Lecture 4
Function Limits and Differentiation

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Computer Science and Mathematics
Preparatory Course

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Motivation

Estimating Velocity by Differentiation
The Vehicle’s Position

0 5 10 15 20 25 30 35 40

0 10 20 30 40 50

0 10 20 30 40 50

position

time
The Vehicle’s Position

![Diagram showing vehicle's position over time](image)
The Vehicle’s Position

![Diagram of vehicle's position over time]

- Time markers: 0, 5, 10, 15, 20, 25, 30, 35, 40
- Position markers: 0, 10, 20, 30, 40, 50
The Vehicle’s Position

0 5 10 15 20 25 30 35 40

0
10
20
30
40
50

position

time
The Vehicle’s Position

```
0 5 10 15 20 25 30 35 40
```

```
0 10 20 30 40 50
```
The Vehicle’s Position
The Vehicle’s Position
The Vehicle’s Position

The graph shows the vehicle’s position over time. The position is marked at intervals of 0, 5, 10, 15, 20, 25, 30, 35, and 40. The graph represents the movement of the vehicle on a timeline.
The Vehicle’s Velocity

- Position vs. Time
- Velocity vs. Time
The Vehicle’s Velocity

[position vs. time graph]

[velocity vs. time graph]
The Vehicle’s Velocity
The Vehicle’s Velocity

Graph 1: Position over time
Graph 2: Velocity over time
The Vehicle’s Velocity

- Position vs. Time
- Velocity vs. Time
The Vehicle’s Velocity
The Vehicle’s Velocity
The Vehicle’s Velocity
Overview

1. Motivation

2. Function Limits
   - Sequences
   - Limit Definition

3. Differentiation
   - Graphical Interpretation
   - Formal Description
   - Rules for Differentiation
   - Numerical Differentiation

4. Tasks
Overview

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4. Tasks
Sequences

Sequence Definition

Functions with the domain $\mathbb{N}$ are called **sequence**. A sequence with the codomain $\mathbb{R}$ is called a sequence of real numbers: $f : \mathbb{N} \to \mathbb{R}$, $n \to f(n)$

Examples:

- **Constant sequence**: $(3)_{n \in \mathbb{N}} = (3, 3, 3, 3, 3, \ldots )$
- **Sequence of natural numbers**: $(n)_{n \in \mathbb{N}} = (1, 2, 3, 4, 5, \ldots )$
- **Harmonic sequence**: $(\frac{1}{n})_{n \in \mathbb{N}} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots )$
- **Geometric sequence**: $(q^n)_{n \in \mathbb{N}} = (q, q^2, q^3, q^4, q^5, \ldots )$
- **Alternating sequence**: $((-1)^n)_{n \in \mathbb{N}} = (-1, 1, -1, 1, -1, \ldots )$
Recursive Sequences

Recursive Sequence Definition

A sequence \((a_n)_{n \in \mathbb{N}}\) may be recursively defined by:

1. The first sequence element: \(a_1\), called **initial value**
2. A recursive rule defining element \(a_{n+1}\) through previous elements \(a_n\)

Example: The Fibonacci Sequence

\[ a_{n+1} = a_n + a_{n-1} = (1, 1, 2, 3, 5, 8, 13, 21, \ldots), \]

with \(a_1 = 1\) and \(a_2 = 1\)
Properties of Sequences

Boundedness

A sequence \((a_n)_{n \in \mathbb{N}}\) has

- an **upper bound**, if there is a \(K \in \mathbb{R}\), such that \(a_n \leq K\) for all \(n \in \mathbb{N}\)

- a **lower bound**, if there is a \(K \in \mathbb{R}\), such that \(a_n \geq K\) for all \(n \in \mathbb{N}\)
Properties of Sequences

Boundedness

A sequence \((a_n)_{n \in \mathbb{N}}\) has

- an **upper bound**, if there is a \(K \in \mathbb{R}\), such that \(a_n \leq K\) for all \(n \in \mathbb{N}\)

- a **lower bound**, if there is a \(K \in \mathbb{R}\), such that \(a_n \geq K\) for all \(n \in \mathbb{N}\)

Monotonicity

A sequence \((a_n)_{n \in \mathbb{N}}\) is:

- (strictly) **monotonically increasing**, if \(a_n (<) \leq a_{n+1}\) for all \(n \in \mathbb{N}\)

- (strictly) **monotonically decreasing**, if \(a_n (>) \geq a_{n+1}\) for all \(n \in \mathbb{N}\)
Convergence and Divergence

**Definitions**

- A sequence \((a_n)_{n \in \mathbb{N}}\) of real numbers **converges** to a real number \(L\), if for all \(\epsilon > 0\), there exists a natural number \(N\):

\[ a_n < L + \epsilon \land a_n > L - \epsilon \text{ for all } n \geq N \]
Convergence and Divergence

Definitions

A sequence \((a_n)_{n \in \mathbb{N}}\) of real numbers **converges** to a real number \(L\), if for all \(\epsilon > 0\), there exists a natural number \(N\):

\[
a_n < L + \epsilon \land a_n > L - \epsilon \text{ for all } n \geq N
\]

**Translation**: A sequence converges to a real number \(L\), if you get closer to \(L\) with each additional element in the sequence.
Convergence and Divergence

Definitions

A sequence \((a_n)_{n \in \mathbb{N}}\) of real numbers **converges** to a real number \(L\), if for all \(\epsilon > 0\), there exists a natural number \(N\):

\[
a_n < L + \epsilon \land a_n > L - \epsilon \text{ for all } n \geq N
\]

**Translation:** A sequence converges to a real number \(L\), if you get closer to \(L\) with each additional element in the sequence.

\(L\) is called the **limit** of a sequence

\[
\lim_{n \to \infty} a_n = L
\]
Convergence and Divergence

Definitions

▶ A sequence \((a_n)_{n \in \mathbb{N}}\) of real numbers converges to a real number \(L\), if for all \(\epsilon > 0\), there exists a natural number \(N\):

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a_n < L + \epsilon \land a_n > L - \epsilon \text{ for all } n \geq N
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Translation: A sequence converges to a real number \(L\), if you get closer to \(L\) with each additional element in the sequence

▶ \(L\) is called the limit of a sequence

\[
\lim_{n \to \infty} a_n = L
\]

▶ A sequence that does not converge is called divergent
**Convergence Example**

The harmonic sequence \( \left( \frac{1}{n} \right)_{n \in \mathbb{N}} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots) \) converges to **Zero**
Convergence Example

The harmonic sequence \( \left( \frac{1}{n} \right)_{n \in \mathbb{N}} = \left( 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \right) \) converges to Zero
Convergence Example

A sequence \((a_n)_{n \in \mathbb{N}}\) of real numbers \textbf{converges} to a real number \(L\), if for all \(\epsilon > 0\), there exists \(N\) : \(a_n < L + \epsilon \land a_n > L - \epsilon\) for all \(n \geq N\)
**Convergence Example**

A sequence \((a_n)_{n \in \mathbb{N}}\) of real numbers **converges** to a real number \(L\), if for all \(\epsilon > 0\), there exists \(N\) : \(a_n < L + \epsilon \land a_n > L - \epsilon\) for all \(n \geq N\)

![Graph showing convergence example](image-url)
Convergence Example

A sequence \((a_n)_{n \in \mathbb{N}}\) of real numbers **converges** to a real number \(L\), if for all \(\varepsilon > 0\), there exists \(N\) : \(a_n < L + \varepsilon \land a_n > L - \varepsilon\) for all \(n \geq N\). 

\[ \varepsilon = 0.28 \]

\[ N = 3 \]
Convergence Example

A sequence \((a_n)_{n \in \mathbb{N}}\) of real numbers converges to a real number \(L\), if for all \(\epsilon > 0\), there exists \(N\) such that \(a_n < L + \epsilon\) and \(a_n > L - \epsilon\) for all \(n \geq N\).
Convergence Example

A sequence \((a_n)_{n \in \mathbb{N}}\) of real numbers \textbf{converges} to a real number \(L\), if for all \(\varepsilon > 0\), there exists \(N\) : 
\[a_n < L + \varepsilon \land a_n > L - \varepsilon\] for all \(n \geq N\). 

\[\varepsilon = 0.36\]

\(N = 2\)
Convergence Example

A sequence \((a_n)_{n \in \mathbb{N}}\) of real numbers **converges** to a real number \(L\), if for all \(\epsilon > 0\), there exists \(N\): \(a_n < L + \epsilon \land a_n > L - \epsilon\) for all \(n \geq N\)

\[\varepsilon = 0.28\]

\[N = 3\]
Properties of Limits

Calculating with Limits

For two converging sequences \((x_n)_{n \in \mathbb{N}}\) and \((y_n)_{n \in \mathbb{N}}\) with limits \(\lim_{n \to \infty} x_n = L_x\) and \(\lim_{n \to \infty} y_n = L_y\) the following holds:

- **Scalar multiplication:** \(\lim_{n \to \infty} (ax_n) = aL_x\) for \(a \in \mathbb{R}\)
- **Addition:** \(\lim_{n \to \infty} (x_n + y_n) = L_x + L_y\)
- **Multiplication:** \(\lim_{n \to \infty} (x_n y_n) = L_x L_y\)
- **Division:** \(\lim_{n \to \infty} \left(\frac{x_n}{y_n}\right) = \frac{L_x}{L_y}\)
- **Norm:** \(\lim_{n \to \infty} (|x_n|) = |L_x|\)
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4. Tasks
A function and its derivative

\[ f(x) = x^2 \quad \quad f'(x) = 2x \]
A function and its derivative

\[ f(x) = x \quad f'(x) = 1 \]
A function and its derivative

\[ f(x) = 0.5 \]
\[ f'(x) = 0 \]
A function and its derivative

\[ f(x) = \sin(x) \]
\[ f'(x) = \cos(x) \]
Derivative as a Tangent

\[ f(x) = \sin(x) \quad f'(x) = \cos(x) \]
Derivative as a Tangent

\[ f(x) = \sin(x) \quad f'(x) = \cos(x) \]
Derivative as a Tangent

\[ f(x) = \sin(x) \quad f'(x) = \cos(x) \]

\[ f'(x) = \cos(x) \]
Formal Definition

Differentiable Function

- A function $f$ with domain $M$ is called differentiable at position $x_0$ if, if the limit value

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}
\]

exists.
Formal Definition

Differentiable Function

- A function $f$ with domain $M$ is called differentiable at position $x_0$ if, if the limit value

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists.

- This limit is called $f'$ or \textbf{derivative of $f$ at position $x_0$}. If $f'$ is defined for all $x_0 \in M$, then $f'$ becomes a new function called the \textbf{derivative of $f$}.
Formal Definition

Differentiable Function

- A function $f$ with domain $M$ is called differentiable at position $x_0$ if, if the limit value

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists.

- This limit is called $f'$ or derivative of $f$ at position $x_0$. If $f'$ is defined for all $x_0 \in M$, then $f'$ becomes a new function called the derivative of $f$.

- Alternate notations:

$$f'(x) = \frac{df}{dx}(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
Differentiation as Limit Example

- **Statement:** The derivative of $f(x) = x^2$ is $f'(x) = 2x$
Differentiation as Limit Example

▶ **Statement:** The derivative of \( f(x) = x^2 \) is \( f'(x) = 2x \)

▶ Applying the formula

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0}
\]
Differentiation as Limit Example

▶ **Statement:** The derivative of \( f(x) = x^2 \) is \( f'(x) = 2x \)

▶ Applying the formula

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0}
\]

▶ Simplifying

\[
\lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = \lim_{x \to x_0} \frac{(\underline{x - x_0})(\underline{x + x_0})}{\underline{x - x_0}} = \lim_{x \to x_0} (x + x_0)
\]
Differentiation as Limit Example

**Statement:** The derivative of \( f(x) = x^2 \) is \( f'(x) = 2x \)

**Applying the formula**

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0}
\]

**Simplifying**

\[
\lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\overline{x - x_0}(x + x_0)}{x - x_0} = \lim_{x \to x_0} (x + x_0)
\]

**Applying the limit:**

\[
\lim_{x \to x_0} (x + x_0) = 2x
\]
Differentiation is a linear operator

Rules

- **Constant Factor**
  \[
  \frac{d}{dx}(af) = a \frac{d}{dx}(f)
  \]

- **Sums**
  \[
  \frac{d}{dx}(f + g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)
  \]

Example:

\[
\frac{d}{dx}(4x^2) = 4 \frac{d}{dx}(x^2) = 4(2x) = 8x
\]
Differentiation is a linear operator

Rules

- **Constant Factor**

  \[
  \frac{d}{dx}(af) = a \frac{d}{dx}(f)
  \]

- **Sums**

  \[
  \frac{d}{dx}(f + g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)
  \]

Example:

\[
\begin{align*}
\frac{d}{dx}(4x^2) &= 4 \frac{d}{dx}(x^2) = 4(2x) = 8x \\
\frac{d}{dx}(4x^2 + x^2) &= 4 \frac{d}{dx}(x^2) + \frac{d}{dx}(x^2) = 4(2x) + 2x = 10x
\end{align*}
\]
Differentiation for Products and Quotients

Rules

- **Multiplication**
  \[ \frac{d}{dx}(fg) = \frac{d}{dx}(f)g + f \frac{d}{dx}(g) \]

- **Exponentiation**
  \[ \frac{d}{dx}(f^n) = n \frac{d}{dx}(f)^{n-1} \]

- **Division**
  \[ \frac{d}{dx} \left( \frac{f}{g} \right) = \frac{\frac{d}{dx}(f)g - f \frac{d}{dx}(g)}{g^2} \]
Examples

- **Multiplication**

\[
\frac{d}{dx}(x^2 \sin(x)) = \frac{d}{dx}(x^2)\sin(x) + x^2 \frac{d}{dx}(\sin(x)) = 2x\sin(x) + x^2\cos(x)
\]
Examples

▶ Multiplication

\[
\frac{d}{dx} (x^2 \sin(x)) = \frac{d}{dx} (x^2) \sin(x) + x^2 \frac{d}{dx} (\sin(x)) = 2x \sin(x) + x^2 \cos(x)
\]

▶ Division

\[
\frac{d}{dx} \left( \frac{1}{x} \right) = \frac{\frac{d}{dx} (1)x - 1 \frac{d}{dx} (x)}{x^2} = \frac{0 - 1}{x^2} = -\frac{1}{x^2}
\]
Exponentiation Rule derives from Multiplication Rule

Example $f'(x^3)$

$$\frac{d}{dx}(x^3) = \frac{d}{dx}(x^2x) = \frac{d}{dx}(x^2)x + x^2 \frac{d}{dx}(x)$$
Exponentiation Rule derives from Multiplication Rule

Example $f'(x^3)$

$$
\frac{d}{dx}(x^3) = \frac{d}{dx}(x^2 \cdot x) = \frac{d}{dx}(x^2) \cdot x + x^2 \frac{d}{dx}(x) = 2xx + x^2 = 3x^2
$$
Exponentiation Rule derives from Multiplication Rule

Example $f'(x^3)$

$$\frac{d}{dx} (x^3) = \frac{d}{dx} (x^2 \cdot x) = \frac{d}{dx} (x^2) \cdot x + x^2 \frac{d}{dx} (x)$$

$$= 2xx + x^2 = 3x^2$$

Example $f'(x^4)$

$$\frac{d}{dx} (x^4) = \frac{d}{dx} (x^2 \cdot x^2) = \frac{d}{dx} (x^2) \cdot x^2 + x^2 \frac{d}{dx} (x^2)$$
Exponentiation Rule derives from Multiplication Rule

Example $f'(x^3)$

\[
\frac{d}{dx}(x^3) = \frac{d}{dx}(x^2x) = \frac{d}{dx}(x^2)x + x^2 \frac{d}{dx}(x)
\]

\[
= 2xx + x^2 = 3x^2
\]

Example $f'(x^4)$

\[
\frac{d}{dx}(x^4) = \frac{d}{dx}(x^2x^2) = \frac{d}{dx}(x^2)x^2 + x^2 \frac{d}{dx}(x^2)
\]

\[
= 2xx^2 + x^22x = 2x^3 + 2x^3 = 4x^3
\]
Special cases

- The derivative of 
  \[ f(x) = e^x \text{ is } f'(x) = e^x \]

- The derivative of 
  \[ f(x) = \ln(x) \text{ is } f'(x) = \frac{1}{x} \]

- The derivative of 
  \[ f(x) = \sin(x) \text{ is } f'(x) = \cos(x) \]
Composite functions

Chain Rule

- Function $h$ is a composition of functions $g$ and $f$
  \[ h(x) = (g \circ f)(x) = g(f(x)) \]
- If $g$ and $f$ are differentiable, $h$ is also differentiable
  \[ \frac{d}{dx}(h(x)) = \frac{d}{dx}(g(y)) \frac{d}{dx}(f(x)), \text{ with } y = f(x) \]
- Verbal rule: **Inner derivative times outer derivative**
Chain Rule Examples

\[ h(x) = 5(7x + 2)^4 = g(f(x)) \]
Chain Rule Examples

\[ h(x) = 5(7x + 2)^4 = g(f(x)) \]

\[ g(x) = 5x^4 \land f(x) = 7x + 2 \]
Chain Rule Examples

\[ h(x) = 5(7x + 2)^4 = g(f(x)) \]

\[ g(x) = 5x^4 \land f(x) = 7x + 2 \]
\[ g'(x) = 20x^3 \land f'(x) = 7 \]
Chain Rule Examples

\[ h(x) = 5(7x + 2)^4 = g(f(x)) \]

\[ g(x) = 5x^4 \land f(x) = 7x + 2 \]
\[ g'(x) = 20x^3 \land f'(x) = 7 \]
\[ h'(x) = 20(7x + 2)^37 = 140(7x + 2)^3 \]
Chain Rule Examples

\( h(x) = 5(7x + 2)^4 = g(f(x)) \)

\[
\begin{align*}
  g(x) &= 5x^4 \land f(x) = 7x + 2 \\
  g'(x) &= 20x^3 \land f'(x) = 7 \\
  h'(x) &= 20(7x + 2)^37 = 140(7x + 2)^3
\end{align*}
\]

\( h(x) = e^{5x} = g(f(x)) \)
Chain Rule Examples

1. \( h(x) = 5(7x + 2)^4 = g(f(x)) \)

\[
g(x) = 5x^4 \land f(x) = 7x + 2
\]

\[
g'(x) = 20x^3 \land f'(x) = 7
\]

\[
h'(x) = 20(7x + 2)^3 \cdot 7 = 140(7x + 2)^3
\]

2. \( h(x) = e^{5x} = g(f(x)) \)

\[
g(x) = e^x \land f(x) = 5x
\]
Chain Rule Examples

\[ h(x) = 5(7x + 2)^4 = g(f(x)) \]

\[ g(x) = 5x^4 \land f(x) = 7x + 2 \]
\[ g'(x) = 20x^3 \land f'(x) = 7 \]
\[ h'(x) = 20(7x + 2)^37 = 140(7x + 2)^3 \]

\[ h(x) = e^{5x} = g(f(x)) \]
\[ g(x) = e^x \land f(x) = 5x \]
\[ g'(x) = e^x \land f'(x) = 5 \]
Chain Rule Examples

1. \( h(x) = 5(7x + 2)^4 = g(f(x)) \)

   \[
   g(x) = 5x^4 \land f(x) = 7x + 2 \\
   g'(x) = 20x^3 \land f'(x) = 7 \\
   h'(x) = 20(7x + 2)^3 \cdot 7 = 140(7x + 2)^3 
   \]

2. \( h(x) = e^{5x} = g(f(x)) \)

   \[
   g(x) = e^x \land f(x) = 5x \\
   g'(x) = e^x \land f'(x) = 5 \\
   h'(x) = e^{5x} \cdot 5 = 5e^{5x} 
   \]
Finding Local Extrema

\[ f(x) = \sin(x) \quad f'(x) = \cos(x) \]
Finding Local Extrema

\[ f(x) = x^2 \]

\[ f'(x) = 2x \]
Calculation of Local Extrema

\[ f(x) = 4x^2 + 6x \]
Calculation of Local Extrema

\[ f(x) = 4x^2 + 6x \]

\[ f'(x) = 8x + 6 \]
Calculation of Local Extrema

\[ f(x) = 4x^2 + 6x \]
\[ f'(x) = 8x + 6 \]
\[ f'(x) = 8x + 6 \overset{!}{=} 0 \]
Calculation of Local Extrema

- \( f(x) = 4x^2 + 6x \)

\[
\begin{align*}
  f'(x) &= 8x + 6 \\
  f'(x) &= 8x + 6 = 0 \\
  \iff 8x &= -6
\end{align*}
\]
Calculation of Local Extrema

\[ f(x) = 4x^2 + 6x \]

\[ f'(x) = 8x + 6 \]

\[ f'(x) = 8x + 6 = 0 \]

\[ \iff 8x = -6 \]

\[ x = -\frac{6}{8} = -\frac{3}{4} \]
Calculation of Local Extrema

\[ f(x) = 4x^2 + 6x \]

\[ f'(x) = 8x + 6 \]

\[ f'(x) = 8x + 6 = 0 \]

\[ \iff \quad 8x = -6 \]

\[ x = \frac{-6}{8} = \frac{-3}{4} \]

\[ f(x) = \sin(x) \]
Calculation of Local Extrema

- \( f(x) = 4x^2 + 6x \)
  \[ f'(x) = 8x + 6 \]
  \[ f'(x) = 8x + 6 \triangleq 0 \]
  \[ \iff 8x = -6 \]
  \[ \iff x = \frac{-6}{8} = \frac{-3}{4} \]

- \( f(x) = \sin(x) \)
  \[ f'(x) = \cos(x) \]
Calculation of Local Extrema

1. **$f(x) = 4x^2 + 6x$**
   
   - $f'(x) = 8x + 6$
   
   - $f'(x) = 8x + 6 \overset{!}{=} 0$
   
   - $\iff 8x = -6$
   
   - $\iff x = \frac{-6}{8} = \frac{-3}{4}$

2. **$f(x) = \sin(x)$**
   
   - $f'(x) = \cos(x)$
   
   - $f'(x) = \cos(x) \overset{!}{=} 0$

   - $x = \cos^{-1}\left(\frac{-3}{4}\right)$
Calculation of Local Extrema

\( f(x) = 4x^2 + 6x \)

\[ f'(x) = 8x + 6 \]

\[ f'(x) = 8x + 6 = 0 \]

\[ \iff 8x = -6 \]

\[ \iff x = \frac{-6}{8} = \frac{-3}{4} \]

\( f(x) = \sin(x) \)

\[ f'(x) = \cos(x) \]

\[ f'(x) = \cos(x) = 0 \]

\[ \iff x = \cos^{-1}(0) \]
Calculation of Local Extrema

- \( f(x) = 4x^2 + 6x \)

  \[
  f'(x) = 8x + 6
  \]

  \[
  f'(x) = 8x + 6 = 0
  \]

  \[
  \iff 8x = -6
  \]

  \[
  \iff x = \frac{-6}{8} = \frac{-3}{4}
  \]

- \( f(x) = \sin(x) \)

  \[
  f'(x) = \cos(x)
  \]

  \[
  f'(x) = \cos(x) = 0
  \]

  \[
  \iff x = \cos^{-1}(0)
  \]

  \[
  \iff x = 90^\circ = \frac{\pi}{2}, 270^\circ = \frac{3\pi}{2}, \ldots
  \]
Differentiability is not given

\[ f(x) = \frac{1}{x} \]

\[ f'(x) = \frac{-1}{x^2} \]
Numerical Differentiation

**Problem:** Only function values \( f(x_0) \) of \( f(x) \) are known, but not the real function \( f \).
Numerical Differentiation

- **Problem**: Only function values \( f(x_0) \) of \( f(x) \) are known, but not the real function \( f \)

- Instead of calculating the derivative of \( f \) analytically, it is possible to approximate \( f'(x) \) using **numerical differentiation**
Numerical Differentiation

- **Problem:** Only function values \( f(x_0) \) of \( f(x) \) are known, but not the real function \( f \)

- Instead of calculating the derivative of \( f \) analytically, it is possible to approximate \( f'(x) \) using **numerical differentiation**

(Simple) Numerical Differentiation

The set \( \mathbb{I} \) describes the computable domain of \( f \) in the given context. It is possible to calculate function value \( f(x_i) \), where \( x_i \in \mathbb{I} \).

\[
f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i},
\]

where \( x_{i+1} \) is the smallest positive distance from \( x_i \) in \( \mathbb{I} \).
Numerical Differentiation Example

From a sensor we receive the following values:

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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$T_{h.ligae}$ derivative at $x_{three}$ equals:

$$f'(x_{three}) = f(x_{three} + 1) - f(x_{three})$$

$$x_{three} + 1 - x_{three} = 1$$

$$f(x_{four}) - f(x_{three})$$

$$zero - two - one - four - one - six - three = zero - two$$

$T_{h.ligae}$ change at position $x_{three}$ is $zero - two$. 

three - two / three / five
Numerical Differentiation Example

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The derivative at $x_3$ equals:

$$f'(x_3) = \frac{f(x_{3+1}) - f(x_3)}{x_{3+1} - x_3}$$
Numerical Differentiation Example

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- The derivative at $x_3$ equals:

$$f'(x_3) = \frac{f(x_{3+1}) - f(x_3)}{x_{3+1} - x_3} \Rightarrow \frac{f(x_4) - f(x_3)}{4 - 3}$$
Numerical Differentiation Example

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The derivative at $x_3$ equals:

$$f'(x_3) = \frac{f(x_{3+1}) - f(x_3)}{x_{3+1} - x_3} \Rightarrow \frac{f(x_4) - f(x_3)}{4 - 3} = \frac{1.6 - 1.4}{1}$$
Numerical Differentiation Example

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$$f'(x_3) = \frac{f(x_{3+1}) - f(x_3)}{x_{3+1} - x_3} \Rightarrow \frac{f(x_4) - f(x_3)}{4 - 3} = \frac{1.6 - 1.4}{1} = 0.2$$
Numerical Differentiation Example

- From a sensor we receive the following values:
  
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- The derivative at $x_3$ equals:

  $$f'(x_3) = \frac{f(x_3 + 1) - f(x_3)}{x_3 + 1 - x_3} \Rightarrow \frac{f(x_4) - f(x_3)}{4 - 3} = \frac{1.6 - 1.4}{1} = 0.2$$

- The change at position $x_3$ is 0.2
Tasks

1. Calculate the derivative of the following functions (on a piece of paper)

1.1 \( f(x) = 7x^4 \)
1.2 \( g(x) = 2x^4 + 3x^3 + x^2 + 10x + 5 \)
1.3 \( h(x) = 4e^{3x} \)
1.4 \( i(x) = (12x^2 + 5)3x^3 \)
1.5 \( j(x) = \frac{3x}{\cos(x)} \)

- First think about the rule you need to use
- Identify the parts of the rule in the equation
- If possible differentiate individual parts first
- Apply the rule
Task Template Braitenberg

- Download the archive `task_template_4.zip` from the course homepage. Extract it into a folder of your choice.

- The archive contains `task_4_1.py`, `task_4_1_student_code.py` and `braitenberg.png`.

- Use `task_4_1.py` to run the program, but edit code only in `task_4_1_student_code.py`.

Explain Task Template!
Tasks

2. Calculate the vehicle’s velocity through numerical differentiation.
   ▶ Open `task_4.1_student_code.py` and implement the function `calc_velocity_from_position`.
   ▶ Use the given list of positions to estimate the vehicle’s velocity using numerical differentiation.
   ▶ Append the resulting velocity values to the `player_velocities_x` list.
   ▶ **Tip:** Use a for-loop that runs through the position values and compares the current list-entry to the preceding one.

3. Write a script that calculates the Fibonacci sequence for an arbitrary number $N$ of elements. Print the numbers to the console.
   ▶ The first two elements of $a_1$ and $a_2$ are always 1
   ▶ Write a loop that runs $N$ times and calculates the Fibonacci number $a_{n+1} = a_n + a_{n-1}$
   ▶ **Tip:** Use variables to store the values for the current value $a_n$ and the previous value $a_{n-1}$ and update them in each loop.