Dynamical systems tutorial:
1. Basic concepts

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Dynamical systems: Tutorial

- the word “dynamics”
  - time-varying measures
  - range of a quantity
  - forces causing/accounting for movement => dynamical systems

- dynamical systems are the universal language of science
  - physics, engineering, chemistry, theoretical biology, economics, quantitative sociology, ...
time-variation and rate of change

• variable $x(t)$;

• rate of change $\frac{dx}{dt}$
time-variation and rate of change

Example:
- variable $x(t) = \text{position}$
- rate of change $\frac{dx}{dt} = \text{velocity}$

Example
- variable $v(t) = \text{velocity}$
- rate of change $\text{?}$
time-variation and rate of change

- trajectory: time course of a dynamical variable
- rate of change: slope of the trajectory
dynamical system: relationship between a variable and its rate of change
linear dynamical system

\[ \frac{dx}{dt} = f(x) \]
solution of linear dynamical systems

\[ \tau \frac{dx}{dt} = -x \]

\[ x(t) = x(0) \exp\left[-\frac{t}{\tau}\right] \]
exponential relaxation to attractors

\[ x(t) = x(0)\exp[-t/\tau] \]

\[ \tau \frac{dx}{dt} = -x \]

\[ \tau \dot{x}(t) = -x(0)\exp[-t/\tau] \]

\[ \tau \ddot{x}(t) = -x(t) \]
exponential relaxation to attractors

\[ x(t) = x(0) \exp\left[ -\frac{t}{\tau} \right] \]

\[ \tau \frac{dx}{dt} = -x \]
(non-linear) dynamical system

\[ \frac{dx}{dt} = f(x) \]
Defining property of dynamical systems

- present determines the future:
  - given an initial condition, predict future time course (or postdict the past)

\[ \frac{dx}{dt} = f(x) \]
Concepts dynamical systems

- $x$: spans the state space (or phase space)
- $f(x)$: is the “dynamics” of $x$ (or vector-field)
- $x(t)$ is a solution of the dynamical systems to the initial condition $x_0$
  - if its rate of change = $f(x)$
  - and $x(0)=x_0$
Different forms of dynamical systems

One-dimensional differential equation: initial value determines the future

\[ \dot{x} = f(x) \]
Different forms of dynamical systems

- vector-valued differential equation
- a vector of initial states determines the future:
  systems of differential equations:

\[ \dot{x} = f(x) \quad \text{where} \quad x = (x_1, x_2, \ldots, x_n) \]
Different forms of dynamical systems

- Continuously many variables $x(y)$ determine the future = an initial function $x(y)$ determines the future
- Partial differential equations
- Integro-differential equations

\[
\dot{x}(y, t) = f \left( x(y, t), \frac{\partial x(y, t)}{\partial y}, \ldots \right)
\]
\[
\dot{x}(y, t) = \int dy' g \left( x(y, t), x(y', t) \right)
\]
Different forms of dynamical systems

- A piece of past trajectory determines the future
- Delay differential equations
- Functional differential equations

\[ \dot{x}(t) = f(x(t - \tau)) \]

\[ \dot{x}(t) = \int_{t'}^{t} dt' f(x(t')) \]