

Lecture 6

Differential Equations

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Computer Science and Mathematics
Preparatory Course

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Overview

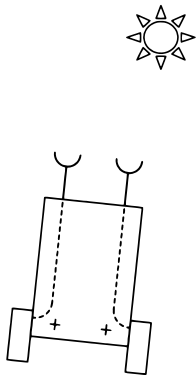
1. Motivation

2. Mathematics

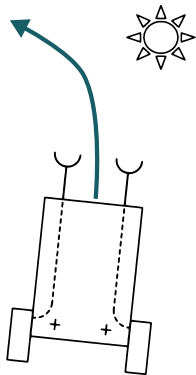
- Solving Differential Equations
- Qualitative Analysis
- Numerical Approximation

3. Tasks

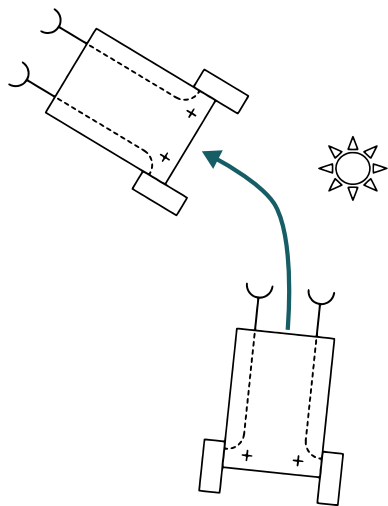
The Vehicle's Behavior as Function of Angle Change



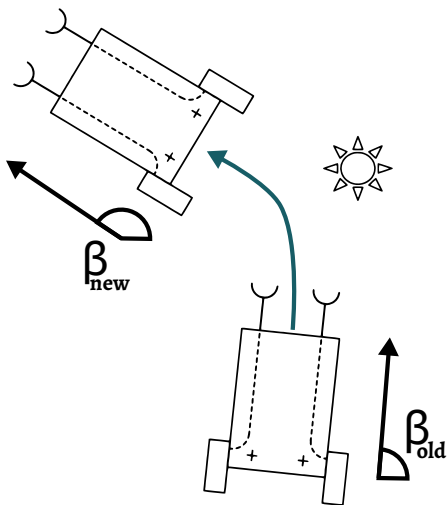
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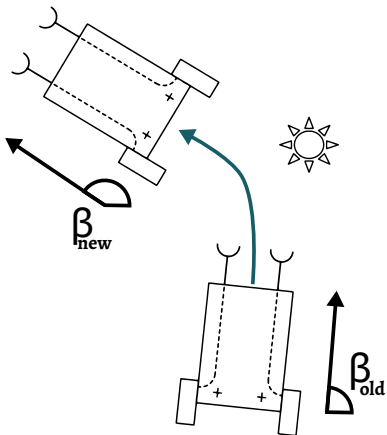
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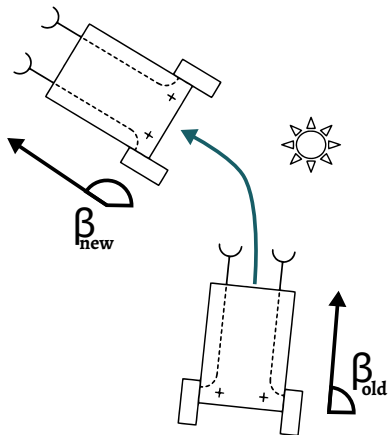


The Vehicle's Behavior as Function of Angle Change



- ▶ The vehicle's change in angle depends on its current sensor input

The Vehicle's Behavior as Function of Angle Change



- ▶ The vehicle's change in angle depends on its current sensor input
- ▶ The following equation may describe its behavior

$$\frac{d\beta}{dt} = -S_L + S_R,$$

where t describes time and S_L, S_R left and right sensor values.

Differential Equation as Rule System

- ▶ A differential equation describes how the rate of change of a system depends on its current state. For example:

$$f'(x) = 4f(x) + 5 = g(f(x)) \quad \text{with} \quad g(x) = 4x + 5$$

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- ▶ A differential equation describes how a system should change in a given state.
- ▶ Brief oversimplification:

A differential equation describes rules for the future

Solving Differential Equations

- ▶ Given a differential equation of the form $f'(x) = g(f(x)) \dots$ the original function $f(x)$ is usually not known.
- ▶ Solving a differential equation describes the process of finding an $f(x)$ that follows the above rule for all x
 - ▶ In the above example:

$$f'(x) = 4f(x) + 5 \Rightarrow f(x) = \frac{e^{4x+c}}{4} - \frac{5}{4} \text{ and } f'(x) = e^{4x+c}$$

- ▶ Differential equations entail two equations
 1. The function $g(f(x))$ governing the rate of change
 2. The function $f(x)$ describing the overall behavior

Derivative vs. Differential equation

▶ $f'(x) = cx$

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Derivative vs. Differential equation

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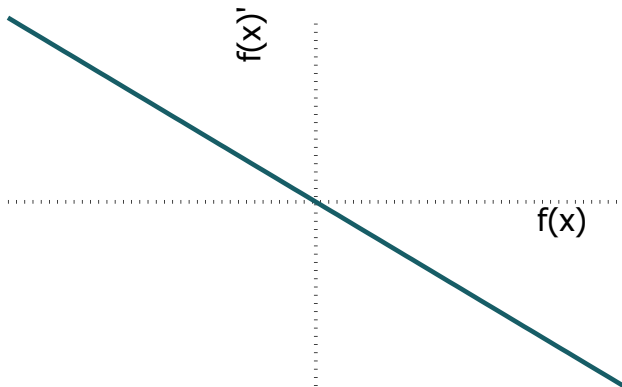
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 $g(f(x)) = cf(x)$
 - ▶ The only function that stays the same when differentiated is the exponential function e^x
 - ▶ Considering the chain rule the derivative of e^{cx} is exactly ce^{cx} therefore
 $f(x) = ce^{cx}$
 - ▶ Usually a differential equation is not that easily solvable

Dynamical Systems Theory

- ▶ Mathematicians want to find solutions to particular differential equations
- ▶ **Dynamical Systems Theory** is concerned with analyzing the qualitative behavior of the system

Qualitative Behavior of Differential Equations

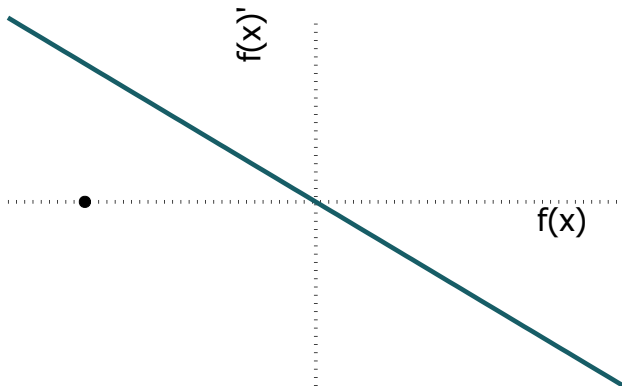
$$f'(x) = y' = \frac{dy}{dx} = -f(x)$$



Phase Plot

Qualitative Behavior of Differential Equations

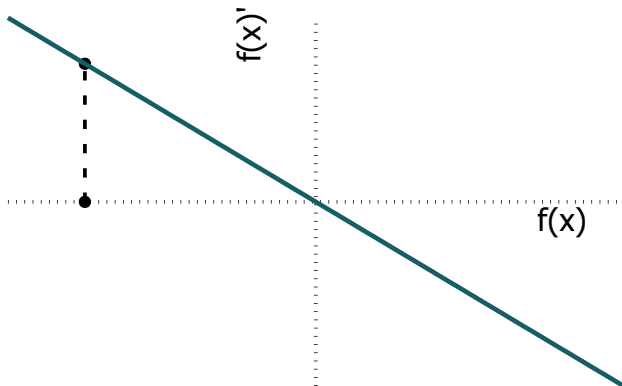
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Phase Plot

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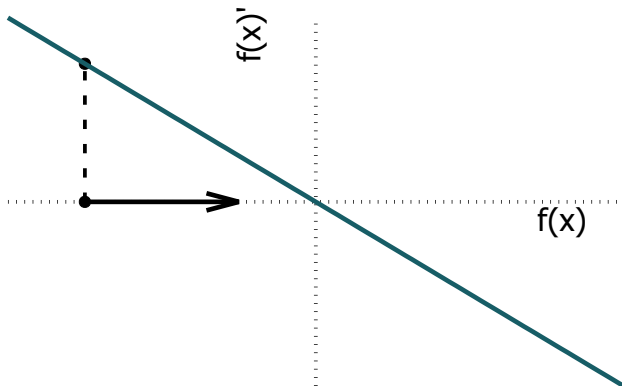
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Phase Plot

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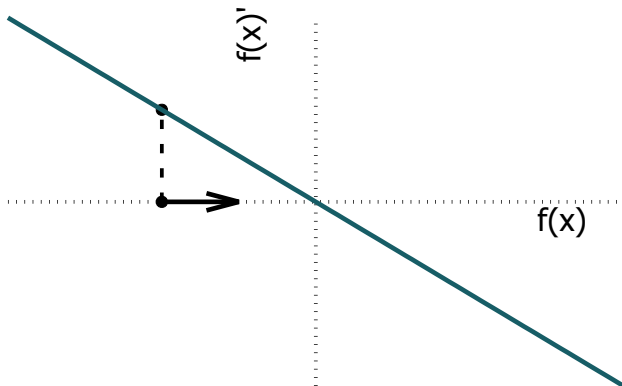
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Phase Plot

Qualitative Behavior of Differential Equations

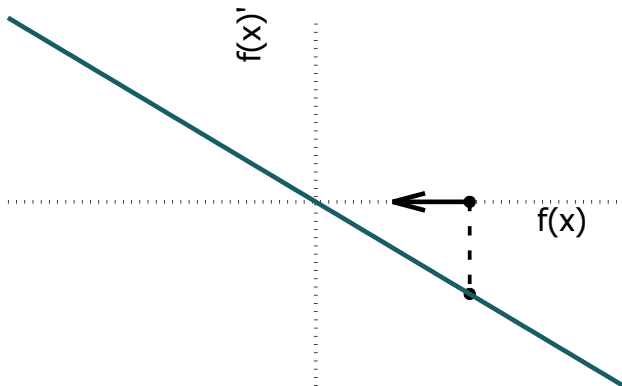
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Phase Plot

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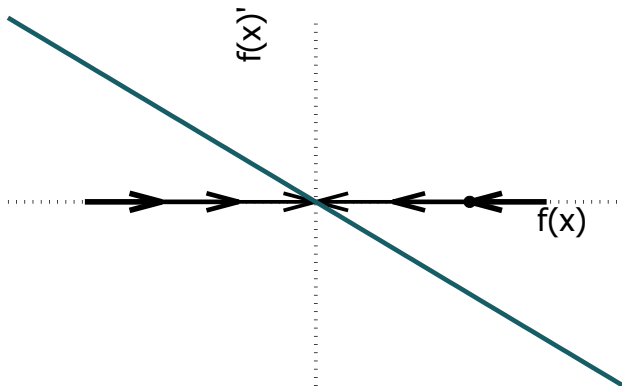
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Phase Plot

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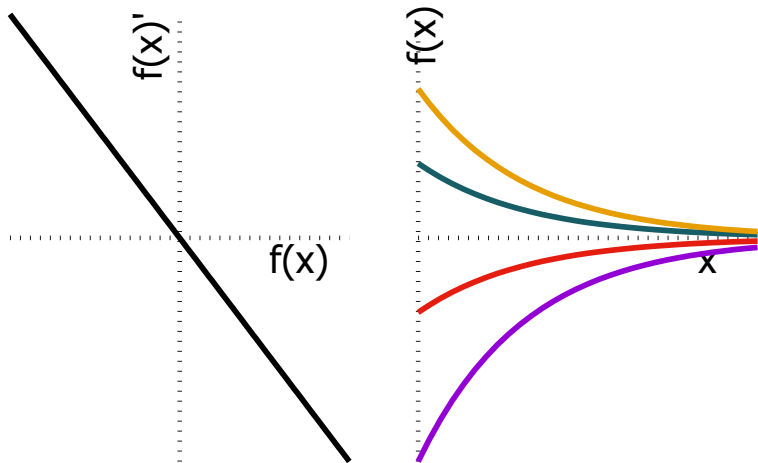
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Phase Plot

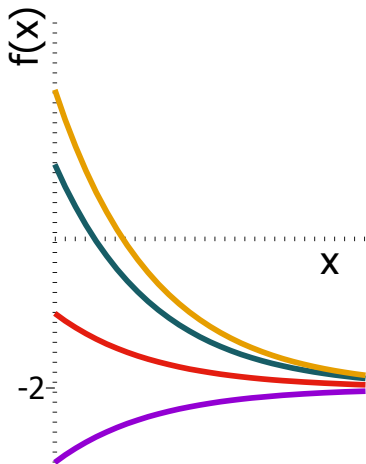
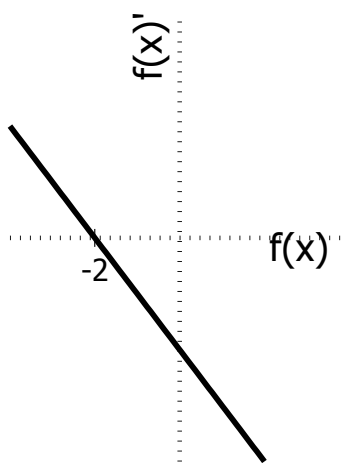
Attractors

$$f'(x) = y' = \frac{dy}{dx} = -f(x)$$



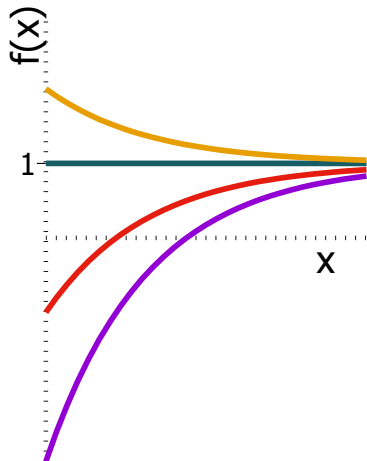
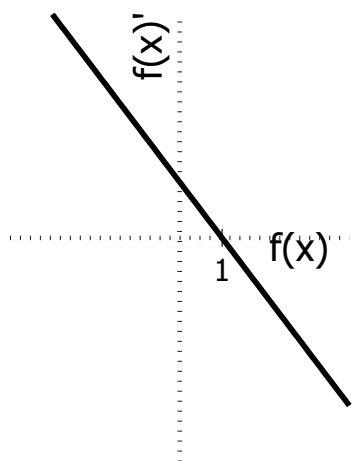
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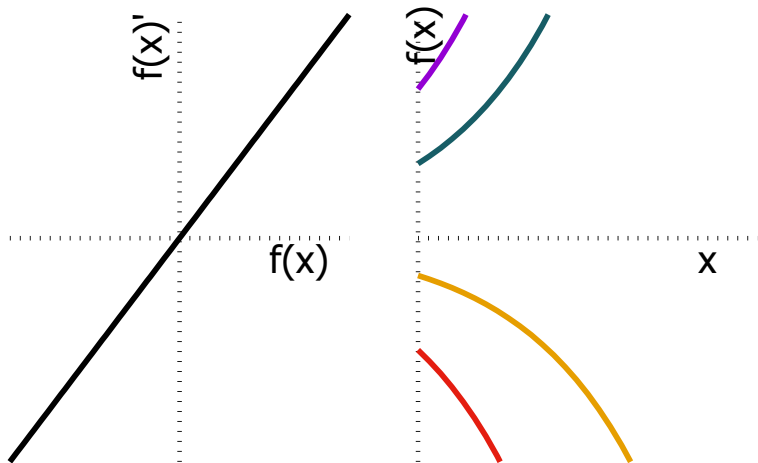
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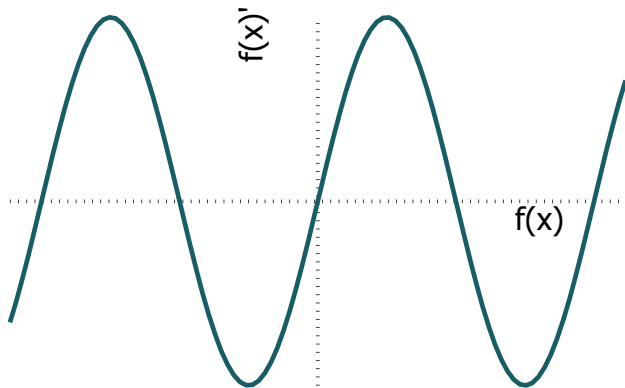
Repellers

$$f'(x) = y' = \frac{dy}{dx} = -f(x)$$



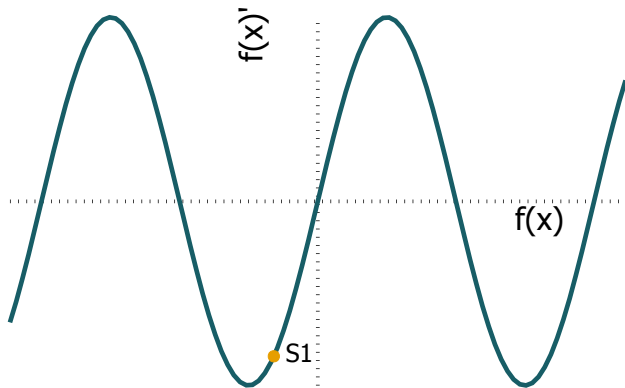
Initial Condition Matters

$$f'(x) = y' = \frac{dy}{dx} = \sin(x)$$



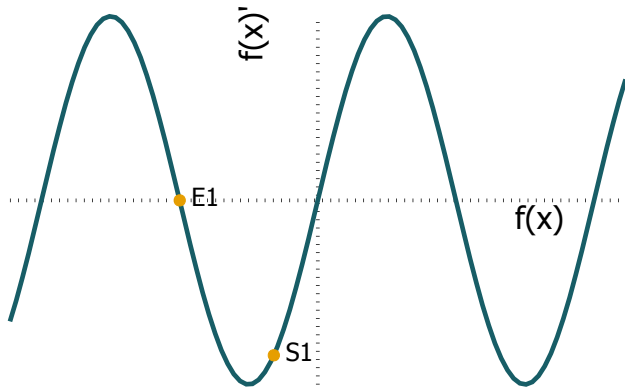
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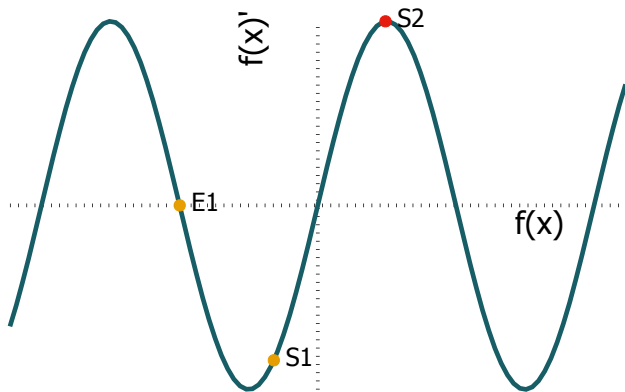
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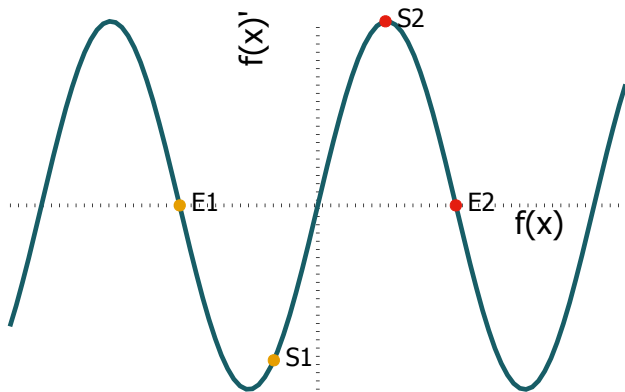
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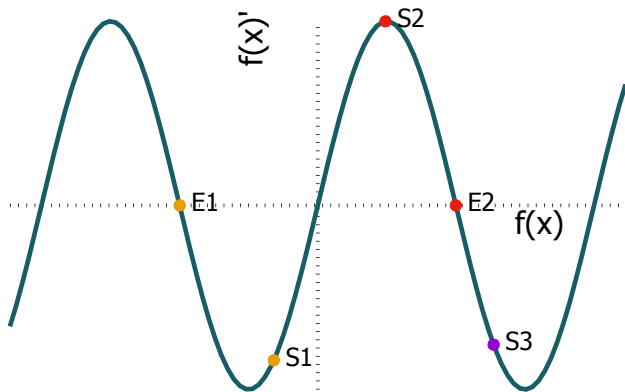
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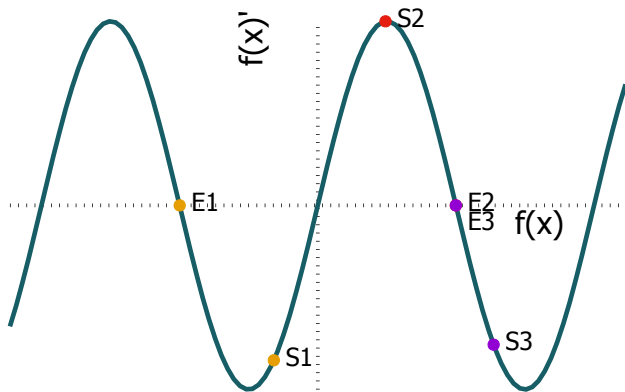
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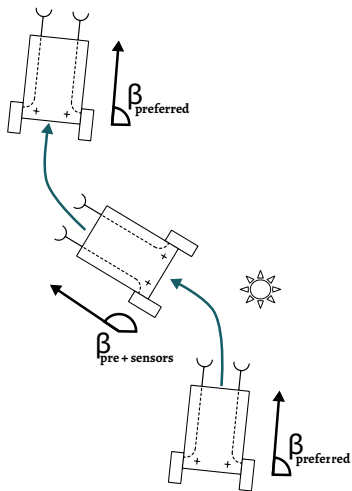


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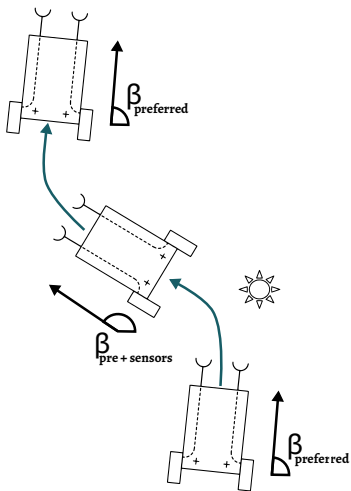
Back to the Braitenberg Vehicle



- We govern the vehicle's behavior with a differential equation

$$\frac{d\beta}{dt} = -\beta - S_L + S_R,$$

Back to the Braitenberg Vehicle



- We govern the vehicle's behavior with a differential equation

$$\frac{d\beta}{dt} = -\beta - S_L + S_R,$$

- Adding an attractor gives the vehicle a preferred orientation

1. Motivation

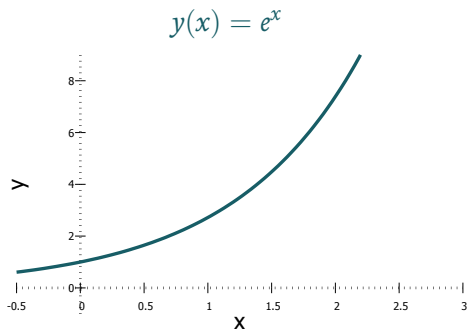
2. Mathematics

- ▶ Solving Differential Equations
- ▶ Qualitative Analysis
- ▶ Numerical Approximation

3. Tasks

Euler Approximation

$$\frac{dy}{dx} = y \quad y(0) = 1$$

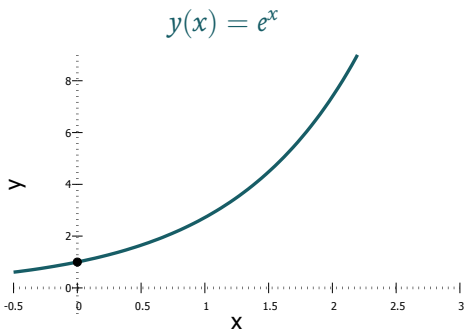


Euler Approximation

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$\Delta x = 1$$

x	y	$\frac{dy}{dx}$
0	1	1

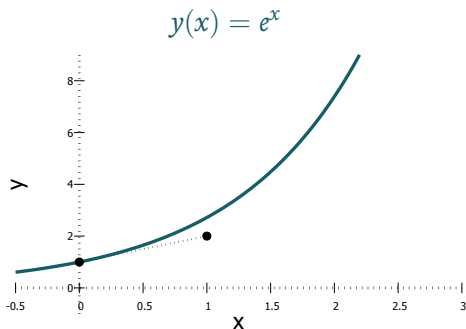


Euler Approximation

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$\Delta x = 1$$

x	y	$\frac{dy}{dx}$
0	1	1
1	2	2

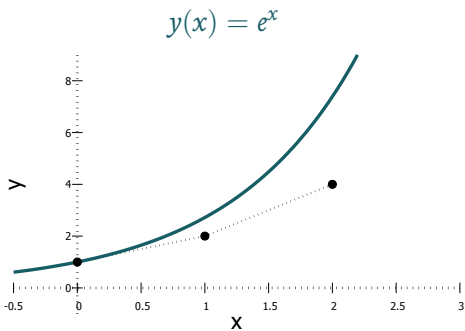


Euler Approximation

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$\Delta x = 1$$

x	y	$\frac{dy}{dx}$
0	1	1
1	2	2
2	4	4

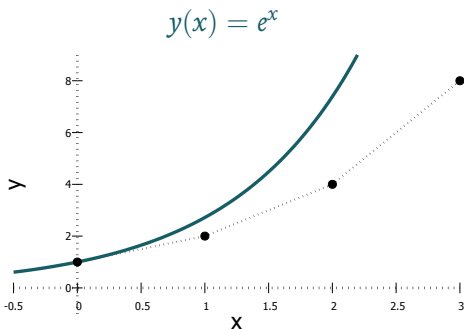


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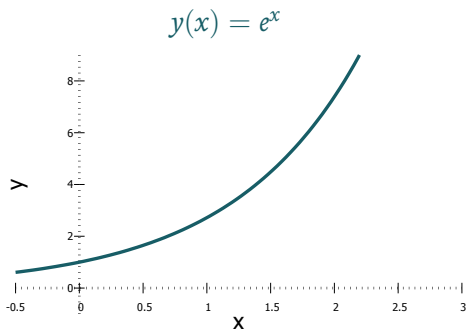
$$\Delta x = 1$$

x	y	$\frac{dy}{dx}$
0	1	1
1	2	2
2	4	4
3	8	8



Varying the stepsize

$$\frac{dy}{dx} = y \quad y(0) = 1$$

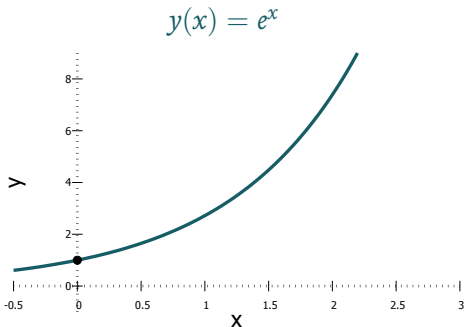


Varying the stepsize

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$\Delta x = 0.5$$

x	y	$\frac{dy}{dx}$
0	1	1

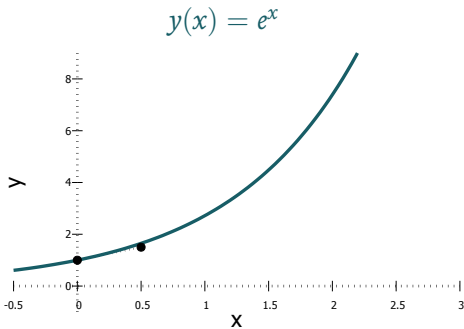


Varying the stepsize

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$\Delta x = 0.5$$

x	y	$\frac{dy}{dx}$
0	1	1
0.5	1.5	1.5

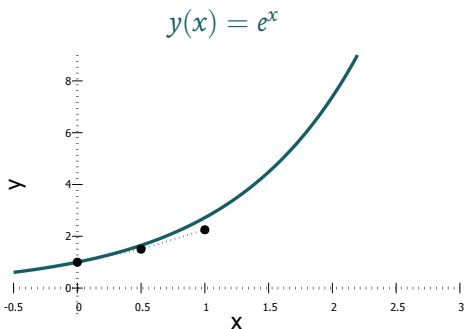


Varying the stepsize

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$\Delta x = 0.5$$

x	y	$\frac{dy}{dx}$
0	1	1
0.5	1.5	1.5
1	2.25	2.25

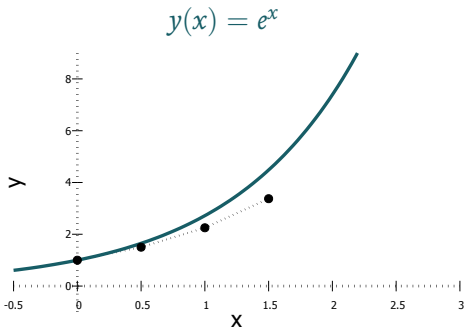


Varying the stepsize

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$\Delta x = 0.5$$

x	y	$\frac{dy}{dx}$
0	1	1
0.5	1.5	1.5
1	2.25	2.25
1.5	3.375	3.375

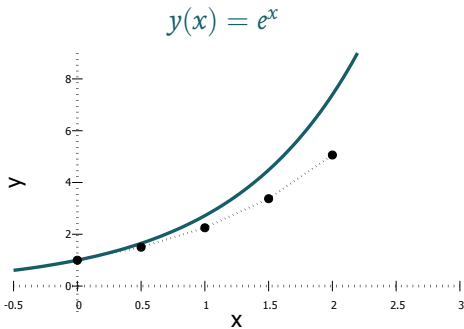


Varying the stepsize

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x	y	$\frac{dy}{dx}$
0	1	1
0.5	1.5	1.5
1	2.25	2.25
1.5	3.375	3.375
2	5.0625	5.0625



Euler Approximation in Words

1. Start with a certain value for x and y and the differential equation $\frac{dy}{dx} = \dots$ you want to approximate
2. Decide for a step size that determines the accuracy of your approximation
3. Repeat as long as you like:
 - 3.1 Use the current y -value to calculate the current rate of change $\frac{dy}{dx}$
 - 3.2 Calculate the next y -value by taking the current y -value and adding to it the rate of change times the step size
 - 3.3 Increase x by the step size

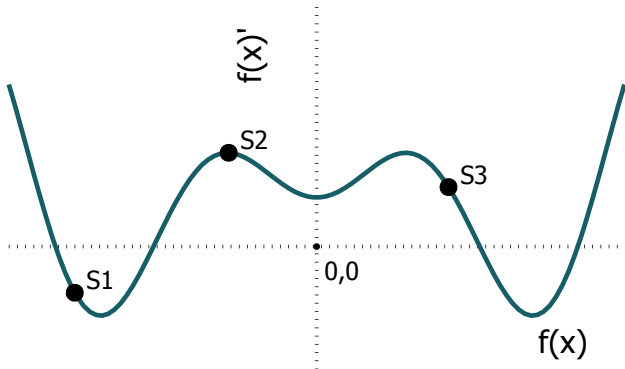
Task Template

- ▶ Download the archive *task_template_6.zip* from the course homepage. Extract it into a folder of your choice.
- ▶ The archive contains *task_61.py*, *student_code_61.py*, and *braitenberg.png*.
- ▶ You only need to edit code in the *student_code* file.

Explain Task Template!

Tasks

1. For each of the three starting values $S1$, $S2$, $S3$ of the function $f(x)$ shown in the plot below determine which function value will be assumed after an infinite amount of time steps (graphically).



Tasks

2. Change the behavior of the vehicle (*graph*) by implementing the function *calc_angle_change*.
 - ▶ *current_angle* is the current orientation of the vehicle in degree.
 - ▶ *left_sensor_values* and *right_sensor_values* are the measured values of the sensors. They increase the closer they are to an obstacle.
 - ▶ First make the angle change dependent on the current sensor values. How can you make the vehicle avoid obstacles? How can you guide it towards obstacles?
 - ▶ *Let your change in the angle depend on the current angle itself. Set an attractor at 45° , such that the vehicle will turn towards 45° degrees in the absence of obstacles.*
 - ▶ *What do you need to change to avoid a particular orientation?*

Tasks

3. Imagine the differential equation $\frac{dy}{dx} = -y + 20$, where y describes the heading of your vehicle. The initial orientation is $y(0) = 40^\circ$.
- ▶ Calculate the y -values up to an x -value of 3 by hand using the Euler approximation method. Use a step size of 0.5.
 - ▶ Implement the Euler approximation method in a python script, which can go to a certain x -value with a certain step size.
 - ▶ Hint: You can reuse a lot of the code from yesterday.
 - ▶ Calculate how long your for-loop has to run depending on the desired x -value and your step size.
 - ▶ Save your results in three different lists. One for the x -values, one for the y -values and one for the $\frac{dy}{dx}$ term.
- * Plot your x -values against your y -values and your y -values against your $\frac{dy}{dx}$ -values. (See the next slide for plotting commands.)

Matplotlib.pyplot

► The pyplot submodule

```
# A submodule can be imported with the . operator
import matplotlib.pyplot as plt
# The as operator allows renaming for convenience
xValues = [1,1,2,3,5,8,13]
yValues = [3,4,7,6,9,10,12]
plt.plot(xValues,yValues) #plots lines
# This generates the plot and .show() displays it
plt.show()
#plots points and lines
plt.plot(xValues,yValues,linestyle = "-", marker="o")
plt.show()
```
