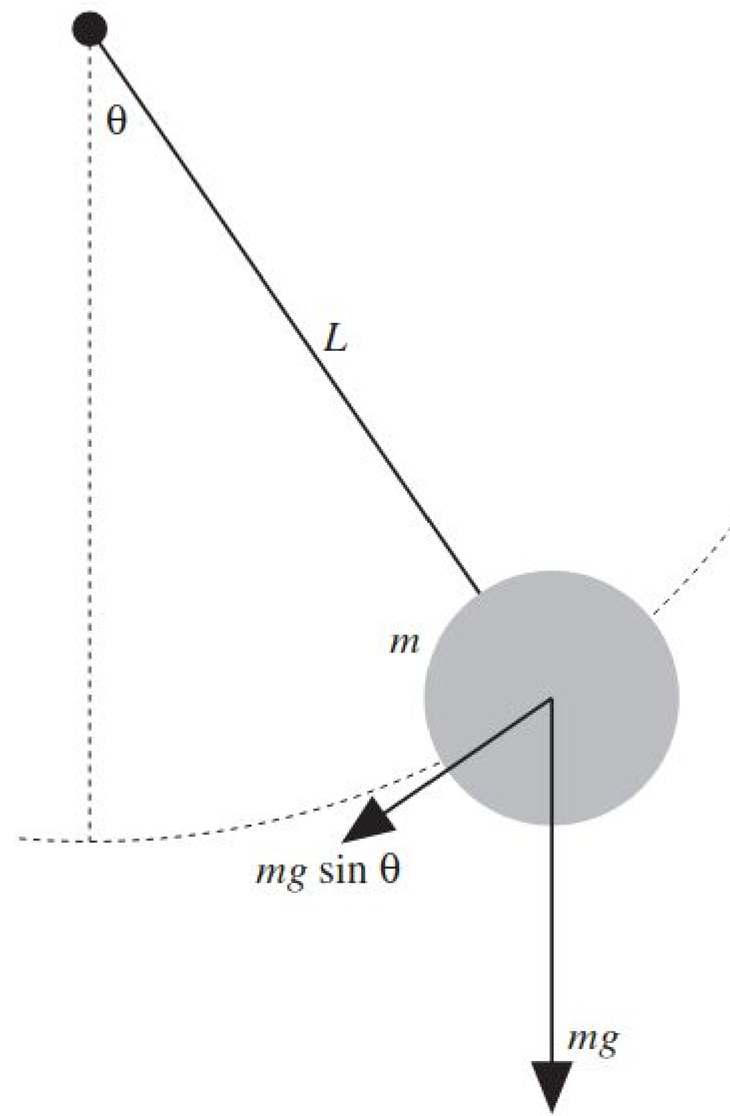
$\dot{\omega} = -\mathrm{g}/L \sin\theta - \delta\omega$

We already know the fixed points!

$(\theta_1, \omega_1) = (0, 0)$ and $(\theta_2, \omega_2) = (\pi, 0)$

Linearize around the fixed points to find

the matrix A!



Example: Pendulum

After some calculus we find our linear systems

around the two fixed points:

$$A_1 = \begin{pmatrix} 0 & 1 \\ -g/L & -\delta \end{pmatrix} \text{for the pendulum hanging down}$$

$$A_2 = \begin{pmatrix} 0 & 1 \\ g/L & -\delta \end{pmatrix} \text{for the "inverted" pendulum}$$

We find that the eigenvalues of A_1 both have a negative real part,

the eigenvalues of A_2 a positive and a negative one!