Dynamical systems tutorial

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“dynamics”

- the word “dynamics”
  - time-varying measures
  - range of a quantity
  - forces causing/accounting for movement => dynamical systems
- dynamical systems are the universal language of science
  - physics, engineering, chemistry, theoretical biology, economics, quantitative sociology, ...
time-variation and rate of change

- variable \( x(t) \);
- rate of change \( \frac{dx}{dt} \)
functional relationship between a variable and its rate of change

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(linear) dynamical system

\[ \tau \frac{dx}{dt} = -x \]
exponential relaxation to attractors

\[ \tau \dot{x} = -x \Rightarrow x(t) = x(0)e^{-t/\tau} \] (check!)

\[ \Rightarrow \text{has a well-defined time scale} \]
(nonlinear) dynamical system

dx/dt = f(x)
dynamical system

- present determines the future
  - given initial condition
- predict evolution (or predict the past)

\[ \frac{dx}{dt} = f(x) \]
dynamical systems

- $\mathbf{x}$: spans the state space (or phase space)
- $f(\mathbf{x})$: is the “dynamics” of $\mathbf{x}$ (or vector-field)
- $\mathbf{x}(t)$ is a solution of the dynamical systems to the initial condition $\mathbf{x}_0$
  - if its rate of change $= f(\mathbf{x})$
  - and $\mathbf{x}(0)=\mathbf{x}_0$
Dynamical systems

differential equation $\dot{x} = f(x)$ in one dimension

=> an initial value of $x$ determines the future
Dynamical systems

- system of differential equations $\dot{x} = f(x)$
- $\Rightarrow$ a vector of initial states, $x = (x_1, x_2, \ldots, x_n)$ determines the future
Dynamical systems

- partial differential equations
  \[ \dot{x}(y, t) = f \left( x(y, t), \frac{\partial x(y, t)}{\partial y}, \ldots \right) \]

- integro-differential equations
  \[ \dot{x}(y, t) = \int dy' f \left( y, y', x(y', t) \right) \]

=> continuously many initial values\(=\)initial function \(x(y)\) determine the future
Dynamical systems

delay differential equations $\dot{x}(t) = f(x(t - \tau))$

functional differential equations

$\dot{x}(t) = \int_{t'}^{t} dt'f(x(t'))$

=> a past piece of trajectory determines the future
Dynamical systems

- Iteration equation in discrete time (map)
  \[ x_{n+1} = g(x_n) \]

- Every dynamical system in continuous time
  \[ \Rightarrow \text{dynamical system in discrete time (Poincaré)} \]

- A dynamical system in discrete time can be lifted to a dynamical system in continuous time (but not uniquely)
Resources

- free online textbook by Scheinermann
  - https://github.com/scheinerman/InvitationToDynamicalSystems
  - send him a postcard (as instructed there)
  - really nice book for beginners…
  - focus on the time-continuous part.
numerics

- Sample time discretely
- Compute solution by iterating through time
- Valid approximation for small time steps...
forward Euler

- $t_i = i \Delta t$ so that $x_i = x(t_i)$
- $\dot{x} = dx/dt \approx \Delta x/\Delta t$ where $\Delta x = x_{i+1} - x_i$
- $\dot{x} = f(x) \Rightarrow x_{i+1} = x_i + \Delta t \, f(x_i)$
- ... valid for small $\Delta t$
- is the “worst” approximation scheme (needs smallest time step to achieve given precision...)
- but useful for real-time embedded (and for stochastic systems)
modern numerics

- Runge-Kutte: error scales with step size to a power (e.g. 4)
- adaptive step size..
- built-into numerical packages… e.g. ode45 in Matlab
=> simulation
qualitative theory of dynamical systems

good source:

qualitative theory of dynamical systems

- The goals is to characterize the ensemble of solutions of the dynamical system (or a family of such).

- = The flow

- … use special invariant solutions to do that… fixed points, their stable/unstable manifolds…
A fixed point, to which neighboring initial conditions converge = attractor

\[
dx/dt = f(x)
\]
A fixed point is a constant solution of the dynamical system

\[ \dot{x} = f(x) \]

\[ \dot{x} = 0 \Rightarrow f(x_0) = 0 \]
stability

Mathematically really: **asymptotic stability**

Defined: A fixed point is asymptotically stable, when solutions of the dynamical system that start nearby converge in time to the fixed point.
the mathematical concept of stability (which we do not use) requires only that nearby solutions stay nearby

defined: a fixed point is **unstable** if it is not stable in that more general sense,

that is: if nearby solutions do not necessarily stay nearby (may diverge)
linear approximation near attractor

non-linearity as a small perturbation/deformation of linear system

=> non-essential non-linearity

\[ \frac{dx}{dt} = f(x) \]
stability in a linear system

- If the slope of the linear system is negative, the fixed point is (asymptotically stable)
stability in a linear system

if the slope of the linear system is positive, then the fixed point is unstable
stability in a linear system

If the slope of the linear system is zero, then the system is indifferent (marginally stable: stable but not asymptotically stable)
stability in linear systems

generalization to multiple dimensions

- if the real-parts of all Eigenvalues are negative: stable
- if the real-part of any Eigenvalue is positive: unstable
- if the real-part of any Eigenvalue is zero: marginally stable in that direction (stability depends on other eigenvalues)
stability in nonlinear systems

- stability is a local property of the fixed point
- => linear stability theory
  - the eigenvalues of the linearization around the fixed point determine stability
  - all real-parts negative: stable
  - any real-part positive: unstable
  - any real-part zero: undecided: now nonlinearity decides (non-hyperpolic fixed point)
stability in nonlinear systems

- all real-parts negative: stable

- any real-part positive: unstable

\[
\frac{d\lambda}{dt} = f(\lambda)
\]
stability in nonlinear systems

- any real-part zero: undecided; now nonlinearity decides (non-hyperpolic fixed point)
bifurcations

- look now at families of dynamical systems, which depend (smoothly) on parameters

- ask: as the parameters change (smoothly), how do the solutions change (smoothly?)

  - smoothly: topological equivalence of the dynamical systems at neighboring parameter values

  - bifurcation: dynamical systems NOT topological equivalent as parameter changes infinitesimally
bifurcation

dx/dt = f(x)

attractor 1

attractor 2
bifurcation

- Bifurcation = qualitative change of dynamics (change in number, nature, or stability of fixed points) as the dynamics changes smoothly.

\[
dx/dt = f(x)
\]

**Diagram:**
- The diagram illustrates the concept of bifurcation with a phase portrait showing the trajectories leading to attractors.
- The arrows indicate the direction of change in the system as it moves towards different attractors.
- Attractor 1 and Attractor 2 demonstrate the qualitative change as the system evolves.
tangent bifurcation

The simplest bifurcation (co-dimension 0): an attractor collides with a repellor and the two annihilate.
local bifurcation

\[ \frac{dx}{dt} = f(x) \]

attractor 1
attractor 2
reverse bifurcation

changing the dynamics in the opposite direction

dx/dt = f(x)

attractor 1  attractor 2
bifurcations are instabilities

- that is, an attractor becomes unstable before disappearing
- (or the attractor appears with reduced stability)
- formally: a zero-real part is a necessary condition for a bifurcation to occur
tangent bifurcation

- Normal form of tangent bifurcation
  \[ \dot{x} = \alpha - x^2 \]

- (=simplest polynomial equation whose flow is topologically equivalent to the bifurcation)

\[ x_0 = \sqrt{\alpha} \]
Hopf theorem

When a single (or pair of complex conjugate) eigenvalue crosses the imaginary axis, one of four bifurcations occur:

- Tangent bifurcation
- Transcritical bifurcation
- Pitchfork bifurcation
- Hopf bifurcation
transcritical bifurcation

\[ \dot{x} = \alpha x - x^2 \]

normal form

\( x = \alpha x - x^2 \)

\( \alpha \) negative

\( \alpha \) positive

\( \alpha = 0 \)

\( x \)

\( dx/dt \)

fixed point

stable

unstable

\( \alpha \)
pitchfork bifurcation

normal form

\[ \dot{x} = \alpha x - x^3 \]

\[ \dot{x} = -2x_0x = -2\sqrt{\alpha}x \]

\( \alpha \) positive

\( \alpha \) negative

\( \alpha = 0 \)
Hopf: need higher dimensions
2D dynamical system: vector-field

\[ \dot{x}_1 = f_1(x_1, x_2) \]
\[ \dot{x}_2 = f_2(x_1, x_2) \]
\[ \dot{x}_1 = f_1(x_1, x_2) \]
\[ \dot{x}_2 = f_2(x_1, x_2) \]
fixed point, stability, attractor

\[ \dot{x}_1 = f_1(x_1, x_2) \]
\[ \dot{x}_2 = f_2(x_1, x_2) \]
Hopf bifurcation

\[ \dot{r} = \alpha r - r^3 \]
\[ \dot{\phi} = \omega \]

**Normal form**

\[ \begin{align*}
\dot{r} &= \alpha - 3r^2 \\
\dot{\phi} &= \alpha \\
\end{align*} \]

Graph showing stable and unstable regions with \( \alpha \) as a parameter.
forward dynamics

- given known equation, determined fixed points / limit cycles and their stability
- more generally: determine invariant solutions (stable, unstable and center manifolds)
inverse dynamics

given solution, find the equation…

design of behavioral dynamics…
inverse dynamics: design

- In the design of behavioral dynamics... you may be given:
- Attractor solutions/stable states
- And how they change as a function of parameters/conditions
- => Identify the class of dynamical systems using the 4 elementary bifurcations
- And use normal form to provide an exemplary representative of the equivalence class of dynamics