

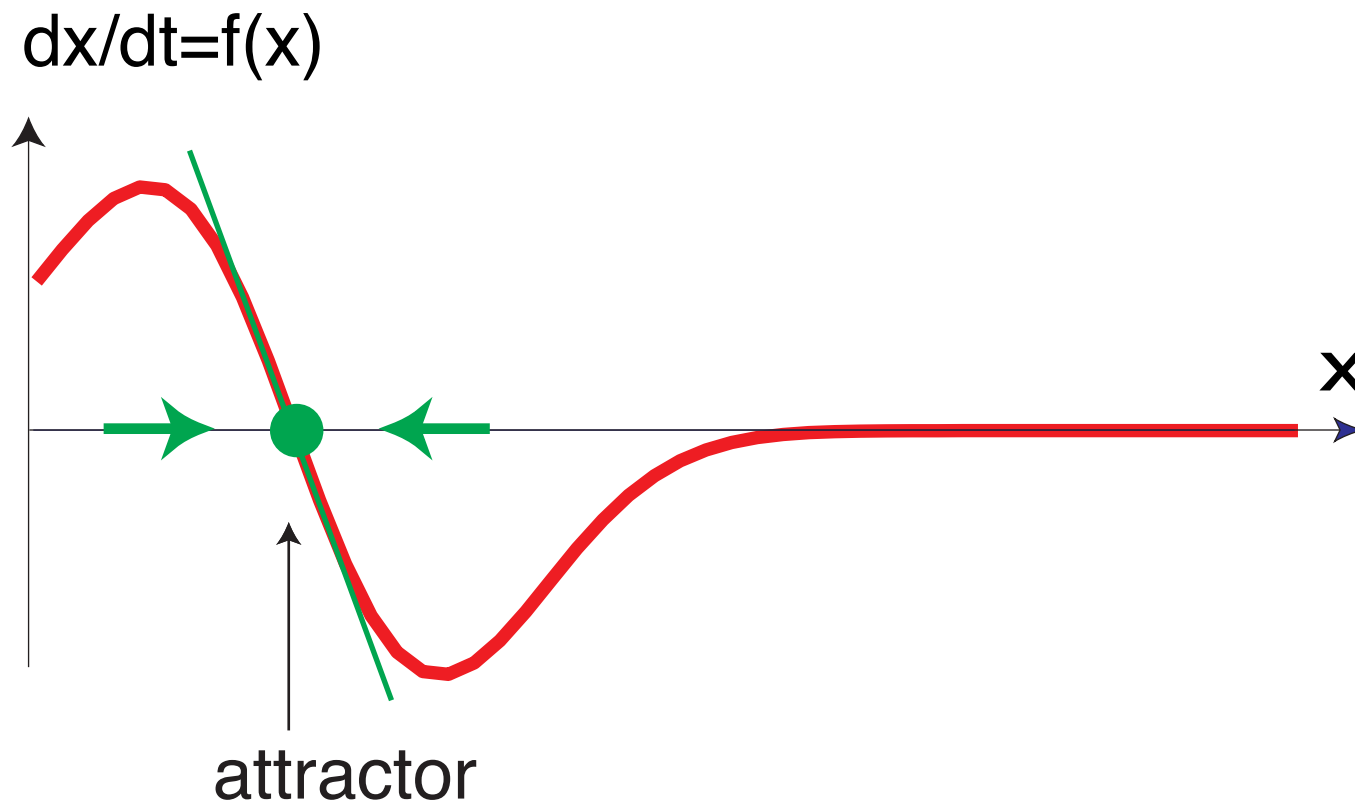
# Dynamical systems tutorial:

## 3. Attractors and instabilities

Gregor Schöner, INI, RUB

# attractor

- **fixed point**, to which neighboring initial conditions converge = **attractor**



# fixed point

■ is a constant solution of the dynamical system

■ 
$$\dot{x} = f(x)$$

$$\dot{x} = 0 \Rightarrow f(x_0) = 0$$

# stability

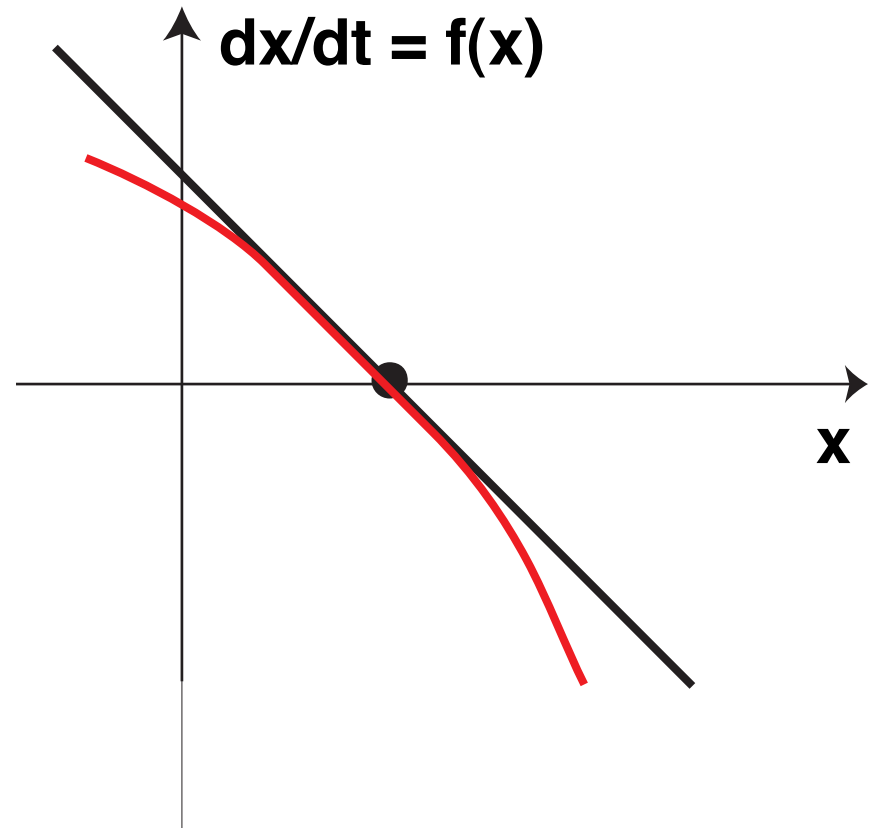
- mathematically really: **asymptotic stability**
- defined: a fixed point is asymptotically stable, when solutions of the dynamical system that start nearby converge in time to the fixed point

# stability

- the mathematical concept of stability (which we do not use) requires only that nearby solutions stay nearby
- Definition: a fixed point is **unstable** if it is not stable in that more general sense,
  - that is: if nearby solutions do not necessarily stay nearby (may diverge)

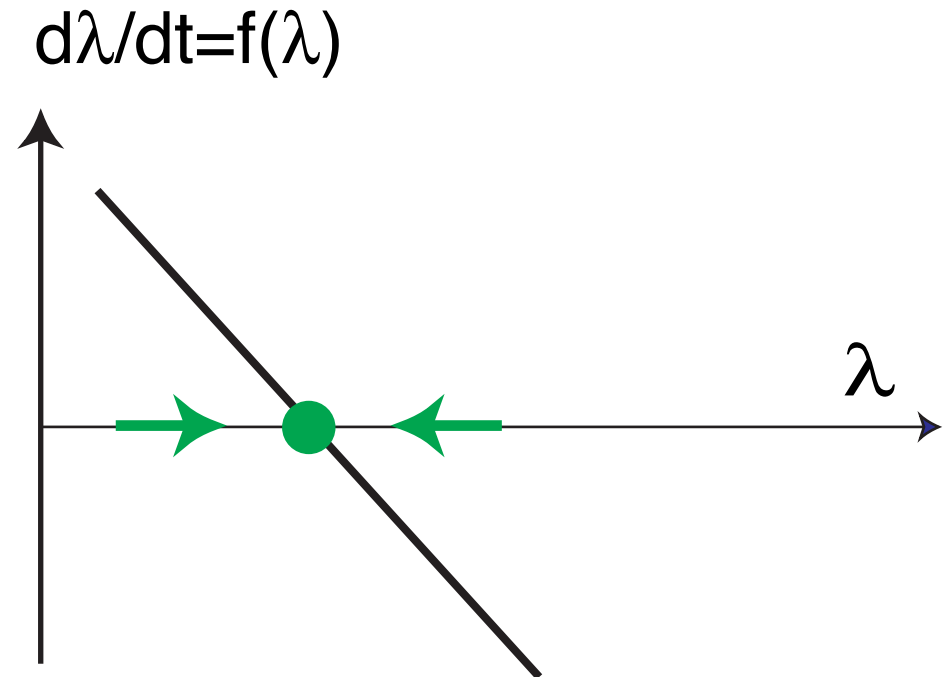
# linear approximation near attractor

- non-linearity as a small perturbation/  
deformation of linear system
- $\Rightarrow$  non-essential non-linearity



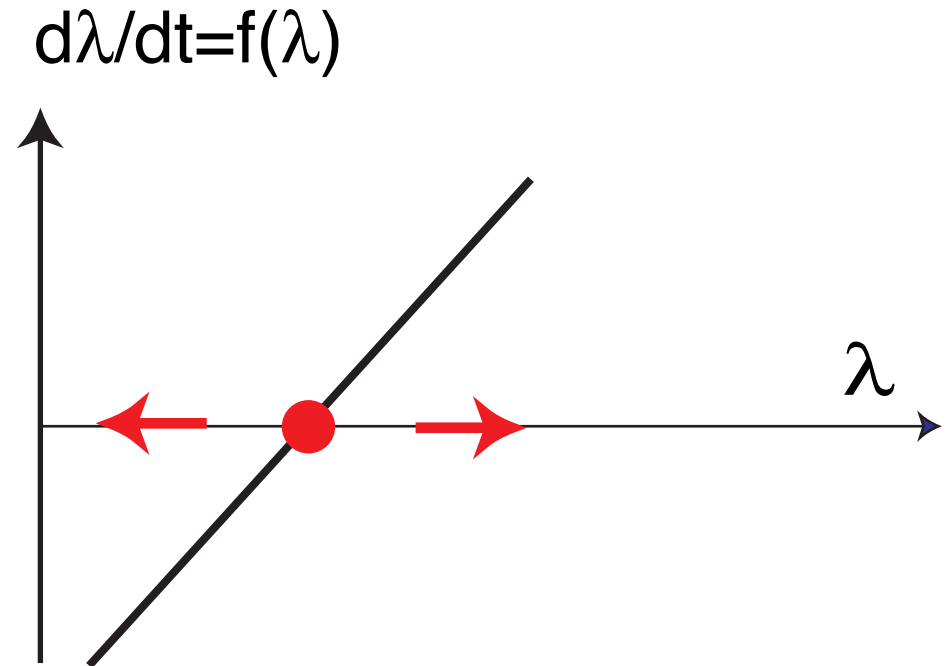
# stability in a linear system

- if the slope of the linear system is negative, the fixed point is (asymptotically stable)



# stability in a linear system

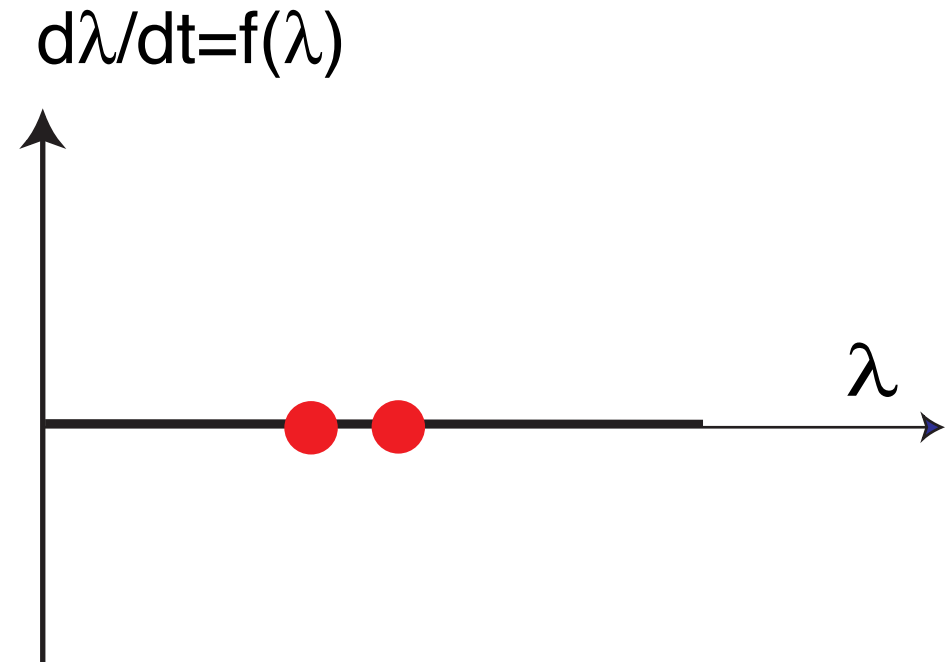
- if the slope of the linear system is positive, then the fixed point is unstable





# stability in a linear system

- if the slope of the linear system is zero, then the system is indifferent (marginally stable: stable but not asymptotically stable)



# stability in linear systems

## ■ generalization to multiple dimensions

- if the real-parts of all Eigenvalues are negative: stable
- if the real-part of any Eigenvalue is positive: unstable
- if the real-part of any Eigenvalue is zero: marginally stable in that direction (stability depends on other eigenvalues)

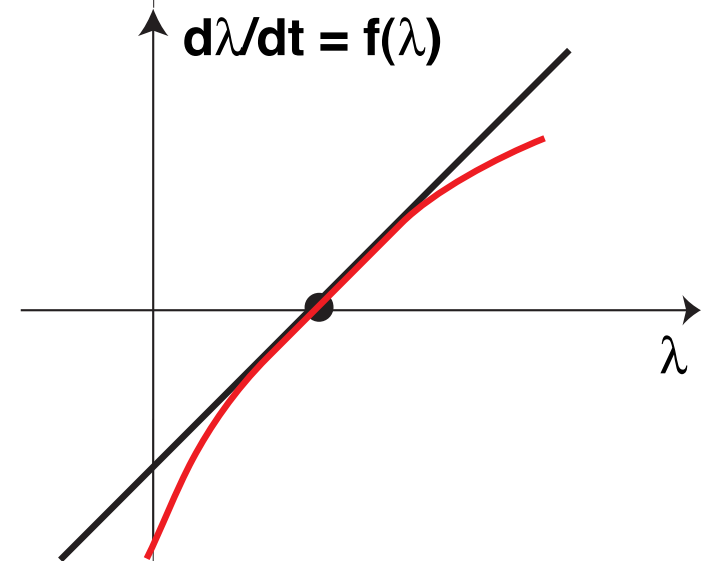
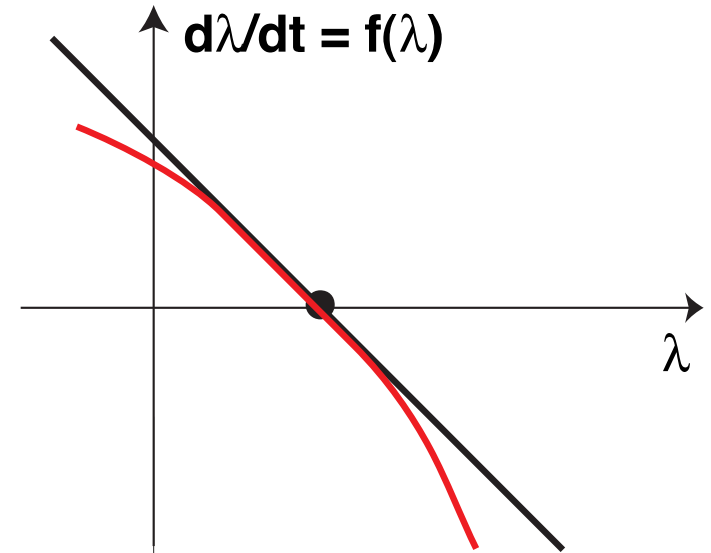
# stability in nonlinear systems

- stability is a local property of the fixed point
- $\Rightarrow$  linear stability theory
  - the eigenvalues of the linearization around the fixed point determine stability
  - all real-parts negative: stable
  - any real-part positive: unstable
  - any real-part zero: undecided: now nonlinearity decides (non-hyperbolic fixed point)

# stability in nonlinear systems

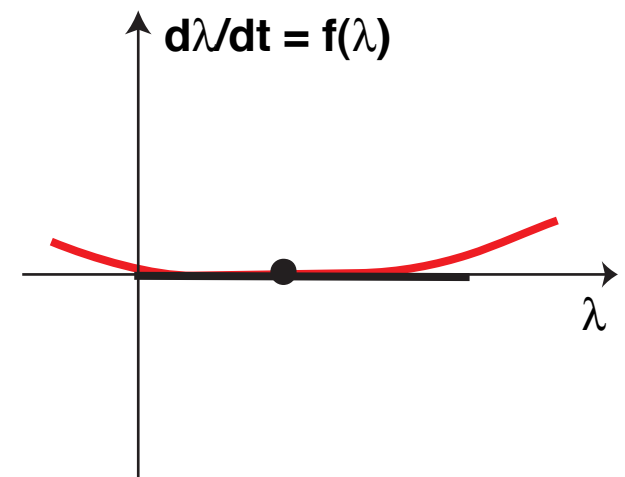
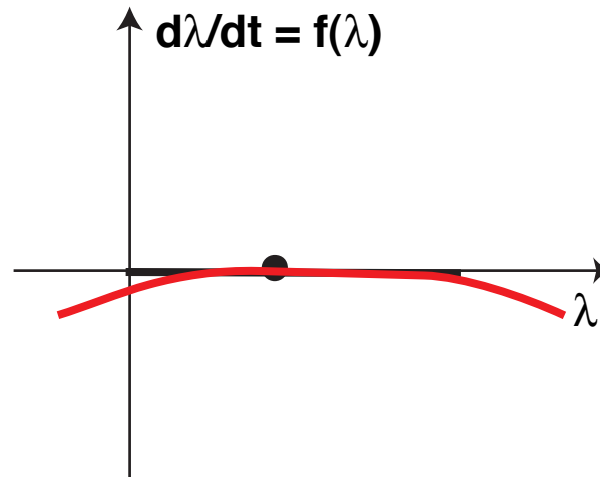
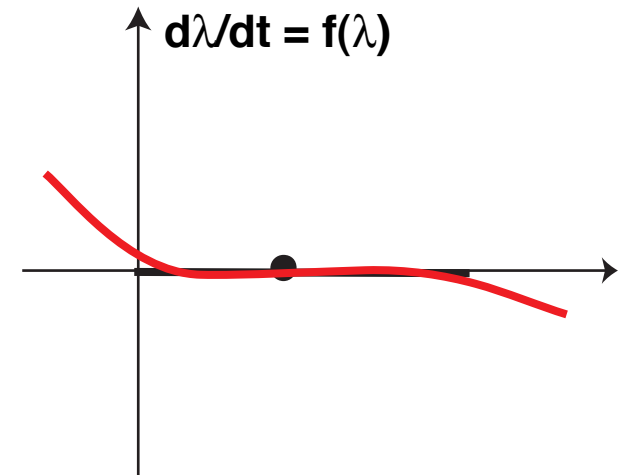
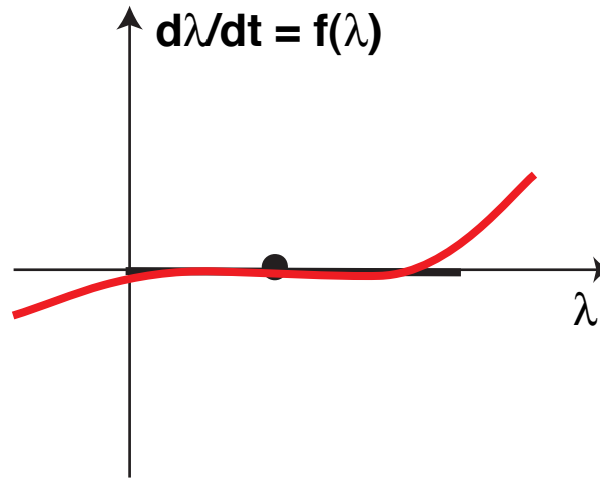
■ all real-parts negative: stable

■ any real-part positive: unstable



# stability in nonlinear systems

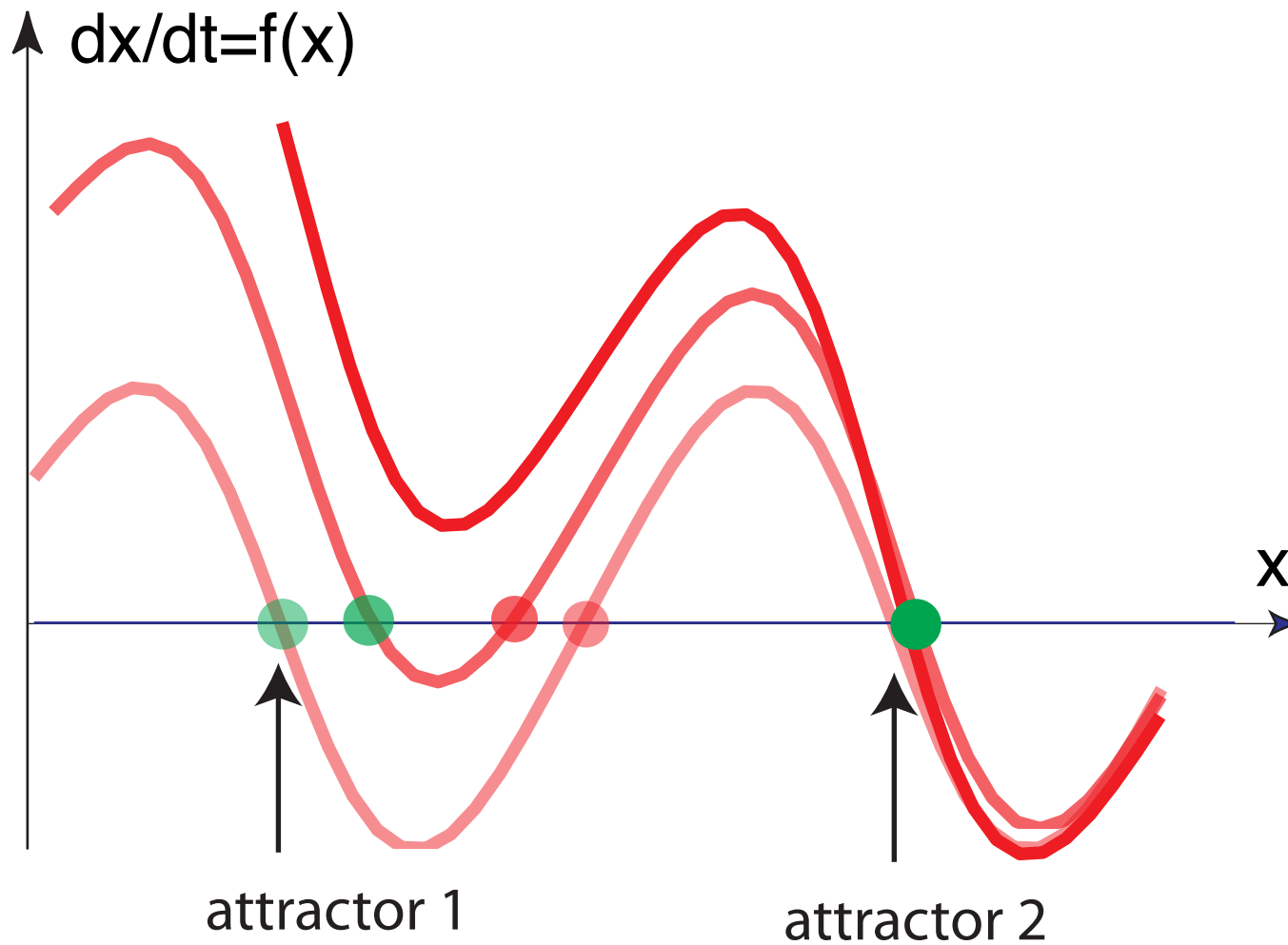
any real-part zero:  
undecided: now  
nonlinearity decides  
(non-hyperbolic fixed  
point)



# bifurcations

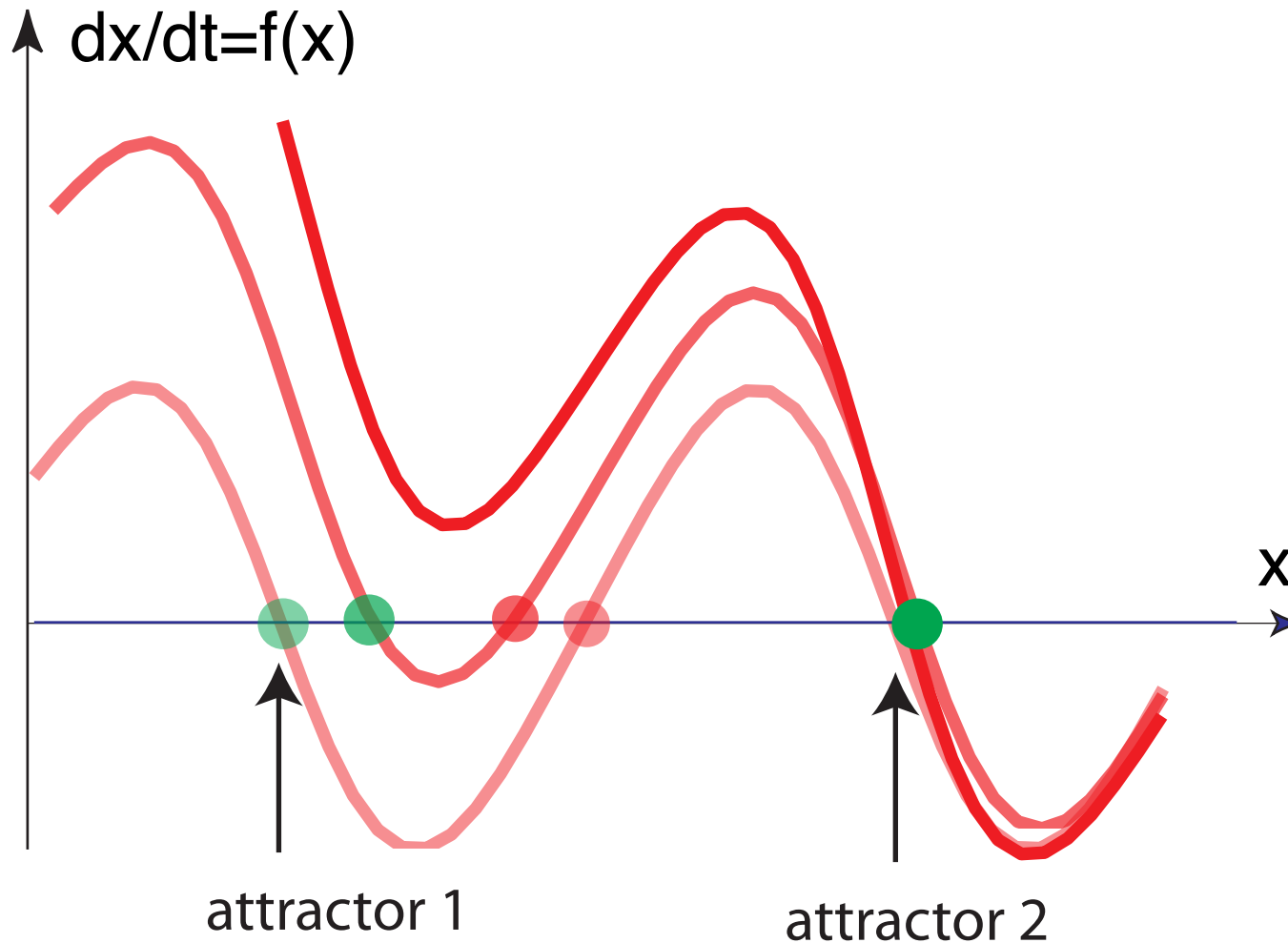
- look now at families of dynamical systems, which depend (smoothly) on parameters
- ask: as the parameters change (smoothly), how do the solutions change (smoothly?)
  - smoothly: topological equivalence of the dynamical systems at neighboring parameter values
  - bifurcation: dynamical systems NOT topological equivalent as parameter changes infinitesimally

# bifurcation



# bifurcation

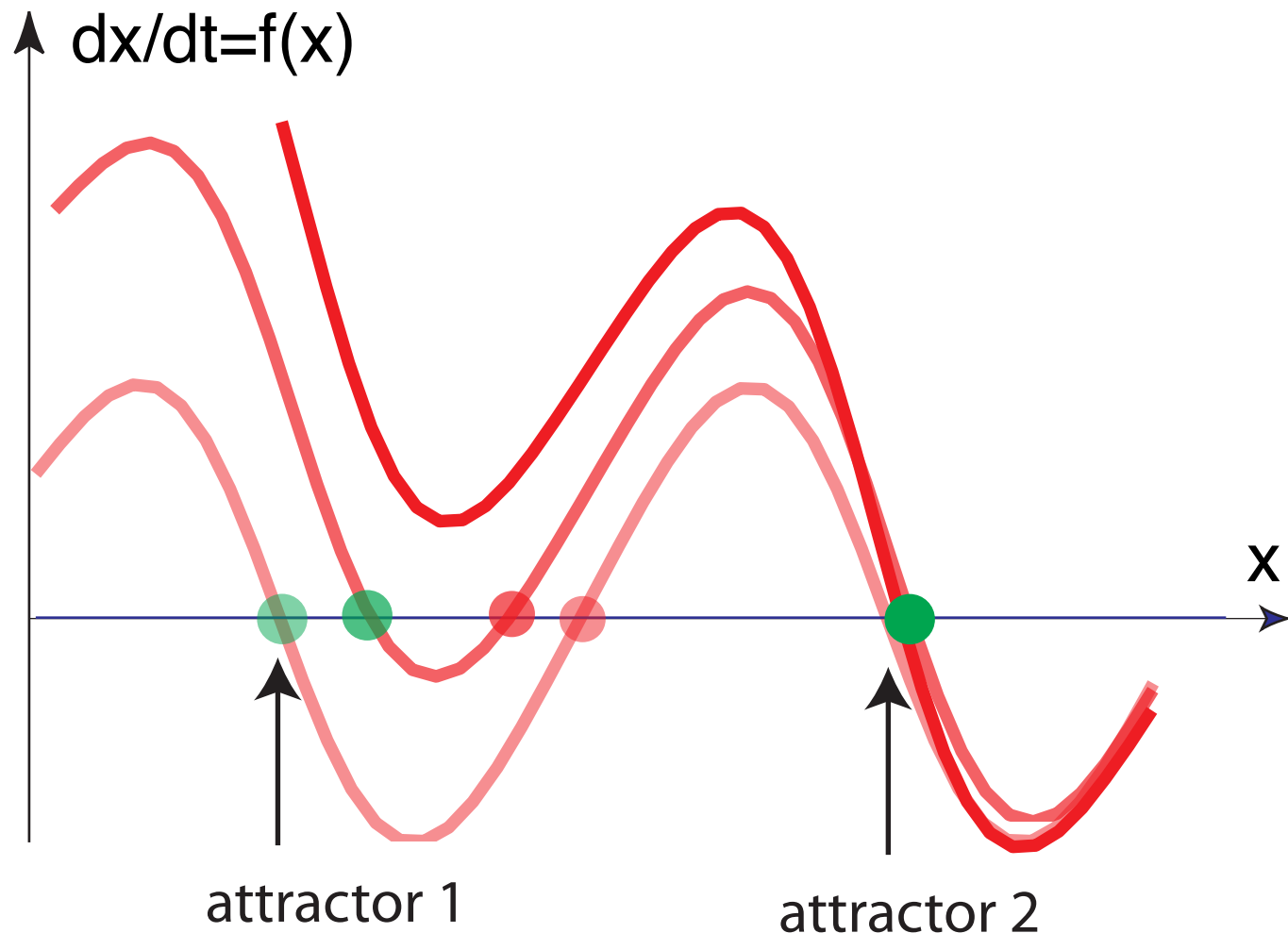
- bifurcation=qualitative change of dynamics (change in number, nature, or stability of fixed points) as the dynamics changes smoothly



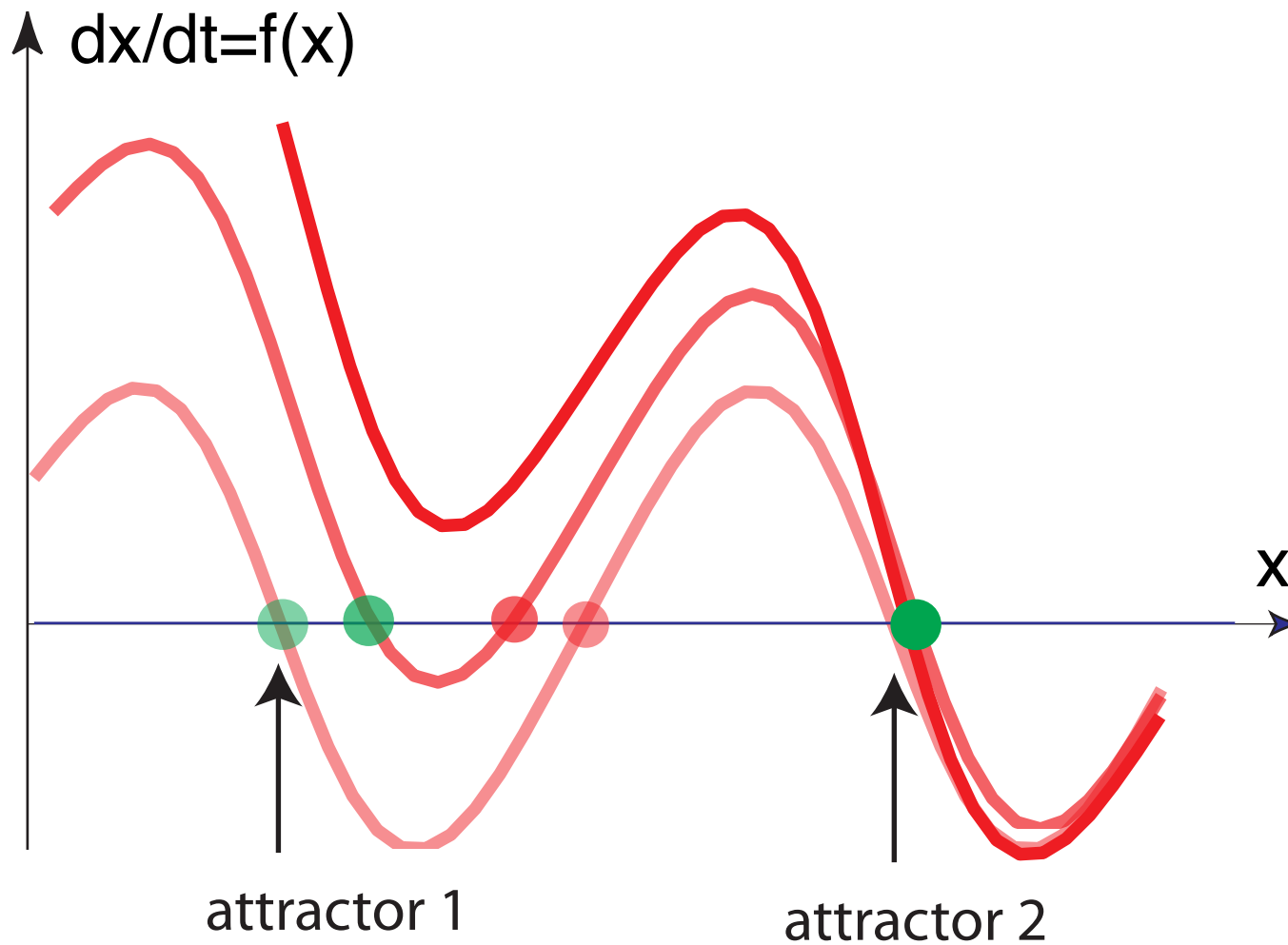


# tangent bifurcation

- the simplest bifurcation (co-dimension 0): an attractor collides with a repeller and the two annihilate

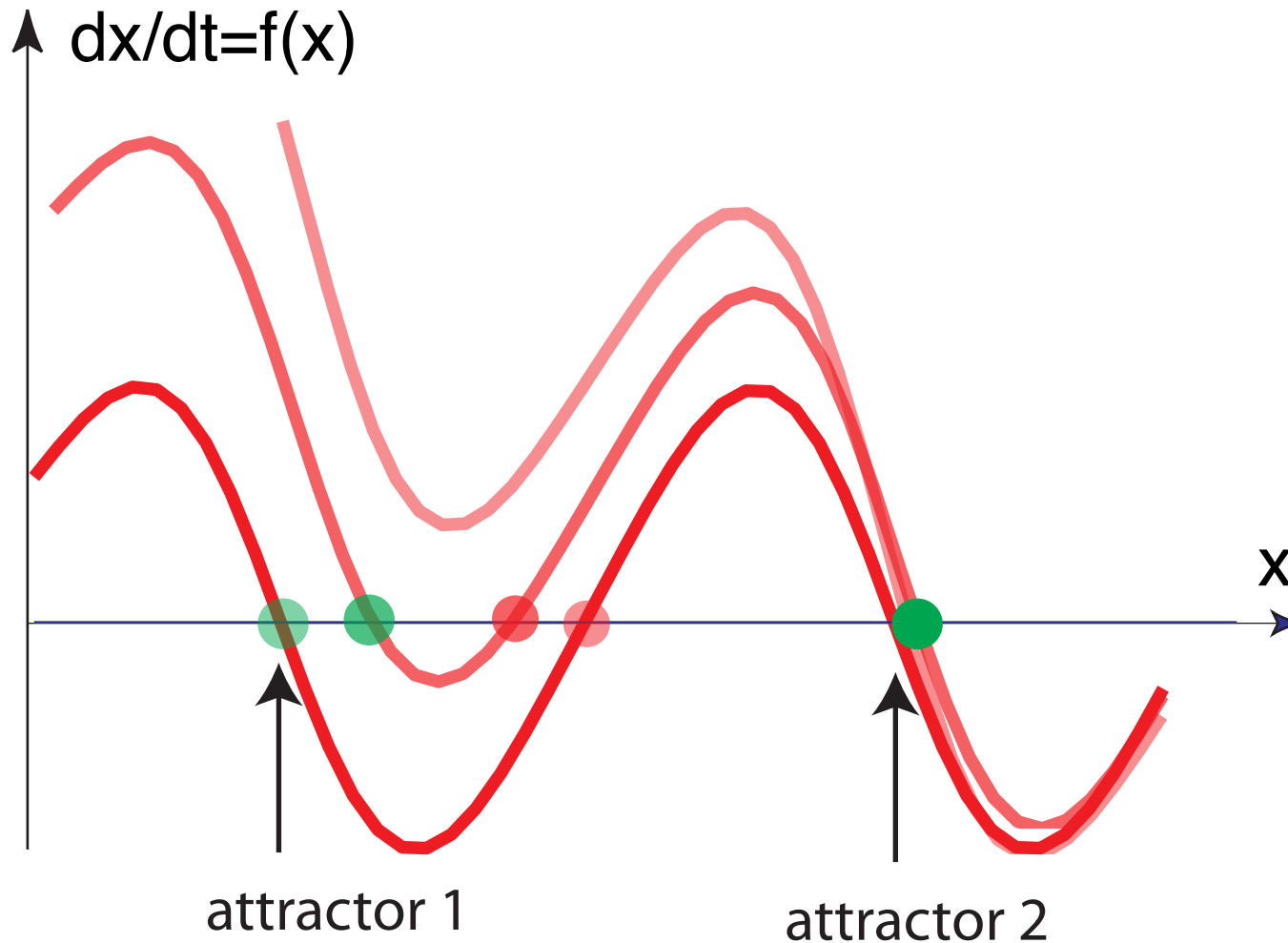


# local bifurcation



# reverse bifurcation

■ changing the dynamics in the opposite direction



# bifurcations are instabilities

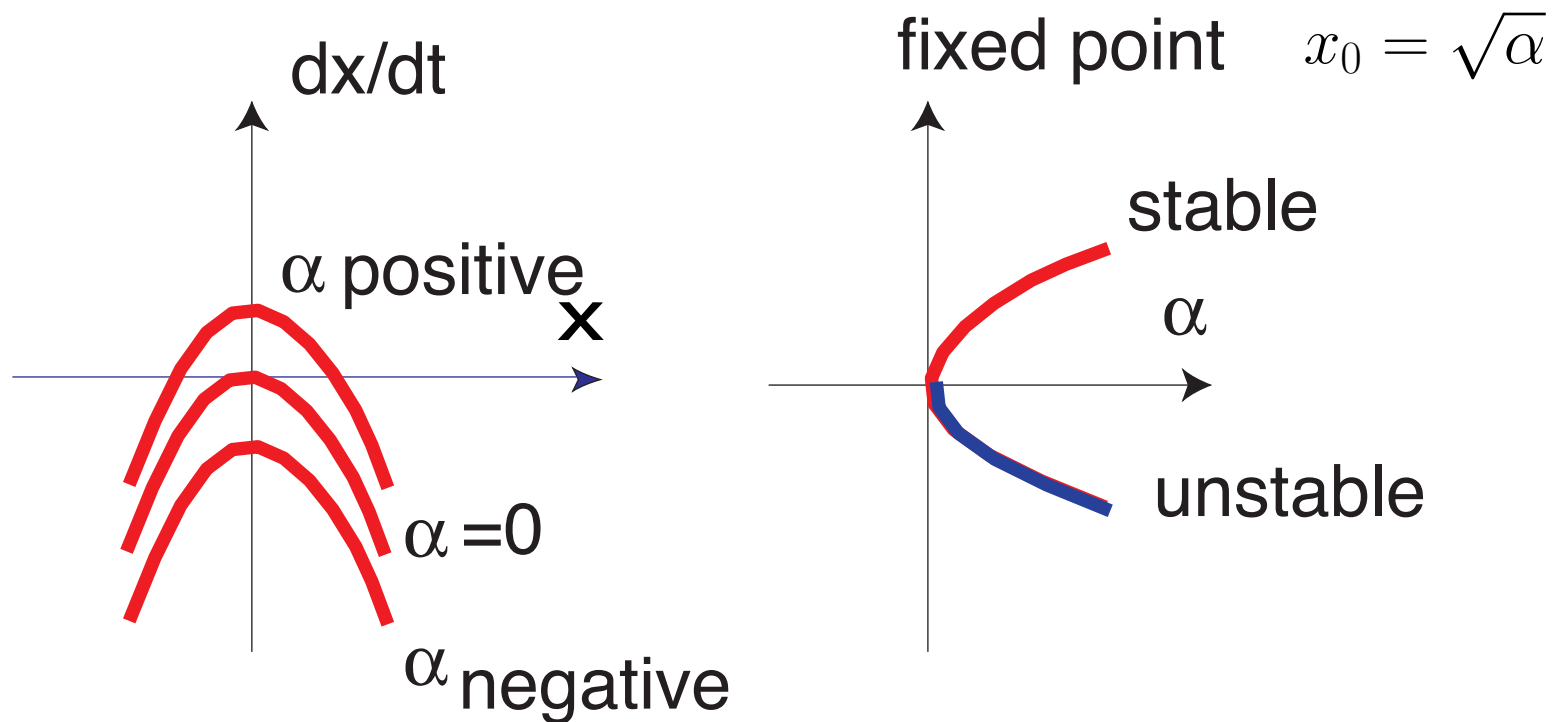
- that is, an attractor becomes unstable before disappearing
- (or the attractor appears with reduced stability)
- formally: a zero-real part is a necessary condition for a bifurcation to occur

# tangent bifurcation

- normal form of tangent bifurcation

$$\dot{x} = \alpha - x^2$$

- (=simplest polynomial equation whose flow is topologically equivalent to the bifurcation)



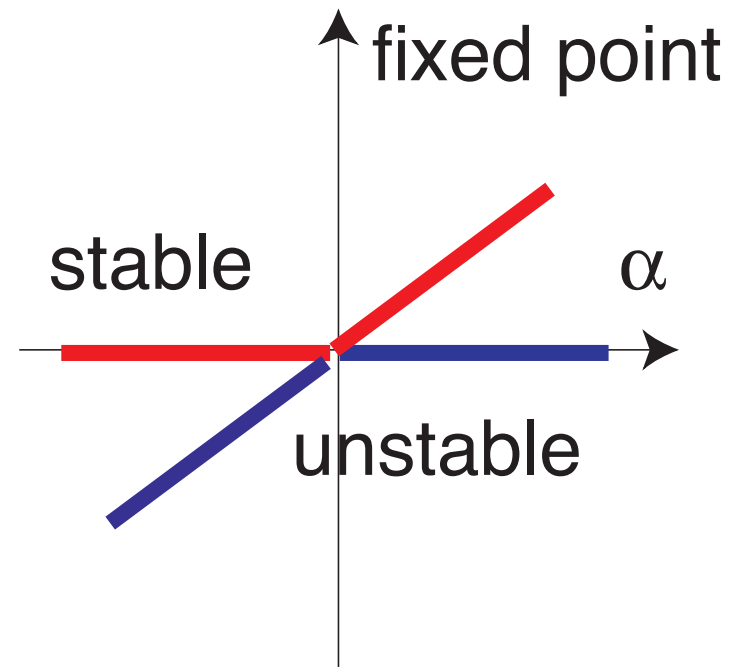
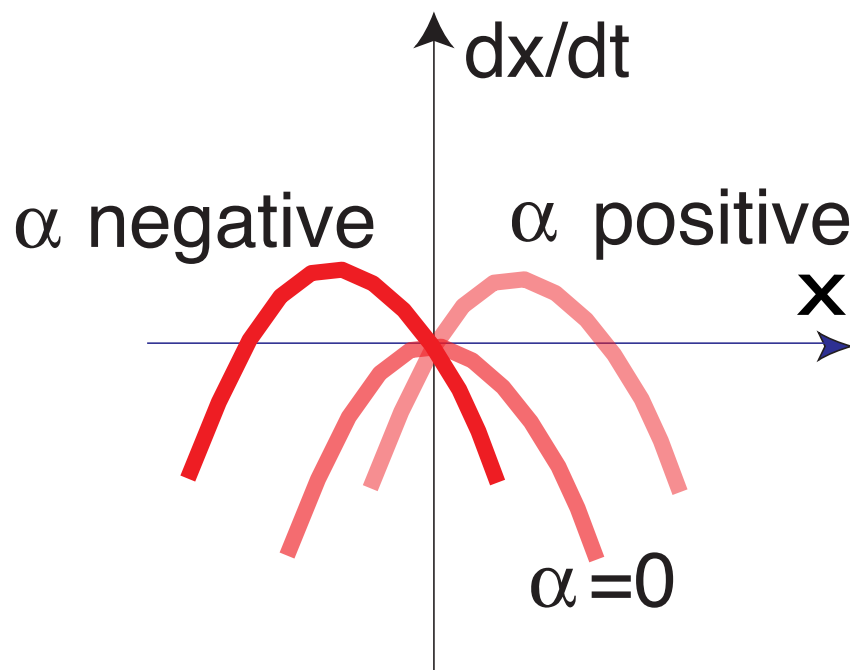
# Hopf theorem

- when a single (or pair of complex conjugate) eigenvalue crosses the imaginary axis, one of four bifurcations occur
  - tangent bifurcation
  - transcritical bifurcation
  - pitchfork bifurcation
  - Hopf bifurcation

# transcritical bifurcation

■ normal form

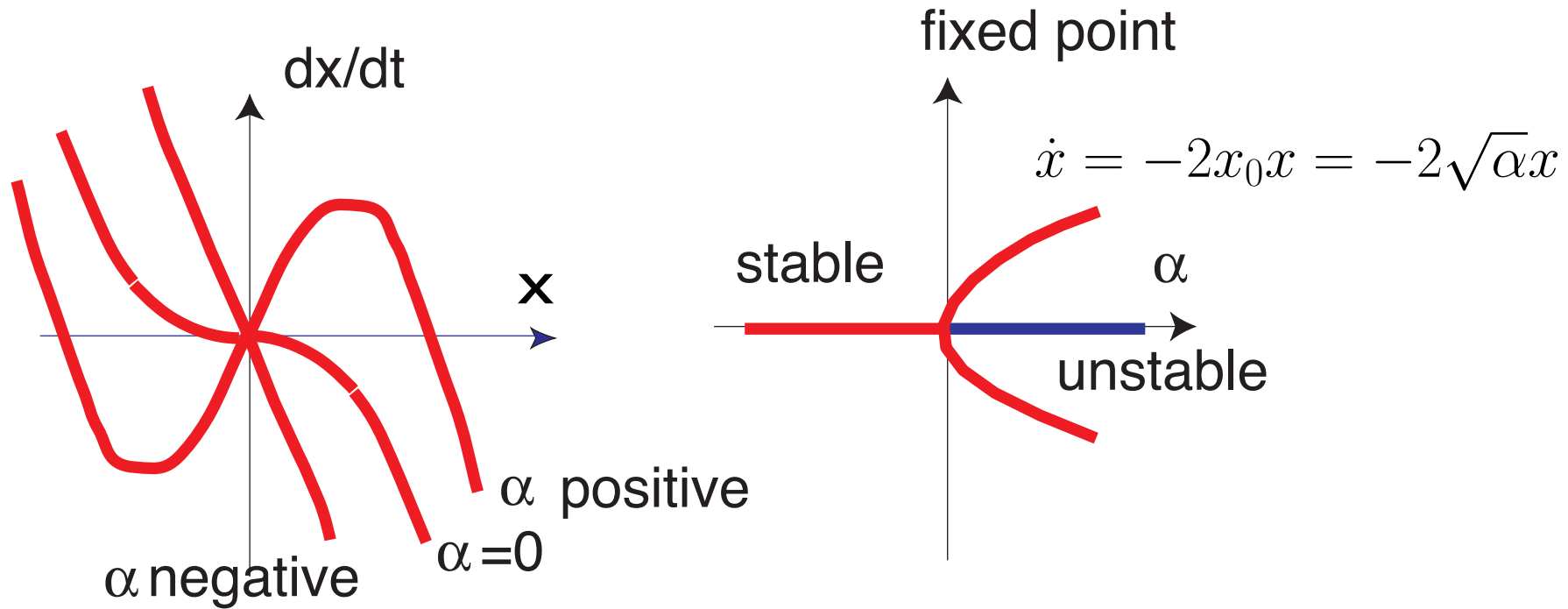
$$\dot{x} = \alpha x - x^2$$



# pitchfork bifurcation

■ normal form

$$\dot{x} = \alpha x - x^3$$

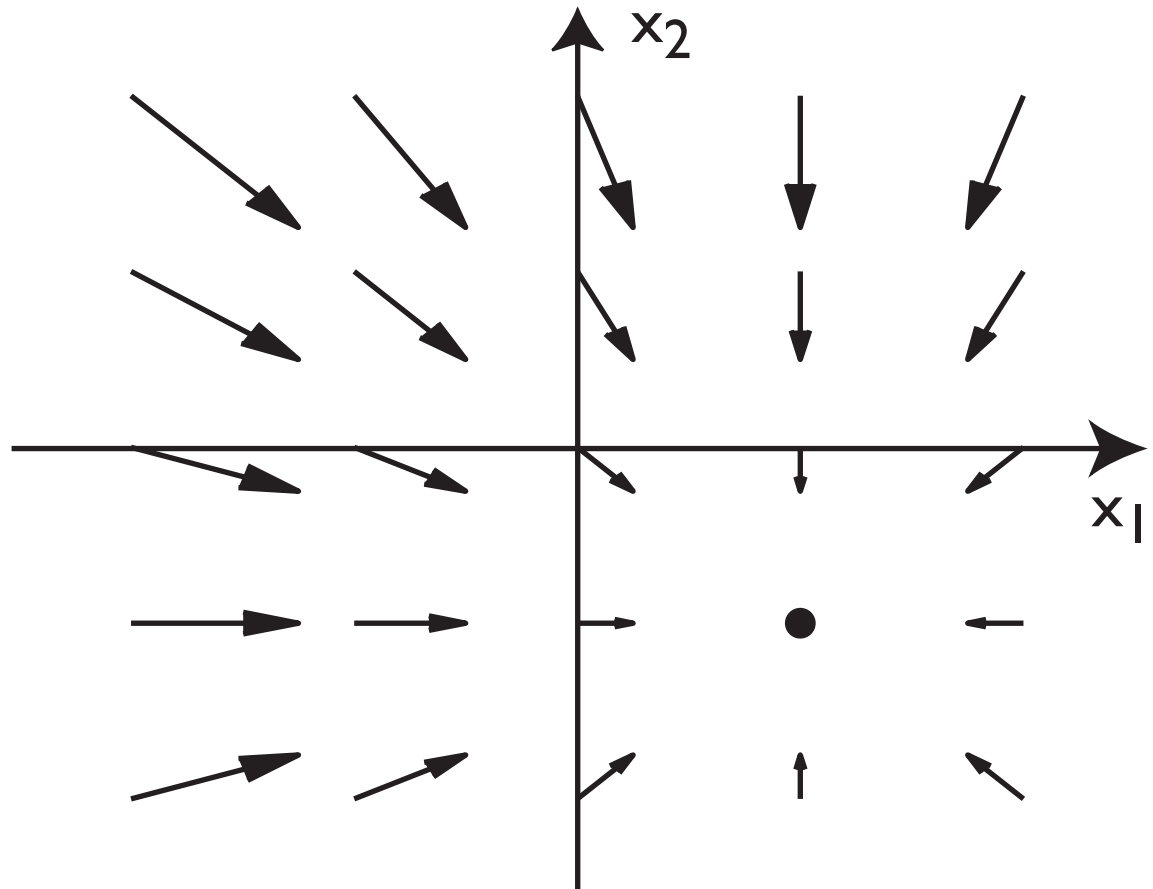




Hopf: need higher dimensions

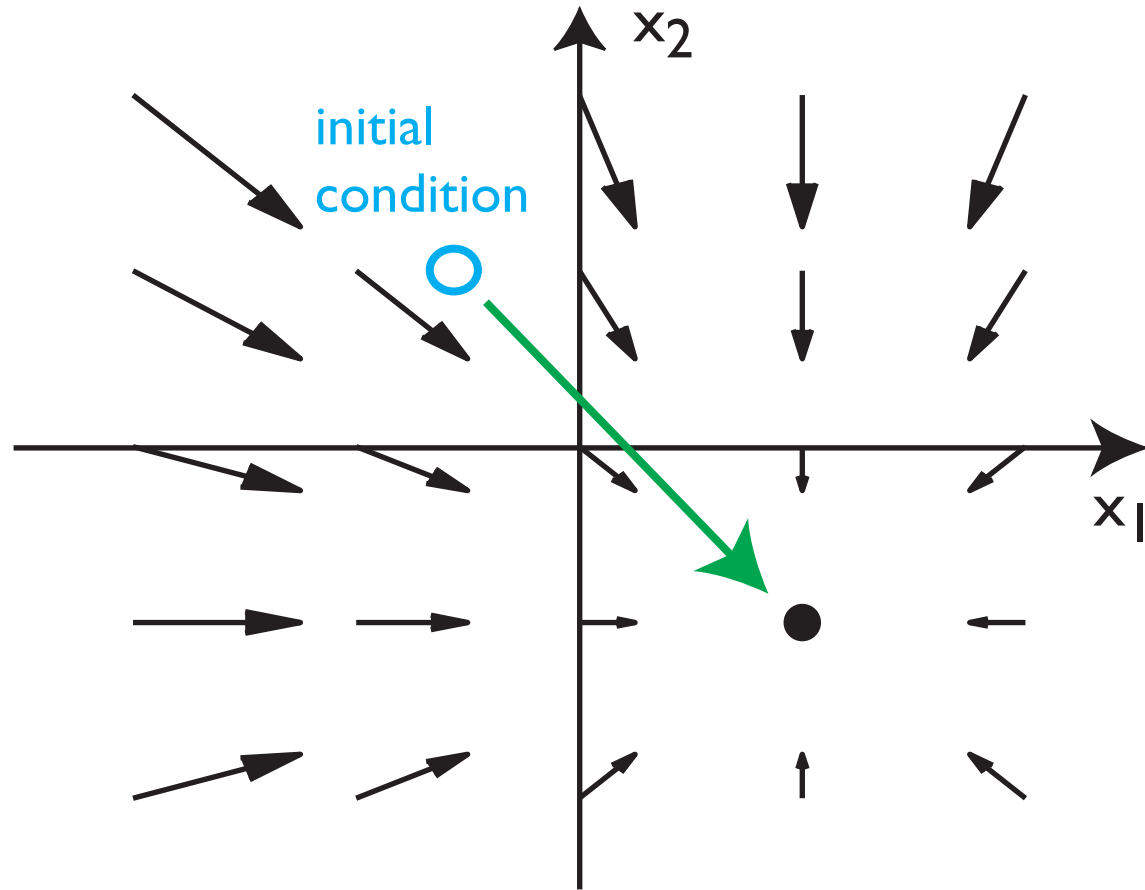
# 2D dynamical system: vector-field

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$



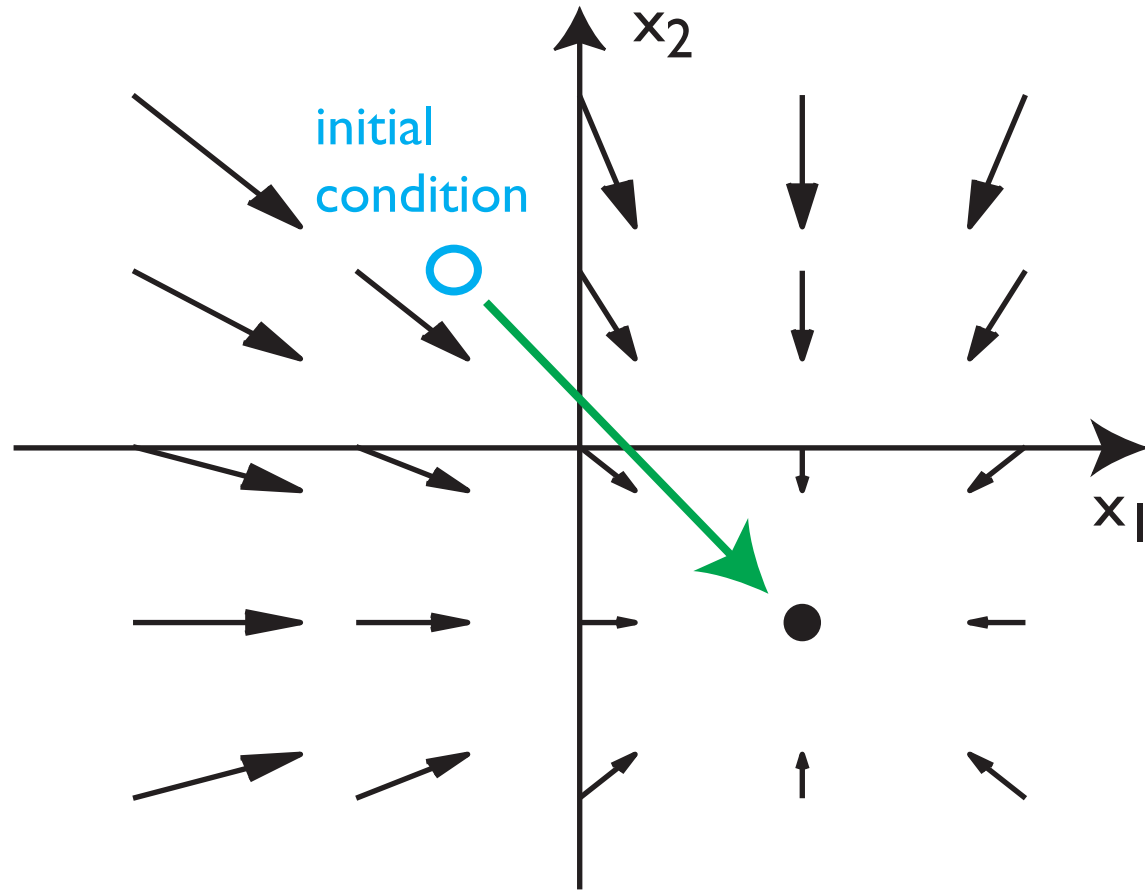
# vector-field

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$



# fixed point, stability, attractor

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

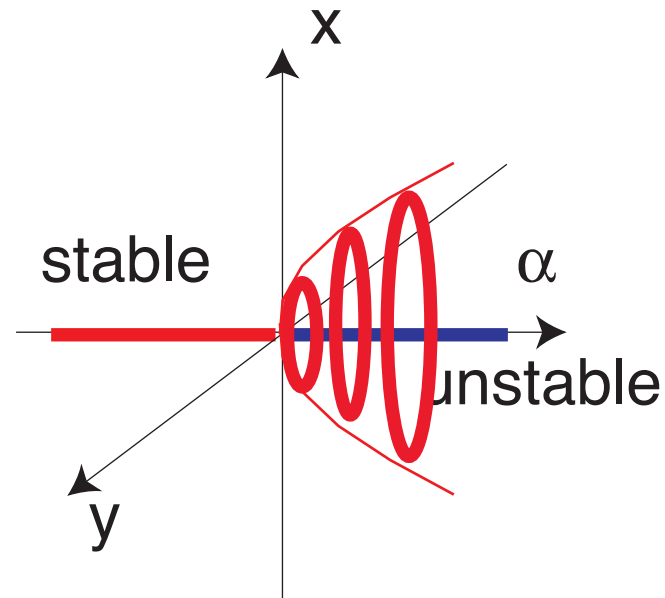
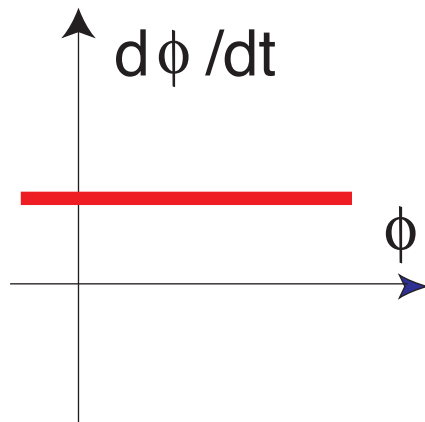
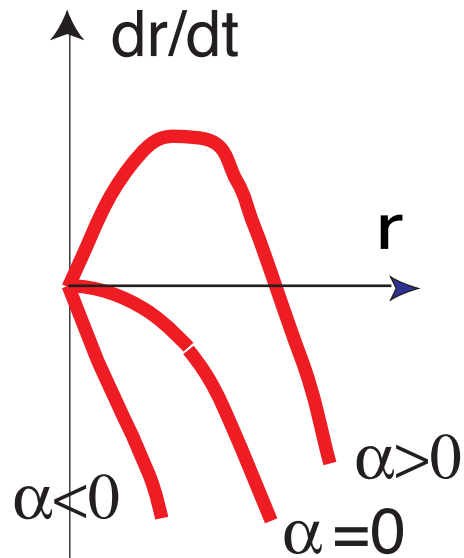


# Hopf bifurcation

$$\dot{r} = \alpha r - r^3$$

$$\dot{\phi} = \omega$$

■ normal form



# forward dynamics

- given known equation, determined fixed points / limit cycles and their stability
- more generally: determine invariant solutions (stable, unstable and center manifolds)

# inverse dynamics

- given solution, find the equation...
- this is the problem faced in design of behavioral dynamics...

# inverse dynamics: design

- in the design of behavioral dynamics... you may be given:
- attractor solutions/stable states
- and how they change as a function of parameters/conditions
- => identify the class of dynamical systems using the 4 elementary bifurcations
- and use normal form to provide an exemplary representative of the equivalence class of dynamics