Dynamical Systems Tutorial

- What makes a system “dynamic”?
- Basic math for describing dynamical systems
- Conceptual understanding
- Stability
- Bifurcation
- Review (?)
Dynamical System

“A dynamical system is a function with an attitude” – E. Scheinerman, Invitation to Dynamical systems

- Variable(s) or state that change over time, $x(t)$
- Rule or function describing how the state evolves with time, $f(x(t))$
- Universal language of science

- physics, engineering, chemistry, biology, economics, ...
Recap: time-variation and rate of change

- Variable or state $x(t)$
- rate of change $\frac{dx}{dt}$
Recap: time-variation and rate of change

- **example:**
  - variable $x(t) = \text{position}$
  - rate of change $\frac{dx}{dt} = \text{velocity}$

- **example:**
  - variable $v(t) = \text{velocity}$
  - rate of change ?
Recap: time-variation and rate of change

- **Time course** $x(t)$
  - Position $x$ in m
  - Time $t$ in s

- **Rate of Change** $dx/dt = $ Slope of $x(t)$
  - Velocity $v$ in m/s
  - Time $t$ in s
Recap: time-variation and rate of change

- trajectory: time course of a dynamical variable
- rate of change: slope of the trajectory
dynamical system: relationship between a variable and its rate of change, \( \frac{du}{dt} = f(u) \)
linear dynamical system

\[ \frac{dx}{dt} = f(x) = ax + b \]
solution of linear dynamical systems

1. Guess:
   A function whose derivative is the function itself but with some factor?

2. Calculating:
   Exercise sheet!
   \[
   \tau \frac{dx}{dt} = -x
   \]
   \[
   dx/dt = -\tau^{-1}x
   \]
   \[
   x(t) = x(0) \exp(-t/\tau)
   \]
   Verify that \(x(t)\) is a solution of the linear differential equation!
solution of linear dynamical systems

\[ \tau \frac{dx}{dt} = -x \]

Phase plot of dynamical system

\[ x(t) = x(0) \exp\left(-\frac{t}{\tau}\right) \]

Time course of the solution \( x(t) \)
exponential relaxation to **attractors** with time scale $\tau$

$\tau$ is the time at which the initial condition $x(0)$ is reduced to $x(0)/e$ (or at any other time point $t$)

→ Exercise sheet!
Dynamical system

\[
\dot{x} = \frac{dx}{dt} = f(x)
\]

Phase plot of the dynamical system
Dynamical system

\[ \dot{x} = \frac{dx}{dt} = f(x) \]

- the present determines the future
Dynamical system

\[ \dot{x} = \frac{dx}{dt} = f(x) \]

- \( x \) spans the **state** space (can be vector-valued or even function valued)

- \( f(x) \) is the "**dynamics**" of \( x \) (or vector-field)

- \( x(t) \) is a **solution** of the dynamical systems with initial condition \( x_0 \) when the rate of change of \( x(t) \) obeys \( \frac{dx}{dt} = f(x) \) and \( x(0) = x_0 \)
Different forms of dynamical systems

- one-dimensional (ordinary) differential equation: initial value determines the future

\[ \dot{x} = f(x) \]

→ E.g. the linear dynamical system we analyzed
Different forms of dynamical systems

- vector-valued (ordinary) differential equation
- a vector of initial states determines the future, systems of differential equations:

\[ \dot{x} = f(x) \quad \text{where} \quad x = (x_1, x_2, \ldots, x_n) \]
Example of vector-valued differential equations: Ideal spring

From “Invitation to Dynamical Systems by E. Scheinerman"
Example of vector-valued differential equations: Ideal spring

Hooke’s law: \( \dot{\mathbf{v}} = -\frac{k}{m} \cdot \mathbf{x} \)

We know: \( \dot{\mathbf{x}} = \mathbf{v} \)

\[
\begin{pmatrix}
\dot{x} \\
\dot{v}
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-k/m & 0
\end{pmatrix}
\begin{pmatrix}
x \\
v
\end{pmatrix}
\]
Example of vector-valued differential equations: Ideal spring

With $x(0) = 1$, verify that a solution is:

$$
\begin{pmatrix}
  x(t) \\
  v(t)
\end{pmatrix}
=
\begin{pmatrix}
  \cos(\sqrt{k/m} \, t) \\
  -\sqrt{k/m} \sin(\sqrt{k/m} \, t)
\end{pmatrix}
$$

Position and velocity of the mass bounce back and forth (without friction)!
Different forms of dynamical systems

- discrete time

- At each state $x_k$, the state at the following time point is given by:

  $x_{k+1} = f(x_k)$ and an initial condition $x_0$
Example of discrete time dynamical system: Bank account

With $x$ being the money in our bank account and $r$ the interest rate which is payed out annually, the state of your bank account in year $k$ can be described by: $x_{k+1} = (1 + r)x_k$ with the initial state $x_0$

Let $r = 4\%$ and $x_0 = $100:

$$x_1 = 1.04 \cdot $100 = $104$$
$$x_2 = 1.04 \cdot $104 = $108.16$$
$$x_3 = 1.04 \cdot $108.16 = $112.49$$
...
Other forms of dynamical systems

- partial differential equations
- integro-differential equations
- delay differential equations
- Functional differential equations
Numerical solutions

- Use the discrete to approximate the continuous:
  compute solution, $x(t_i)$, by iterating through time, $t_i = i \Delta t$, $i=0,1,...,N$

- for example: (forward Euler)

$$x_{i+1} = x_i + \Delta t \ f(x_i)$$
=> code / simulation
Fixed point

- is a constant solution of the dynamical system
- that is a state with zero rate of change:

\[
\dot{x} = f(x) \quad \text{and} \quad \dot{x} = 0 \quad \Rightarrow \quad f(x_0) = 0
\]

Zero crossings in the phase plot!
**attractor**

- **fixed point**, to which neighboring initial conditions converge = **attractor**
stability

- mathematically really: asymptotic stability
- defined: a fixed point is asymptotically stable, when solutions of the dynamical system that start nearby converge in time to the fixed point
stability

- the mathematical concept of stability (which we do not use) requires only that nearby solutions stay nearby

- Definition: a fixed point is **unstable** if it is not stable in that more general sense,

  - that is: if nearby solutions do not necessarily stay nearby (may diverge)
How to tell whether a fixed point is stable or unstable?
linear approximation near fixed point

- non-linearity as a small perturbation/deforation of linear system
- Only need to analyze the linearized system!
stability in a (one-dimensional) linear system

- if the slope of the linear system is negative, the fixed point is (asymptotically stable)
stability in a linear system

- If the slope of the linear system is positive, then the fixed point is unstable.
stability in a linear system

- If the slope of the linear system is zero, then the system is indifferent (marginally stable: stable but not asymptotically stable)
stability in linear systems

- generalization to multiple dimensions:
  - if the real-parts of all Eigenvalues are negative: stable
  - if the real-part of any Eigenvalue is positive: unstable
  - if the real-part of any Eigenvalue is zero: marginally stable in that direction (stability depends on other eigenvalues)
Eigenvectors and Eigenvalues

The eigenvectors $\mathbf{v}$ of a matrix $A$ are:

$$A\mathbf{v} = \lambda \mathbf{v}$$

with the scalar $\lambda$, which is the eigenvalue associated with the eigenvector $\mathbf{v}$

The eigenvalue $\lambda$ denotes the factor by which the eigenvector $\mathbf{v}$ is scaled
Solution of the linear system

\[ \dot{x} = Ax \]
\[ \rightarrow x(t) = x(0) \exp(At) \]

e raised to a matrix?
Solution of the linear system

\[ \dot{x} = Ax \]
\[ \rightarrow x(t) = x(0) \exp(At) \]

e raised to a matrix? Yes, if A is diagonal!

With \( A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \) follows: \( \exp(At) = \begin{pmatrix} \exp(\lambda_1 t) & 0 \\ 0 & \exp(\lambda_2 t) \end{pmatrix} \)

We can use the eigenvalues to diagonalize the matrix A
Why is only the real part important?

Eigenvalues may be complex, $\lambda = a + i \, b$

With Euler’s formula:

$\exp((a + i b)t) = \exp(at) [\cos(bt) + i \sin(bt)]$

So that the amplitude is defined by the real part of the eigenvalue!

https://en.wikipedia.org/wiki/Euler's_formula
stability in nonlinear systems

- stability is a local property of the fixed point

- => linear stability theory
  
  - the eigenvalues of the linearization around the fixed point determine stability
  
  - all real-parts negative: stable
  
  - any real-part positive: unstable
  
  - any real-part zero: undecided: now nonlinearity decides (non-hyperbolic fixed point)
stability in nonlinear systems

- all real-parts negative: stable
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stability in nonlinear systems

- any real-part zero: undecided: now nonlinearity decides (non-hyperbolic fixed point)
Example: Pendulum

Nonlinear dynamical system:
\[
\dot{\theta} = \omega \\
\dot{\omega} = -\frac{g}{L} \sin \theta - \delta \omega
\]

We already know the fixed points!
\[(\theta_1, \omega_1) = (0, 0) \text{ and } (\theta_2, \omega_2) = (\pi, 0)\]
Linearize around the fixed points to find the matrix A!
Example: Pendulum

After some calculus we find our linear systems around the two fixed points:

\[
A_1 = \begin{pmatrix}
0 & 1 \\
-g/L & -\delta \\
\end{pmatrix}
\] for the pendulum hanging down

\[
A_2 = \begin{pmatrix}
0 & 1 \\
g/L & -\delta \\
\end{pmatrix}
\] for the "inverted" pendulum

We find that the eigenvalues of \(A_1\) both have a negative real part, the eigenvalues of \(A_2\) a positive and a negative one!