Lecture 4
Function Limits and Differentiation

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Computer Science and Mathematics
Preparatory Course

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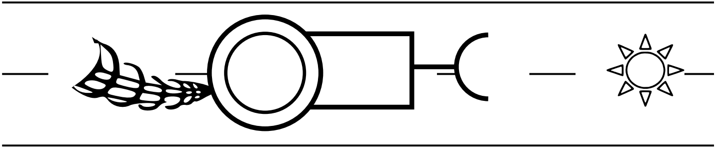
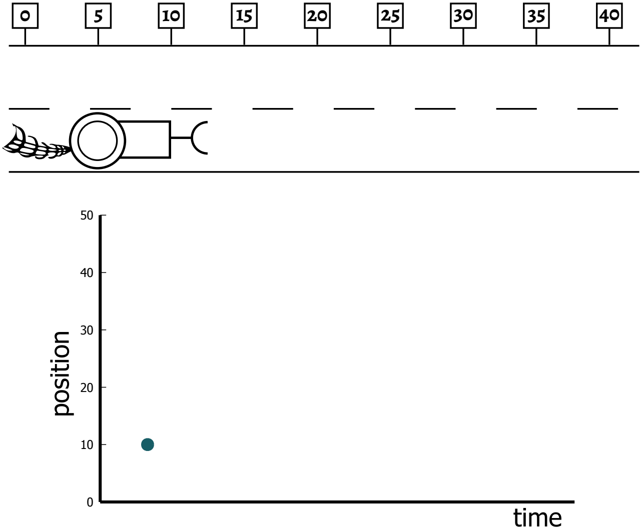
Motivation

Estimating Velocity by Differentiation



The Vehicle's Position



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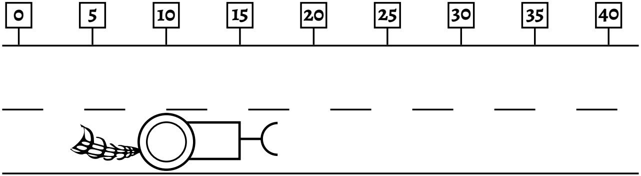

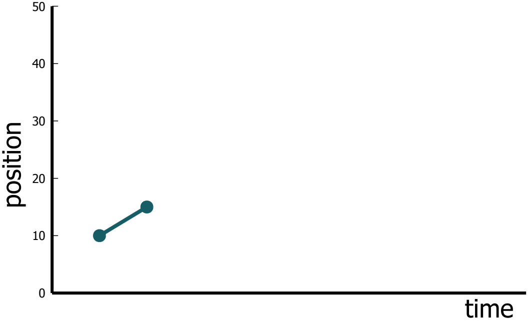

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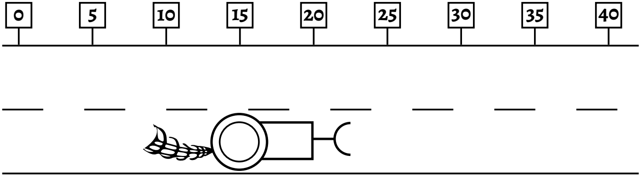

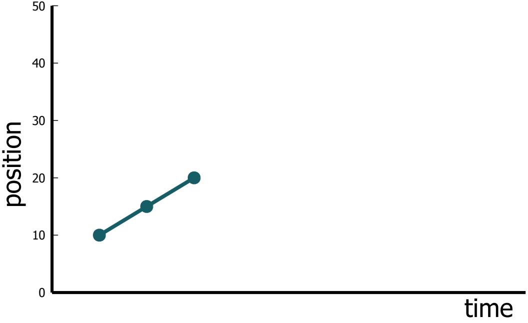

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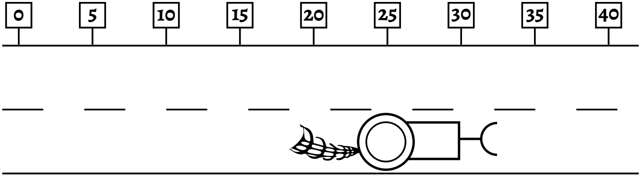

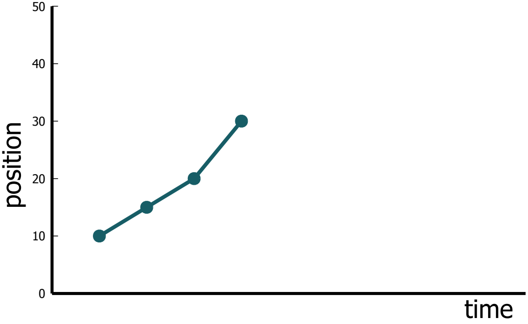

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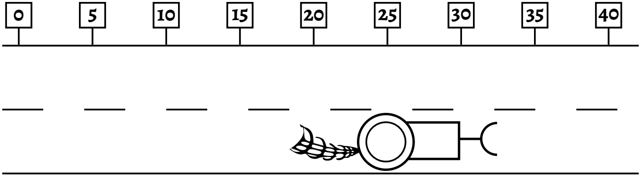

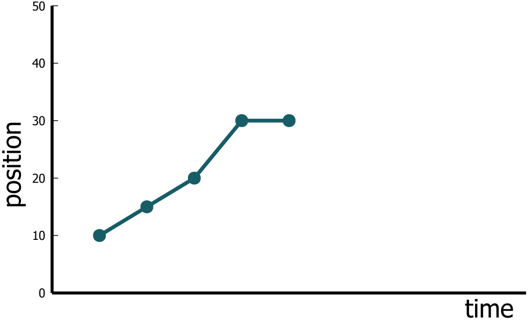

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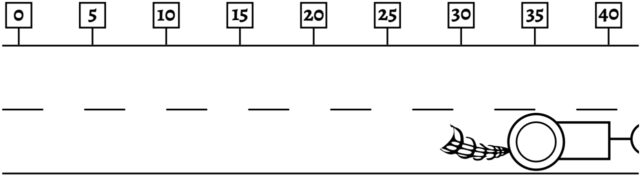

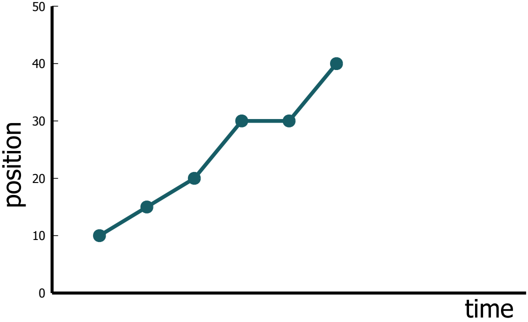

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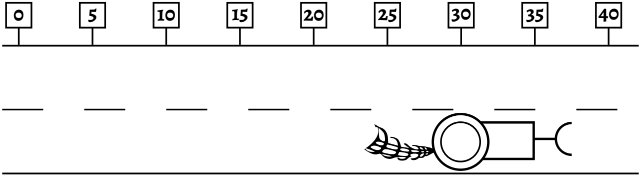

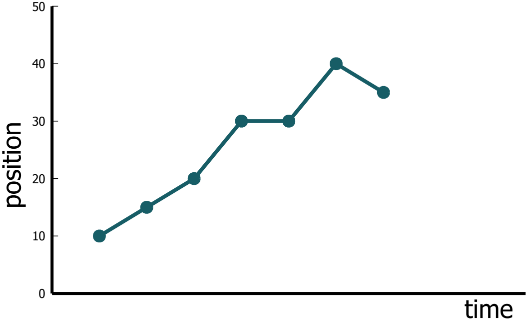

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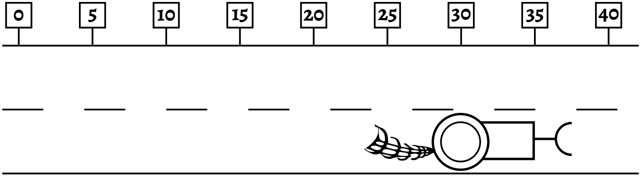
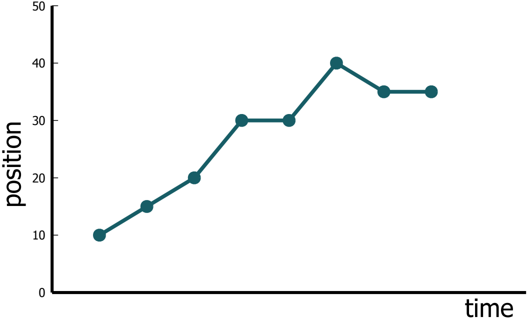
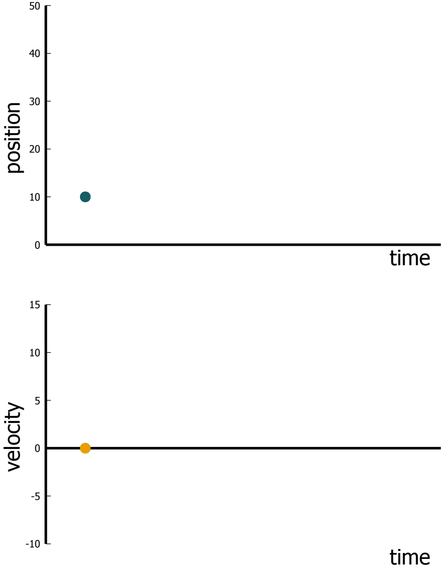




The Vehicle's Velocity



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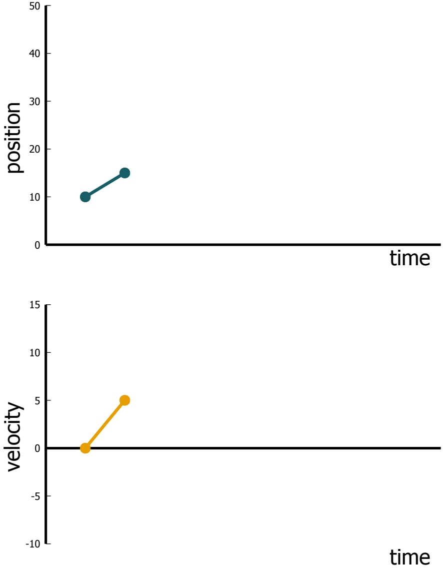

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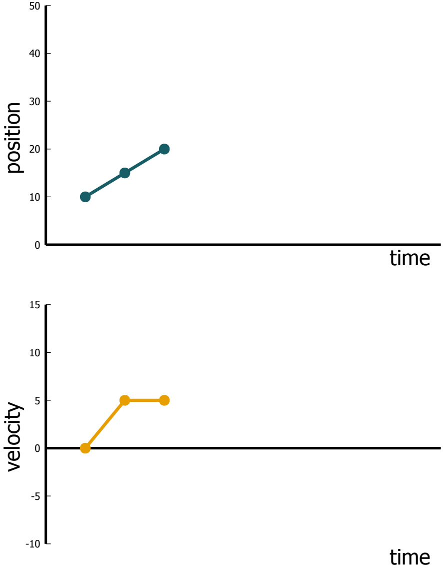

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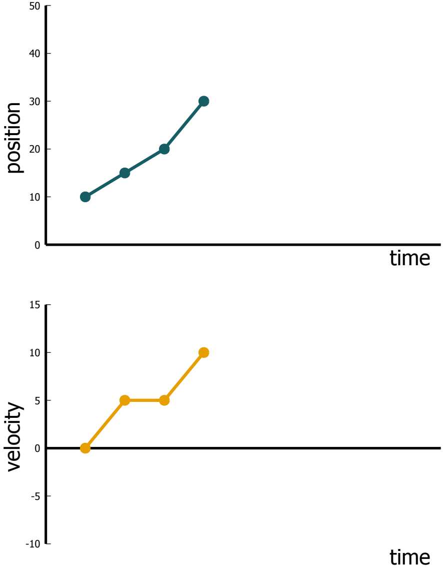

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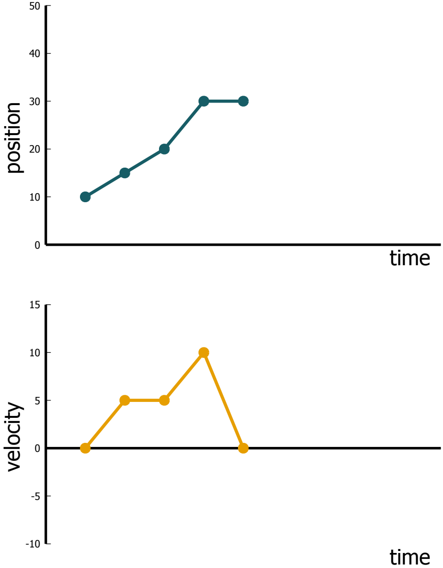

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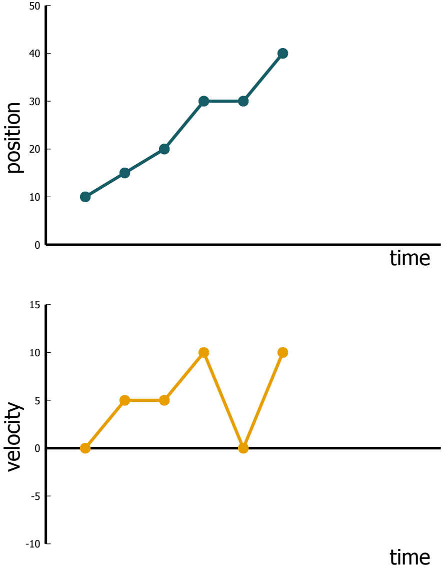

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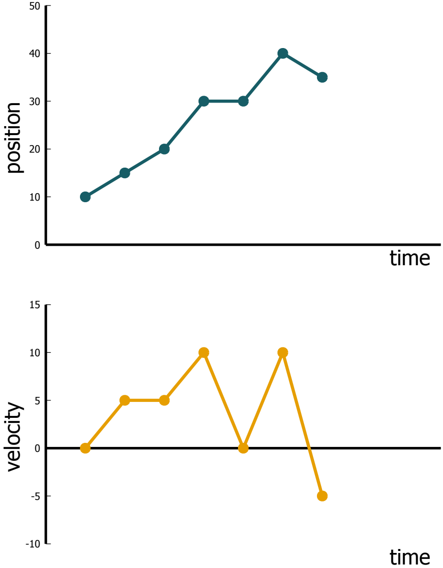

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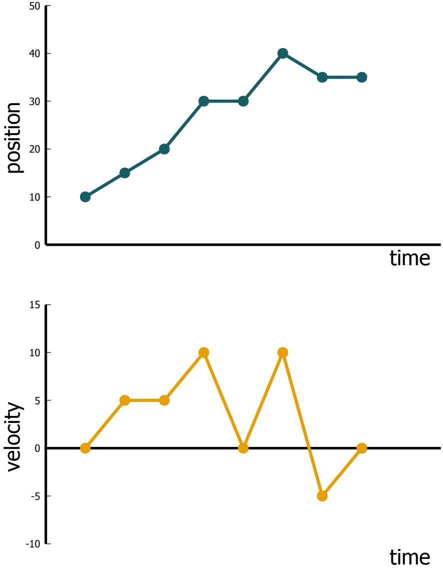

Overview

Overview

Sequences

Sequence Definition

Functions with the domain $\mathbb{N}$ are called **sequence**. A sequence with the codomain $\mathbb{R}$ is called a sequence of real numbers: $f : \mathbb{N} \to \mathbb{R}, n \to f(n)$

Examples:

- Constant sequence: $(3)_{n\in\mathbb{N}} = (3, 3, 3, 3, 3, \ldots)$
- Sequence of natural numbers: $(n)_{n\in\mathbb{N}} = (1, 2, 3, 4, 5, \ldots)$
- Harmonic sequence: $(\frac{1}{n})_{n\in\mathbb{N}} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots)$
- Geometric sequence: $(q^n)_{n\in\mathbb{N}} = (q, q^2, q^3, q^4, q^5, \ldots)$
- Alternating sequence: $((-1)^n)_{n\in\mathbb{N}} = (-1, 1, -1, 1, -1, \ldots)$

Recursive Sequences

Recursive Sequence Definition

A sequence $(a_n)_{n\in\mathbb{N}}$ may be recursively defined by:

1. The first sequence element : a_1, called **initial value**
2. A recursive rule defining element a_{n+1} through previous elements a_n

Example: The Fibonacci Sequence

$$a_{n+1} = a_n + a_{n-1} = (1, 1, 2, 3, 5, 8, 13, 21, \dots),$$
$$\text{with } a_1 = 1 \text{ and } a_2 = 1$$

Properties of Sequences

Boundedness

A sequence $(a_n)_{n\in\mathbb{N}}$ has

- an **upper bound**, if there is a $K \in \mathbb{R}$, such that $a_n \leq K$ for all $n \in \mathbb{N}$
- a **lower bound**, if there is a $K \in \mathbb{R}$, such that $a_n \geq K$ for all $n \in \mathbb{N}$

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Monotonicity

A sequence $(a_n)_{n\in\mathbb{N}}$ is :

- **(strictly) monotonically increasing**, if $a_n (<) \leq a_{n+1}$ for all $n \in \mathbb{N}$
- **(strictly) monotonically decreasing**, if $a_n (>) \geq a_{n+1}$ for all $n \in \mathbb{N}$

Convergence and Divergence

Definitions

- A sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers **converges** to a real number L, if for all $\epsilon > 0$, there exists a natural number N:

$$a_n < L + \epsilon \ \wedge \ a_n > L - \epsilon \text{ for all } n \geq N$$

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$$\lim_{n\to\infty} a_n = L$$

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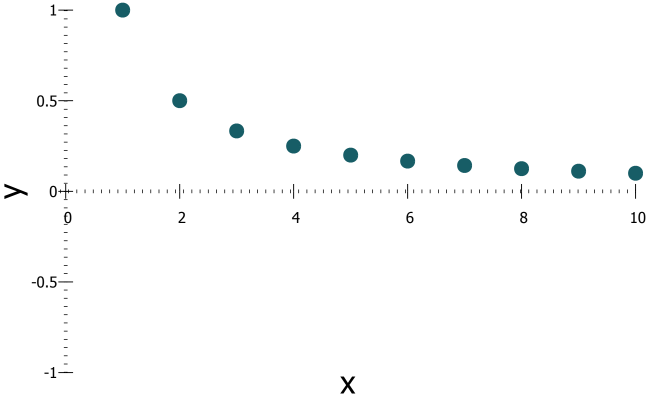
- A sequence that does not converge is called **divergent**

Convergence Example

The harmonic sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right)$ converges to **Zero**



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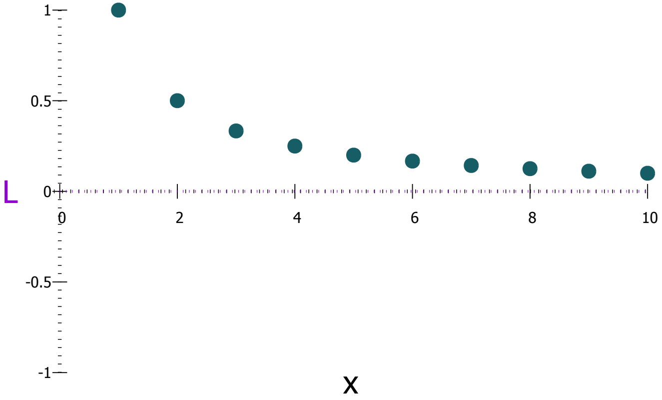

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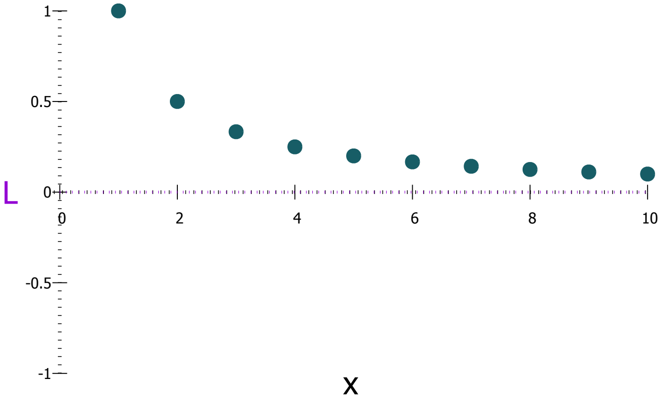

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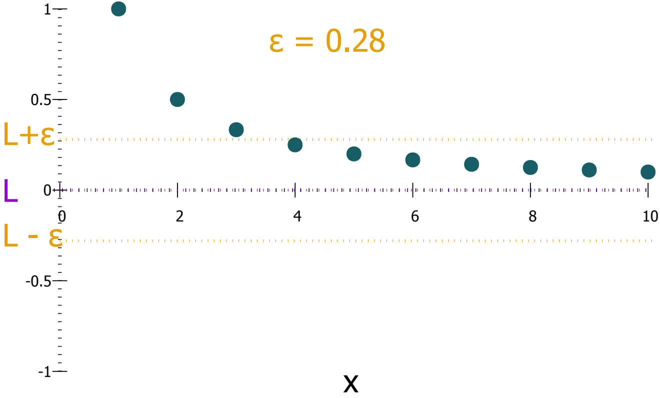

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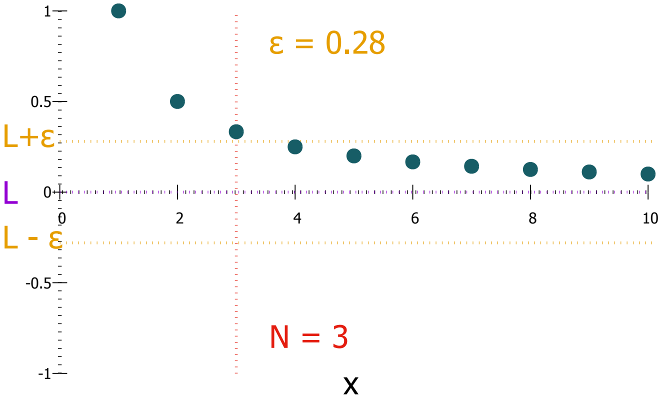

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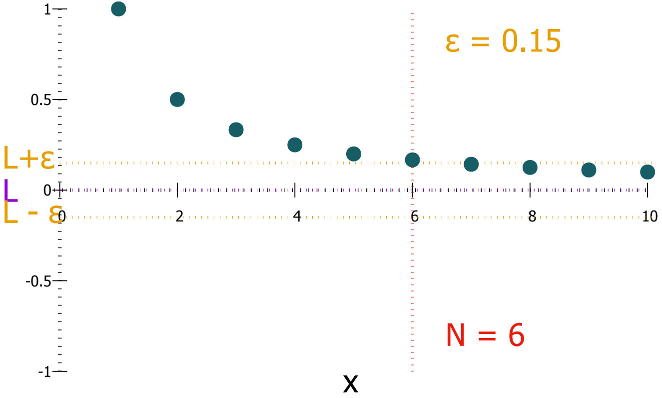

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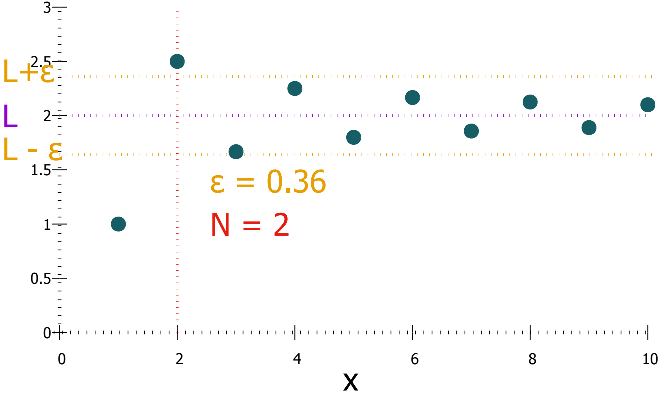

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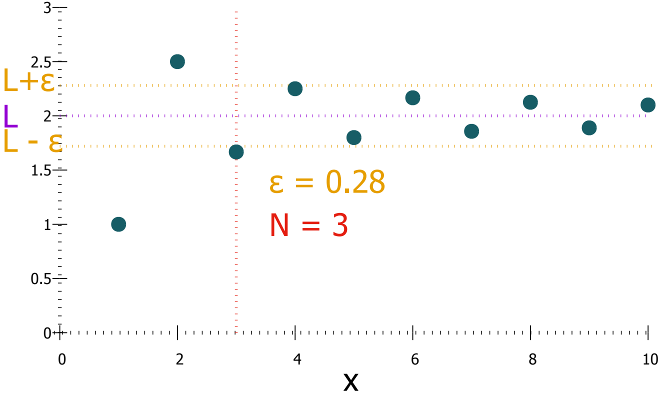


Properties of Limits

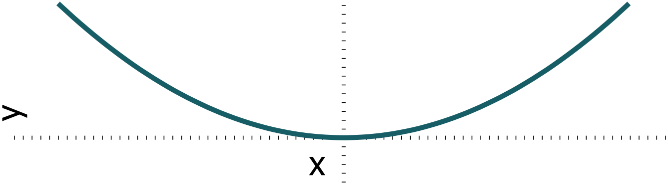
Calculating with Limits

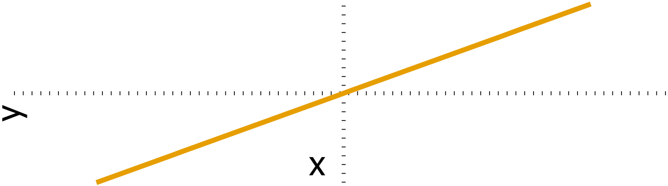
For two converging sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ with limits $\lim_{n\to\infty} x_n = L_x$ and $\lim_{n\to\infty} y_n = L_y$ the following holds:

- **Scalar multiplication:** $\lim_{n\to\infty}(ax_n) = aL_x$ for $a \in \mathbb{R}$
- **Addition:** $\lim_{n\to\infty}(x_n + y_n) = L_x + L_y$
- **Multiplication:** $\lim_{n\to\infty}(x_n y_n) = L_x L_y$
- **Division:** $\lim_{n\to\infty}(\frac{x_n}{y_n}) = \frac{L_x}{L_y}$
- **Norm:** $\lim_{n\to\infty}(|x_n|) = |L_x|$

A function and its derivative

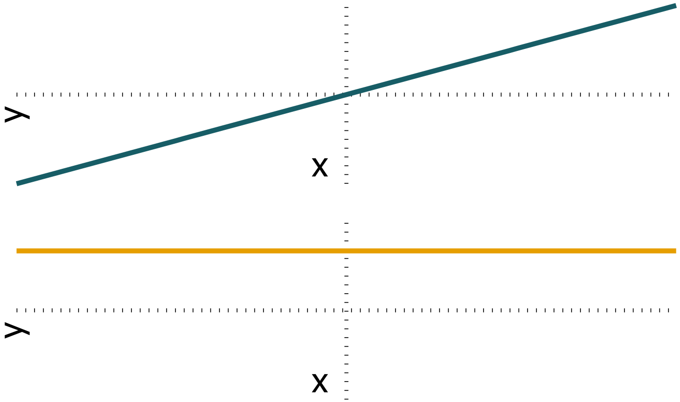
$f(x) = x^2$ $\qquad$ $f'(x) = 2x$





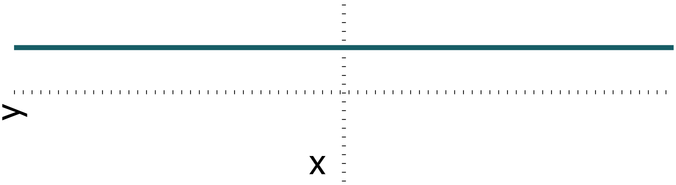
A function and its derivative

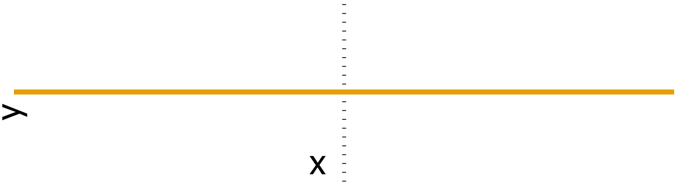
$f(x) = x$ $\quad$ $f'(x) = 1$



A function and its derivative

$f(x) = 0.5$ $\qquad$ $f'(x) = 0$





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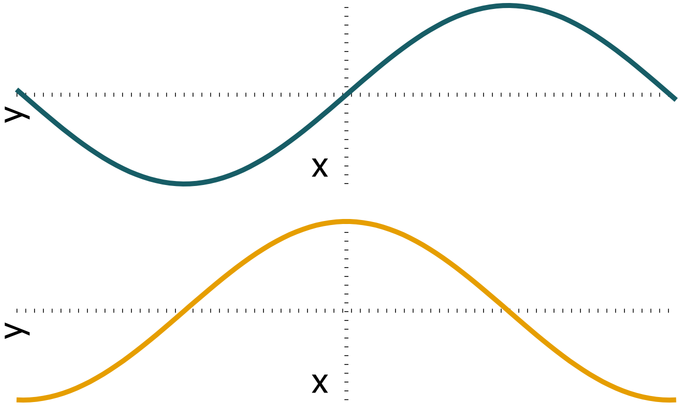
$f(x) = sin(x)$ $f'(x) = cos(x)$



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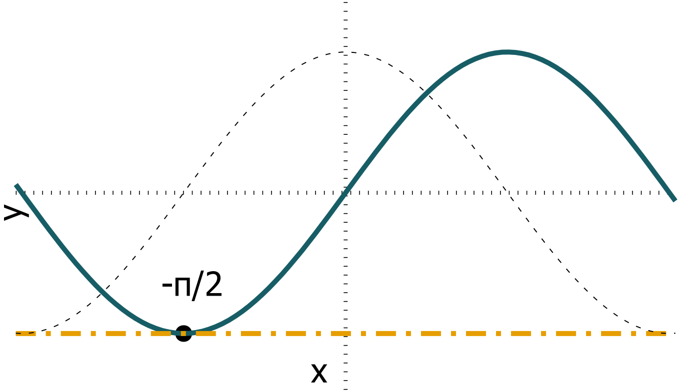

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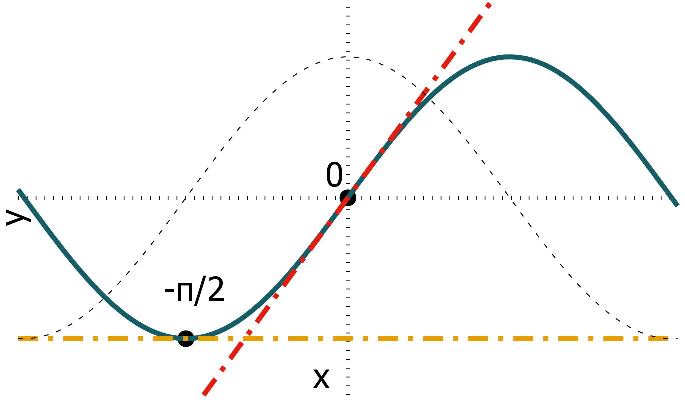

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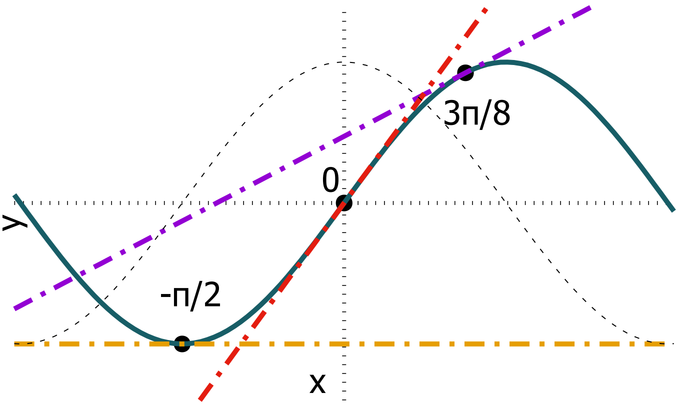

Formal Definition

Differentiable Function

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$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists.

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- Alternate notations:

$$f'(x) = \frac{df}{dx}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation as Limit Example

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$$\lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\cancel{(x - x_0)}(x + x_0)}{\cancel{x - x_0}} = \lim_{x \to x_0} (x + x_0)$$

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- Applying the limit:

$$\lim_{x \to x_0} (x + x_0) = 2x$$

Differentiation is a linear operator

Rules

- **Constant Factor**

$$\frac{d}{dx}(af) = a\frac{d}{dx}(f)$$

- **Sums**

$$\frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)$$

Example:

$$\frac{d}{dx}(4x^2) = 4\frac{d}{dx}(x^2) = 4(2x) = 8x$$

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$$\frac{d}{dx}(4x^2 + x^2) = 4\frac{d}{dx}(x^2) + \frac{d}{dx}(x^2) = 4(2x) + 2x = 10x$$

Differentiation for Products and Quotients

Rules

- **Multiplication**

$$\frac{d}{dx}(fg) = \frac{d}{dx}(f)g + f\frac{d}{dx}(g)$$

- **Exponentiation**

$$\frac{d}{dx}(f^n) = n\frac{d}{dx}(f)^{n-1}$$

- **Division**

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{d}{dx}(f)g - f\frac{d}{dx}(g)}{g^2}$$

Examples

- **Multiplication**

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- **Division**

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{\frac{d}{dx}(1)x - 1\frac{d}{dx}(x)}{x^2} = \frac{0-1}{x^2} = \frac{-1}{x^2}$$

Exponentiation Rule derives from Multiplication Rule

- Example $f'(x^3)$

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$$= 2xx^2 + x^2 2x = 2x^3 + 2x^3 = 4x^3$$

Special cases

- The derivative of

$$f(x) = e^x \text{ is } f'(x) = e^x$$

- The derivative of

$$f(x) = ln(x) \text{ is } f'(x) = \frac{1}{x}$$

- The derivative of

$$f(x) = sin(x) \text{ is } f'(x) = cos(x)$$

Composite functions

Chain Rule

- Function h is a composition of functions g and f

$$h(x) = (g \circ f)(x) = g(f(x))$$

- If g and f are differentiable, h is also differentiable

$$\frac{d}{dx}(h(x)) = \frac{d}{dx}(g(y))\frac{d}{dx}(f(x)), \text{ with } y = f(x)$$

- Verbal rule: **Inner derivative times outer derivative**

Chain Rule Examples

- $h(x) = 5(7x + 2)^4 = g(f(x))$

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Chain Rule Examples

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\begin{aligned}
g(x) &= 5x^4 \wedge f(x) = 7x + 2 \\
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h'(x) &= 20(7x+2)^3 7 = 140(7x+2)^3
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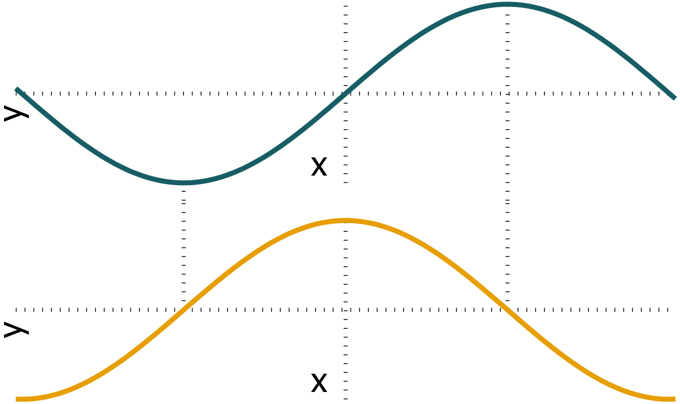
$$h'(x) = e^{5x} 5 = 5e^{5x}$$

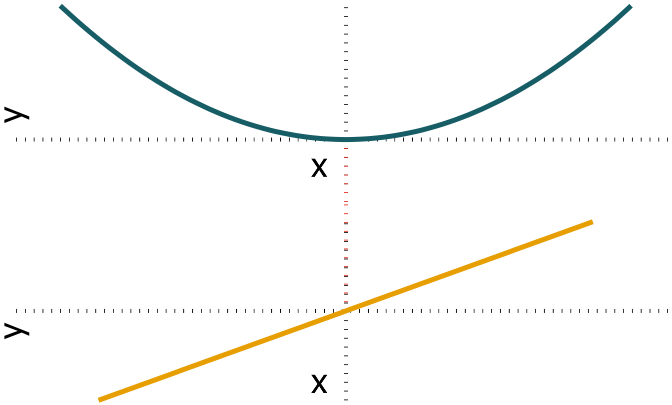
Finding Local Extrema

$f(x) = sin(x)$ $f'(x) = cos(x)$



Finding Local Extrema

$f(x) = x^2$ $f'(x) = 2x$



Calculation of Local Extrema

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  $$f'(x) = cos(x)$$

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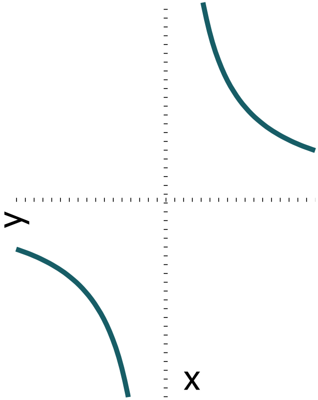
  $$\iff x = cos^{-1}(0)$$

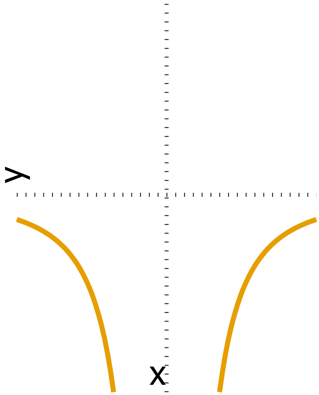
  $$\iff x = 90^\circ = \frac{\pi}{2}, 270^\circ = \frac{3\pi}{2}, \ldots$$

Differentiability is not given

$f(x) = \frac{1}{x}$



$f'(x) = \frac{-1}{x^2}$



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(Simple) Numerical Differentiation

The set $\mathbb{I}$ describes the computable domain of f in the given context. It is possible to calculate function value $f(x_i)$, where $x_i \in \mathbb{I}$.

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i},$$

where x_{i+1} is the smallest positive distance from x_i in $\mathbb{I}$.

Numerical Differentiation Example

- From a sensor we receive the following values:

x_i	0	1	2	3	4	5	6	7	8	9
$f(x_i)$	3.1	2.9	2.4	1.4	1.6	3	3.1	3.3	3.5	4.2

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- The change at position x_3 is 0.2

Tasks

1. Calculate the derivative of the following functions (on a piece of paper)
   1.1 $f(x) = 7x^4$
   1.2 $g(x) = 2x^4 + 3x^3 + x^2 + 10x + 5$
   1.3 $h(x) = 4e^{3x}$
   1.4 $i(x) = (12x^2 + 5)3x^3$
   1.5 $j(x) = \frac{3x}{\cos(x)}$

   - First think about the rule you need to use
   - Identify the parts of the rule in the equation
   - If possible differentiate individual parts first
   - Apply the rule

Task Template Braitenberg

- Download the archive *task_template_4.zip* from the course homepage. Extract it into a folder of your choice.
- The archive contains *task_4_1.py*, *task_4_1_student_code.py* and *braitenberg.png*.
- Use *task_4_1.py* to run the program, but edit code only in *task_4_1_student_code.py*.

Explain Task Template!

Tasks

2. Calculate the vehicle's velocity through numerical differentiation.
   - Open *task_4_1_student_code.py* and implement the function *calc_velocity_from_position*.
   - Use the given list of positions to estimate the vehicles velocity using numerical differentiation.
   - Append the resulting velocity values to the *player_velocities_x* list.
   - **Tip**: Use a for-loop that runs through the position values and compares the current list-entry to the preceding one.

3. Write a script the calculates the Fibonacci sequence for an arbitrary number N of elements. Print the numbers to the console.
   - The first two elements of a_1 and a_2 are always 1
   - Write a loop that runs N times and calculates the Fibonacci number $a_{n+1} = a_n + a_{n-1}$
   - **Tip:** Use variables to store the values for the current value a_n and the previous value a_{n-1} and update them in each loop.