Lecture 6
Differential Equations

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Computer Science and Mathematics
Preparatory Course

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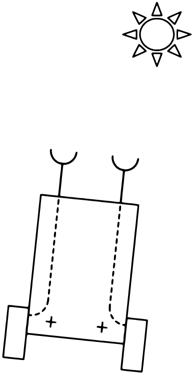
Overview

1. **Motivation**
2. **Mathematics**
   - Solving Differential Equations
   - Qualitative Analysis
   - Numerical Approximation
3. **Tasks**

The Vehicle's Behavior as Function of Angle Change



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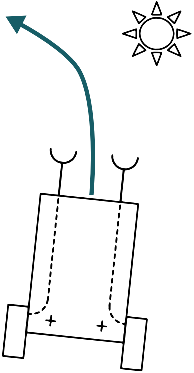

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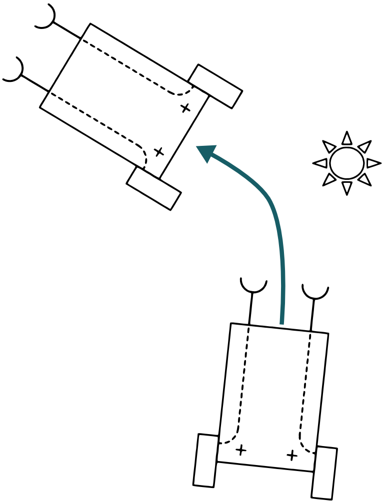

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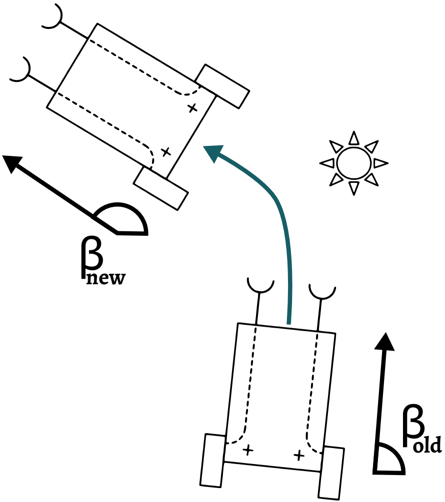

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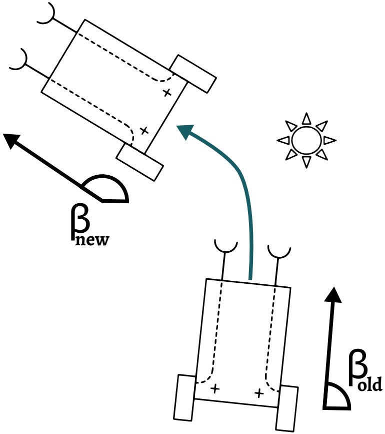

- The vehicle's change in angle depends on its current sensor input

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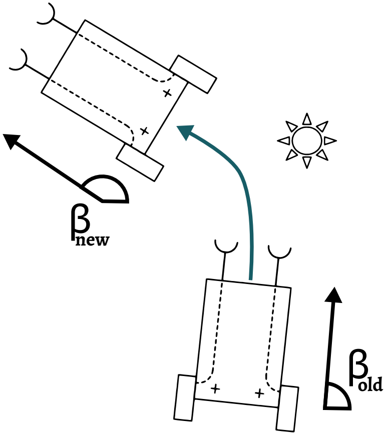

- The vehicle's change in angle depends on its current sensor input
- The following equation may describe its behavior

$$\frac{d\beta}{dt} = -S_L + S_R,$$

where t describes time and S_L, S_R left and right sensor values.

Differential Equation as Rule System

- A differential equation describes how the rate of change of a system depends on its current state. For example:

$$f'(x) = 4f(x) + 5 = g(f(x)) \quad \text{with} \quad g(x) = 4x + 5$$

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- A differential equation describes how a system should change in a given state.
- Brief oversimplification:

A differential equation describes rules for the future

Solving Differential Equations

- Given a differential equation of the form $f'(x) = g(f(x)) \ldots$ the original function $f(x)$ is usually not known.
- Solving a differential equation describes the process of finding an $f(x)$ that follows the above rule for all x
- Differential equations entail two equations
  1. The function $g(f(x))$ governing the rate of change
  2. The function $f(x)$ describing the overall behavior

Derivative vs. Differential equation

- $f'(x) = cx$

- $f'(x) = cf(x)$

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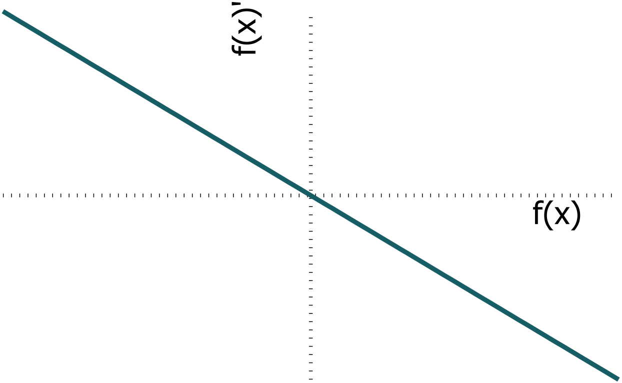
  - The only function that stays the same when differentiated is the exponential function e^x
  - Considering the chain rule the derivative of e^{cx} is exactly ce^{cx} therefore $f(x) = ce^{cx}$
  - Usually a differential equation is not that easily solvable

Dynamical Systems Theory

- Mathematicians want to find solutions to particular differential equations
- **Dynamical Systems Theory** is concerned with analyzing the qualitative behavior of the system

Qualitative Behavior of Differential Equations

$$f'(x) = y' = \frac{dy}{dx} = -f(x)$$



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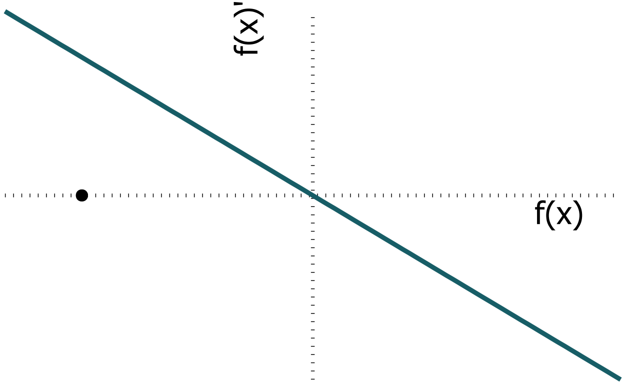

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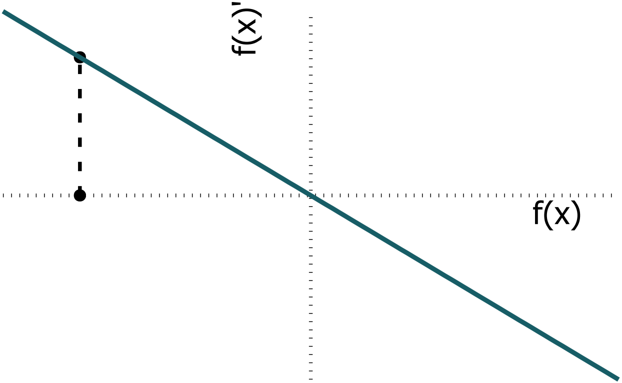

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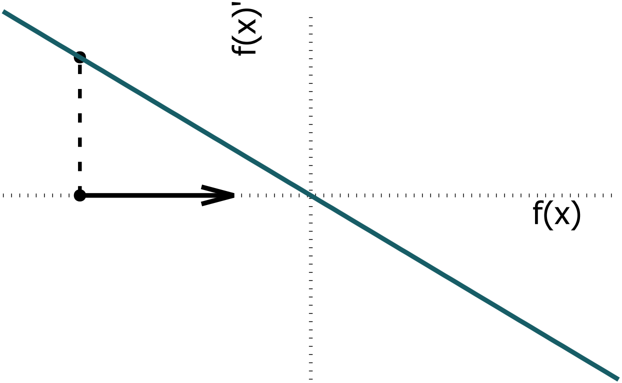

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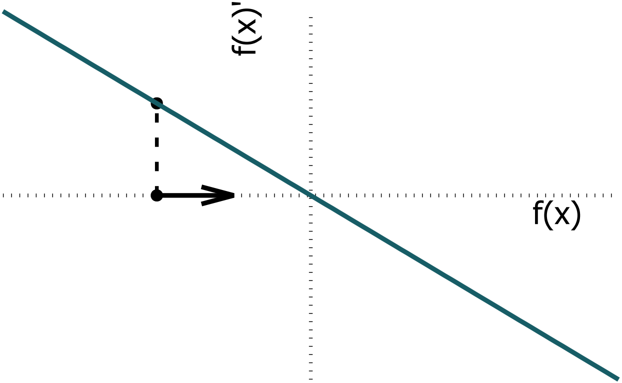

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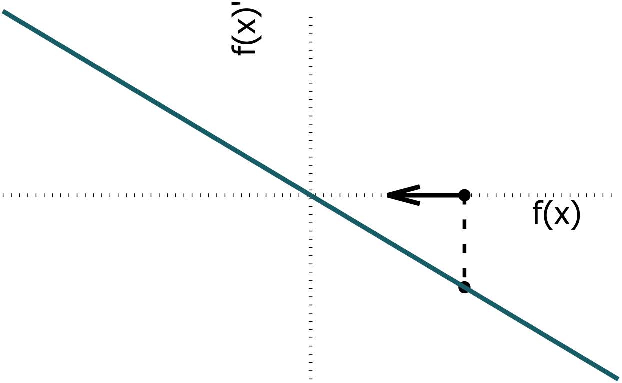

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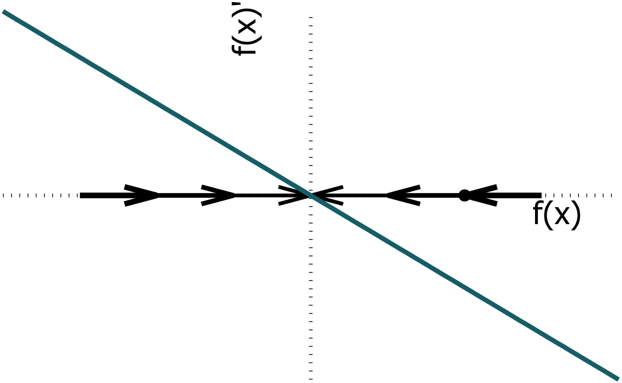
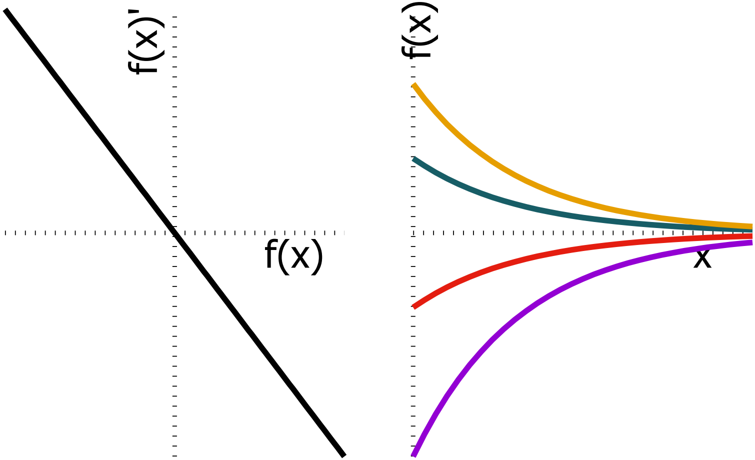


Attractors

$$f'(x) = y' = \frac{dy}{dx} = -f(x)$$



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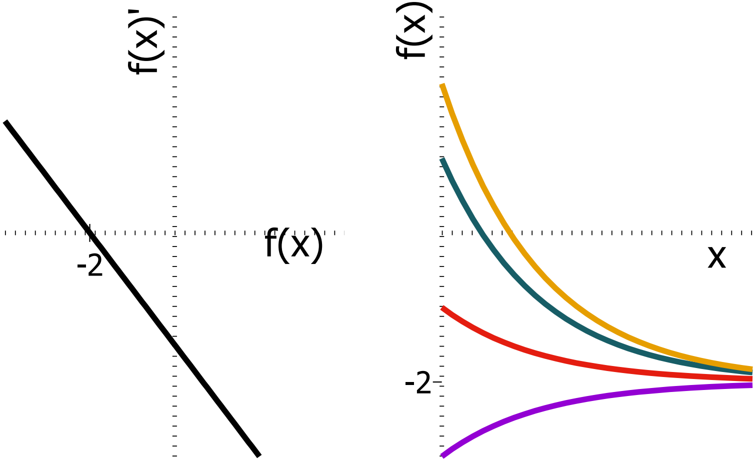

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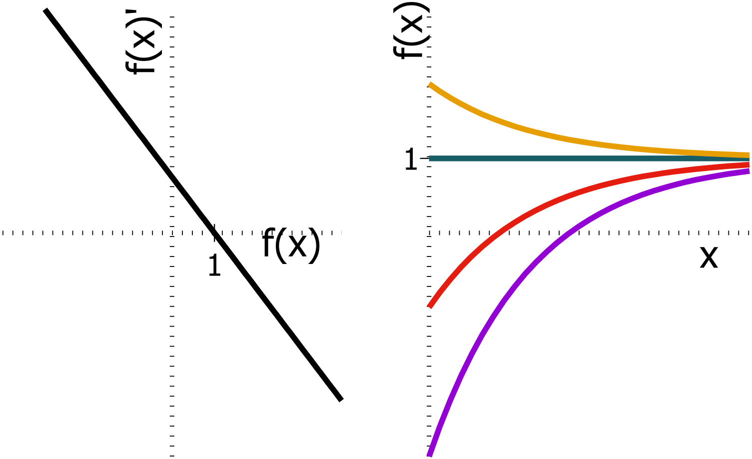
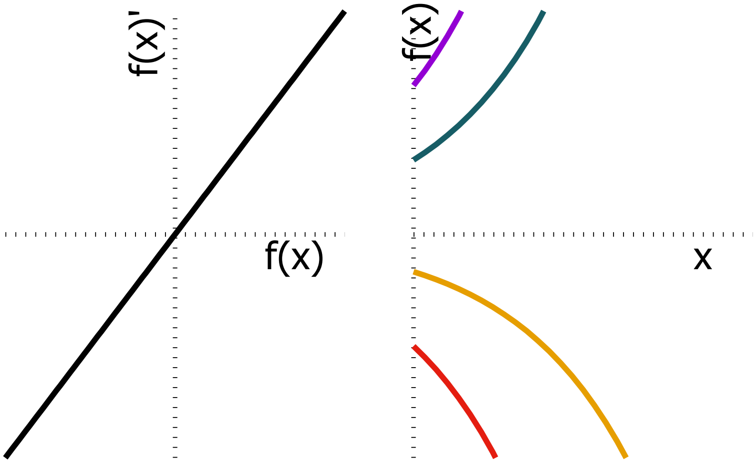


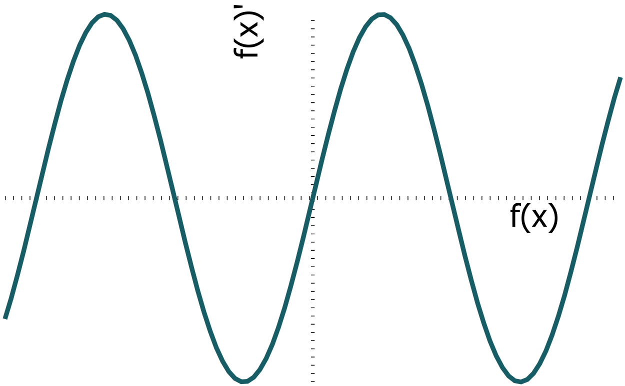
Repellors

$$f'(x) = y' = \frac{dy}{dx} = -f(x)$$



Initial Condition Matters

$$f'(x) = y' = \frac{dy}{dx} = sin(x)$$



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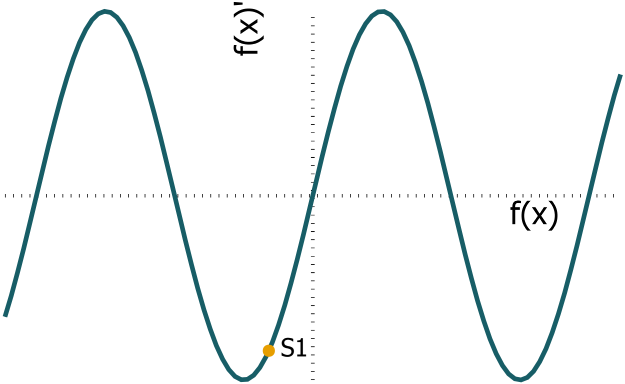

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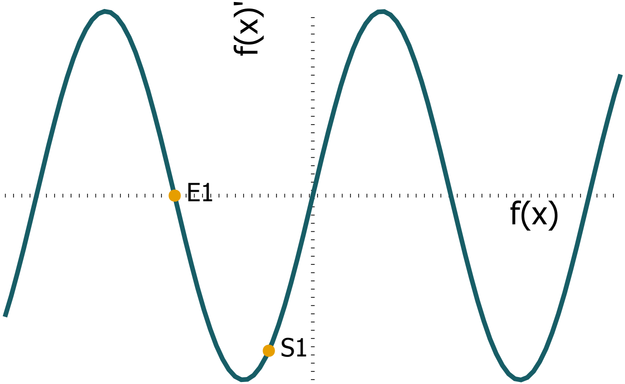

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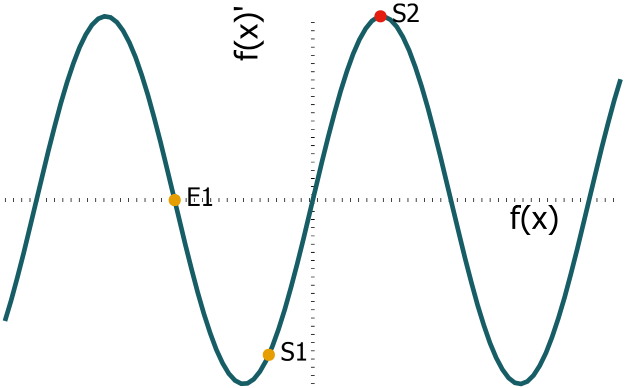

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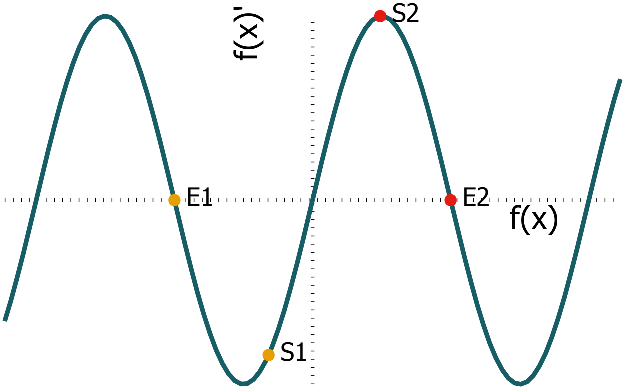

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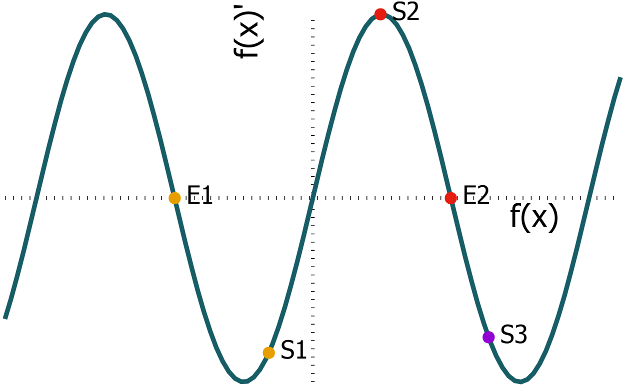

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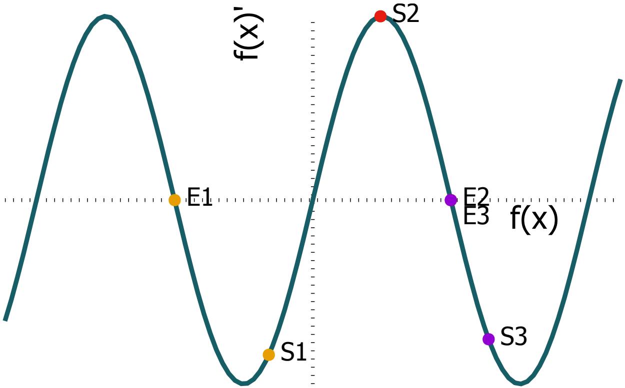
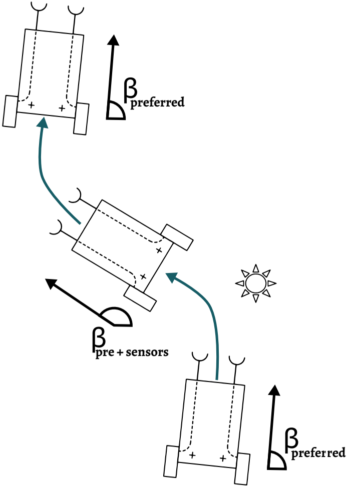


Back to the Braitenberg Vehicle



- We govern the vehicles behavior with a differential equation

$$\frac{d\beta}{dt} = -\beta - S_L + S_R,$$

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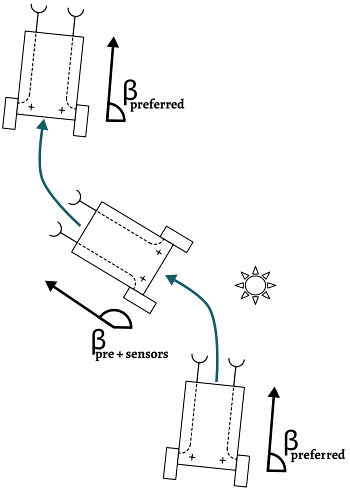

- We govern the vehicles behavior with a differential equation

$$\frac{d\beta}{dt} = -\beta - S_L + S_R,$$

- Adding an attractor gives the vehicle a preferred orientation

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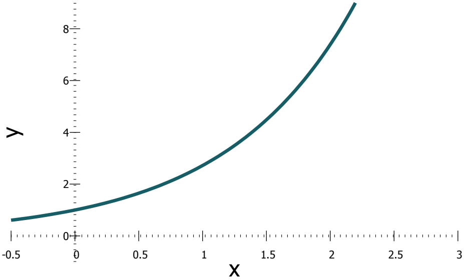
2. Mathematics
- Solving Differential Equations
- Qualitative Analysis
- Numerical Approximation

3. Tasks

Euler Approximation

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$y(x) = e^x$$



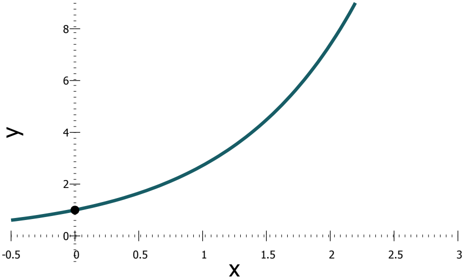
Euler Approximation

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$\Delta x = 1$$

x	y	$\frac{dy}{dx}$
0	1	1

$y(x) = e^x$

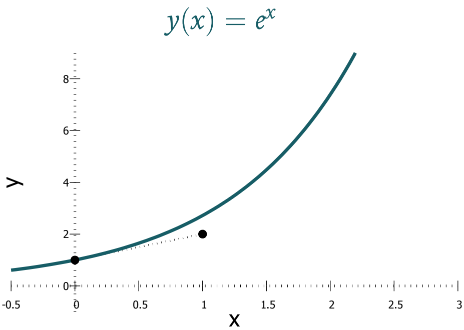


Euler Approximation

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$\Delta x = 1$$

x	y	$\frac{dy}{dx}$
0	1	1
1	2	2



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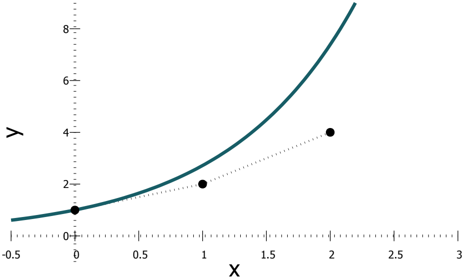
$$\Delta x = 1$$

x	y	$\frac{dy}{dx}$
0	1	1
1	2	2
2	4	4

$y(x) = e^x$



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$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$\Delta x = 1$$

x	y	$\frac{dy}{dx}$
0	1	1
1	2	2
2	4	4
3	8	8

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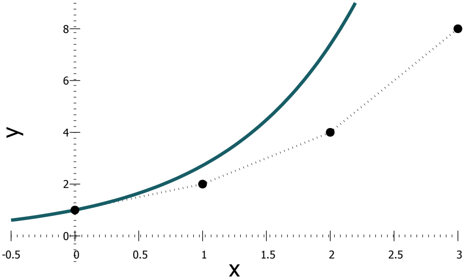
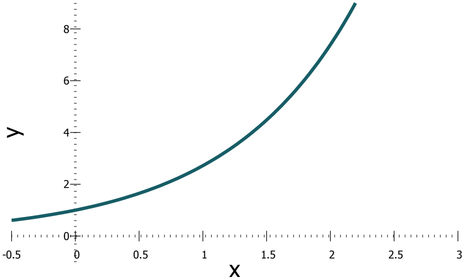


Varying the stepsize

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$y(x) = e^x$



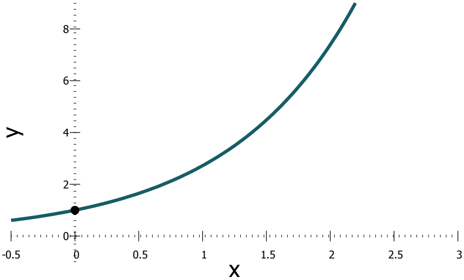
Varying the stepsize

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$\Delta x = 0.5$$

x	y	$\frac{dy}{dx}$
0	1	1

$y(x) = e^x$

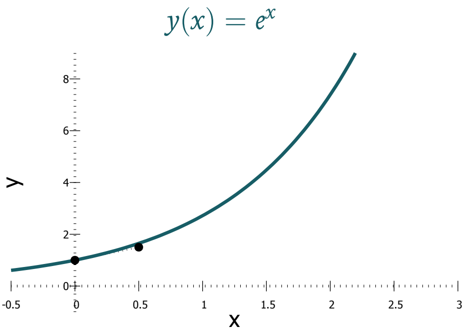


Varying the stepsize

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$\mathbf{\Delta x = 0.5}$$

x	y	$\frac{dy}{dx}$
0	1	1
0.5	1.5	1.5



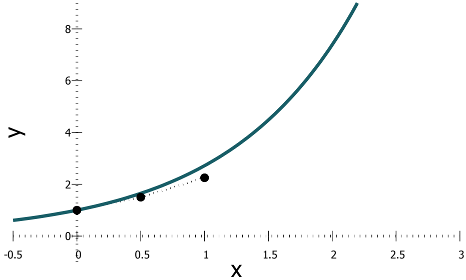
Varying the stepsize

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$\boldsymbol{\Delta x = 0.5}$$

x	y	$\frac{dy}{dx}$
0	1	1
0.5	1.5	1.5
1	2.25	2.25

$y(x) = e^x$



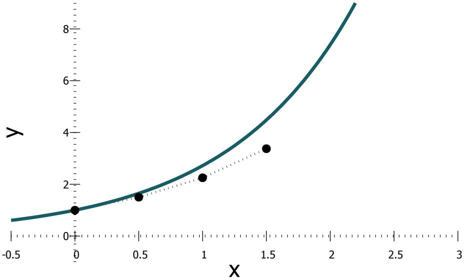
Varying the stepsize

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$\boldsymbol{\Delta x = 0.5}$$

x	y	$\frac{dy}{dx}$
0	1	1
0.5	1.5	1.5
1	2.25	2.25
1.5	3.375	3.375

$y(x) = e^x$



Varying the stepsize

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1	2.25	2.25
1.5	3.375	3.375
2	5.0625	5.0625

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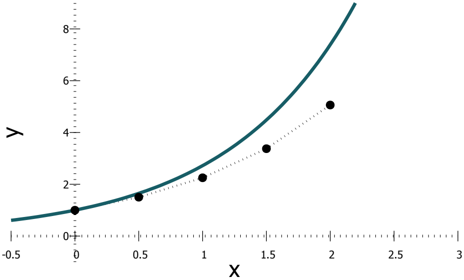


Euler Approximation in Words

1. Start with a certain value for x and y and the differential equation $\frac{dy}{dx} = \ldots$ you want to approximate
2. Decide for a step size that determines the accuracy of your approximation
3. Repeat as long as you like:
   - 3.1 Use the current y-value to calculate the current rate of change $\frac{dy}{dx}$
   - 3.2 Calculate the next y-value by taking the current y-value and adding to it the rate of change times the step size
   - 3.3 Increase x by the step size

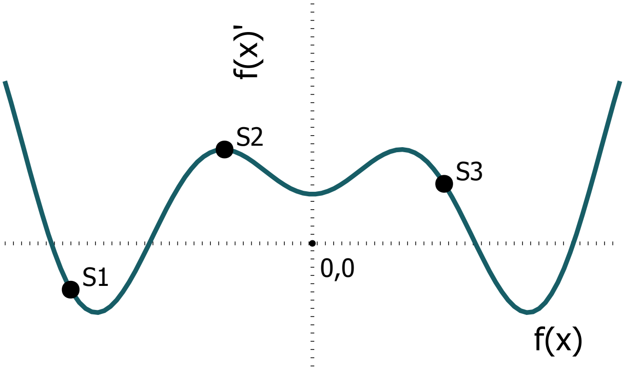
Task Template

- Download the archive *task_template_6.zip* from the course homepage. Extract it into a folder of your choice.
- The archive contains *task_61.py*, *student_code_61.py*, and *braitenberg.png*.
- You only need to edit code in the *student_code* file.

Explain Task Template!

Tasks

1. For each of the three starting values $S1$, $S2$, $S3$ of the function $f(x)$ shown in the plot below determine which function value will be assumed after an infinite amount of time steps (graphically).



Tasks

2. Change the behavior of the vehicle (*graph*) by implementing the function *calc_angle_change*.
    - *current_angle* is the current orientation of the vehicle in degree.
    - *left_sensor_values* and *right_sensor_values* are the measured values of the sensors. They increase the closer they are to an obstacle.
    - First make the angle change dependent on the current sensor values. How can you make the vehicle avoid obstacles? How can you guide it towards obstacles?
    - *Let your change in the angle depend on the current angle itself. Set an attractor at 45°, such that the vehicle will turn towards 45° degrees in the absence of obstacles.*
    - *What do you need to change to avoid a particular orientation?*

Tasks

3. Imagine the differential equation $\frac{dy}{dx} = -y + 20$, where y describes the heading of your vehicle. The initial orientation is $y(0) = 40°$.
   - Calculate the y-values up to an x-value of 3 by hand using the Euler approximation method. Use a step size of 0.5.
   - Implement the Euler approximation method in a python script, which can go to a certain x-value with a certain step size.
   - Hint: You can reuse a lot of the code from yesterday.
   - Calculate how long your for-loop has to run depending on the desired x-value and your step size.
   - Save your results in three different lists. One for the x-values, one for the y-values and one for the $\frac{dy}{dx}$ term.
   - Plot your x-values against your y-values and your y-values against your $\frac{dy}{dx}$-values. (See the next slide for plotting commands.)

Matplotlib.pyplot

- The pyplot submodule

```
# A submodule can be imported with the . operator
import matplotlib.pyplot as plt
# The as operator allows renaming for convenience
xValues = [1,1,2,3,5,8,13]
yValues = [3,4,7,6,9,10,12]
plt.plot(xValues,yValues) #plots lines
# This generates the plot and .show() displays it
plt.show()
#plots points and lines
plt.plot(xValues,yValues,linestyle = "-", marker="o")
plt.show()
```