# Motor control

Gregor Schöner

# Motor control

- is about the processes of bringing about the physical movement of an arm (robot or human)
- this entails
  - the mechanical dynamics of an arm
  - control principles
  - actuators

# Resources

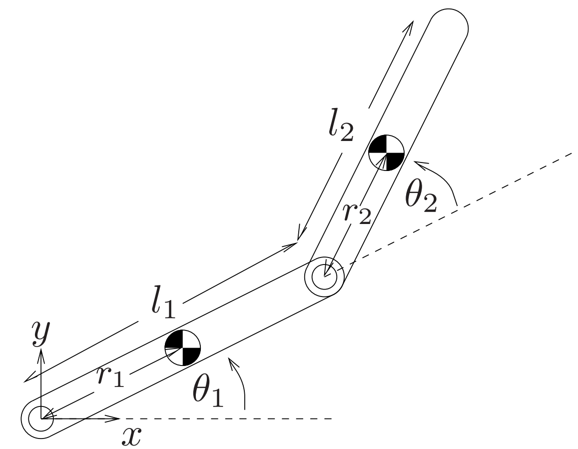
- R M Murray, Z Li, S S. Sastry: A mathematical introduction to robotic manipulation. CRC Press, 1994
- K M Lynch, F C Park: Modern Robotics: Mechanics, Planning, and Control. Cambridge University Press, 2017
- online version of both available…

# Newton's law

- for a mass, m, described by a variable, x, in an inertial frame: $m\ddot{x} = f(x, t)$ where f is a force
- in non-inertial frames, e.g. rotating or accelerating frames:
  - centripetal forces
  - Coriolis forces

# Rigid bodies: constraints

- constraints reduce the effective numbers of degrees of freedom. . .

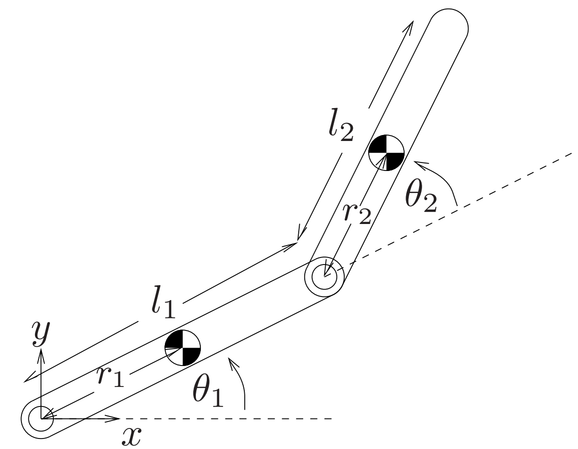



$$F_i = m_i \ddot{r}_i \qquad r_i \in \mathbb{R}^3, i = 1, \ldots, n.$$

$$g_j(r_1, \ldots, r_n) = 0 \qquad j = 1, \ldots, k.$$

# Rigid bodies: constraints

- generalized coordinates capture the remaining, free degrees of freedom




$$r_i = f_i(q_1, \ldots, q_m) \qquad \Longleftrightarrow \qquad g_j(r_1, \ldots, r_n) = 0$$
$$i = 1, \ldots, n \qquad\qquad\qquad j = 1, \ldots, k.$$

# Lagrangian mechanics

- The Lagrangian framework makes it possible to capture dynamics in generalized coordinates that reflect constraints
- Lagrange function L = kinetic- potential energy $L(q,\dot{q}) = T(q,\dot{q}) - V(q),$
- Least action principle: The integral of L over time=action is minimal

$$\delta A = \delta \int L(q,\dot{q},t)dt = 0$$

[Murray, Sastry, Li, 94]

# Lagrangian mechanics

- Least action principle: The integral of L over time=action is minimal $\delta A = \delta \int L(q, \dot{q}, t) dt = 0$

[Murray, Sastry, Li, 94]

# Euler-Lagrange equation

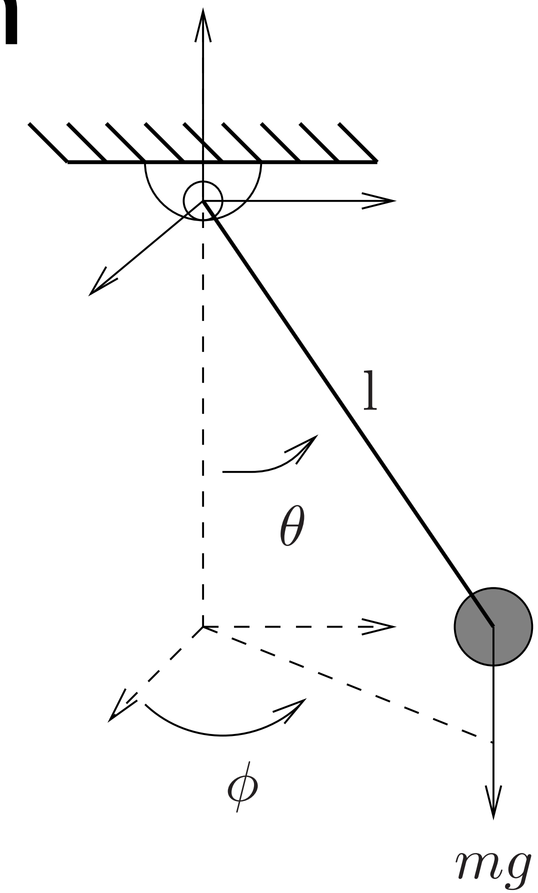
- $\delta A = \int (\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q})dt = 0$
- with $\delta \dot{q} = d\delta q/dt$
- and with partial integration
- $\delta A = [\frac{\partial L}{\partial \dot{q}}\delta q] + \int \left( \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) \delta q \ dt = 0$
- first term vanishes: no variation at start/end points

# Euler-Lagrange equation

- => $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$
- …plus generalized external forces, $\gamma$
- $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \gamma$
- in component form:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \Upsilon_i \qquad i = 1, \ldots, m,$$

# Example: pendulum




- generalized coordinates: $\theta, \phi$
- $$T = \frac{1}{2} m l^2 \|\dot{r}\|^2 = \frac{1}{2} m l^2 \left(\dot{\theta}^2 + (1 - \cos^2\theta)\dot{\phi}^2\right)$$
- $$V = -mgl\cos\theta,$$
- $$L(q, \dot{q}) = \frac{1}{2} m l^2 \left(\dot{\theta}^2 + (1 - \cos^2\theta)\dot{\phi}^2\right) + mgl\cos\theta$$

position relative to base $r(\theta, \phi) = \begin{bmatrix} l\sin\theta\cos\phi \\ l\sin\theta\sin\phi \\ -l\cos\theta \end{bmatrix}$

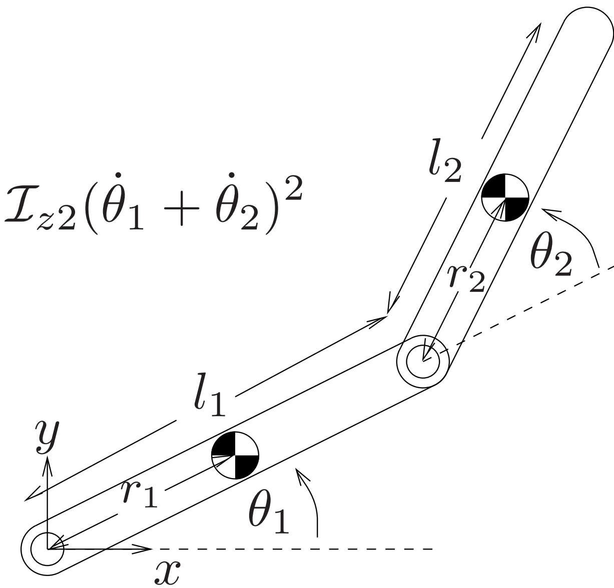
# Example: pendulum

- $$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = \frac{d}{dt}\left(ml^2\dot\theta\right) = ml^2\ddot\theta$$
  $$\frac{\partial L}{\partial \theta} = ml^2 \sin\theta\cos\theta\,\dot\phi^2 - mgl\sin\theta$$
  $$\frac{d}{dt}\frac{\partial L}{\partial \dot\phi} = \frac{d}{dt}\left(ml^2\sin^2\theta\dot\phi\right) = ml^2\sin^2\theta\,\ddot\phi + 2ml^2\sin\theta\cos\theta\,\dot\theta\dot\phi$$
  $$\frac{\partial L}{\partial \phi} = 0$$

- $$\underbrace{\begin{bmatrix} ml^2 & 0 \\ 0 & ml^2\sin^2\theta \end{bmatrix}\begin{bmatrix} \ddot\theta \\ \ddot\phi \end{bmatrix}}_{\text{inertial}} + \underbrace{\begin{bmatrix} -ml^2\sin\theta\cos\theta\,\dot\phi^2 \\ 2ml^2\sin\theta\cos\theta\,\dot\theta\dot\phi \end{bmatrix}}_{\text{centrifugal (Coriolis)}} + \underbrace{\begin{bmatrix} mgl\sin\theta \\ 0 \end{bmatrix}}_{\text{gravitational}} = 0.$$

# Example: two-link planar robot

- generalized coordinates: $\theta_1, \theta_2$
- $$T(\theta,\dot\theta) = \frac{1}{2}m_1(\dot{\bar{x}}_1^2 + \dot{\bar{y}}_1^2) + \frac{1}{2}\mathcal{I}_{z1}\dot\theta_1^2 + \frac{1}{2}m_2(\dot{\bar{x}}_2^2 + \dot{\bar{y}}_2^2) + \frac{1}{2}\mathcal{I}_{z2}(\dot\theta_1+\dot\theta_2)^2$$
$$= \frac{1}{2}\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix}^T \begin{bmatrix}\alpha+2\beta c_2 & \delta+\beta c_2\\ \delta+\beta c_2 & \delta\end{bmatrix}\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix},$$




- where $s_i = \sin(\theta_i), c_i = \cos(\theta_i)$

$$\begin{bmatrix}\alpha+2\beta c_2 & \delta+\beta c_2\\ \delta+\beta c_2 & \delta\end{bmatrix}\begin{bmatrix}\ddot\theta_1\\ \ddot\theta_2\end{bmatrix} + \begin{bmatrix}-\beta s_2\dot\theta_2 & -\beta s_2(\dot\theta_1+\dot\theta_2)\\ \beta s_2\dot\theta_1 & 0\end{bmatrix}\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix} = \begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}$$

inertial

centrifugal/Coriolis

active torques

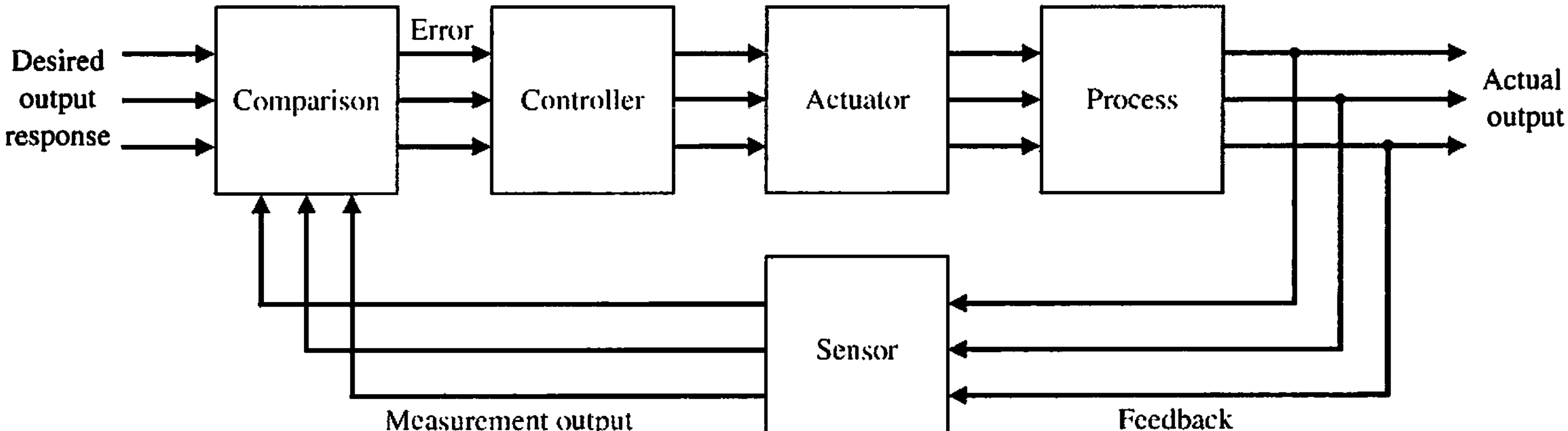
# Open-chain manipulator

$$\boxed{M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + N(\theta, \dot{\theta}) = \tau}$$

| | | | |
|---|---|---|---|
| inertial | centrifugal/ Coriolis | gravitational | active torques |

# Control systems

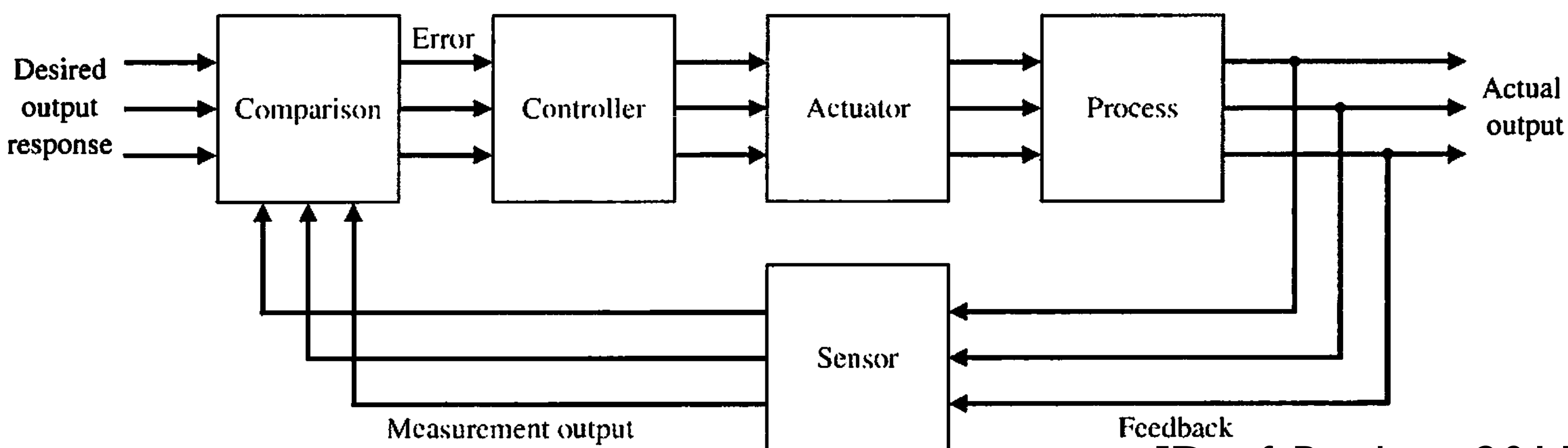
- robotic motion as a special case of control




[Dorf, Bischop, 2011]

# Control systems

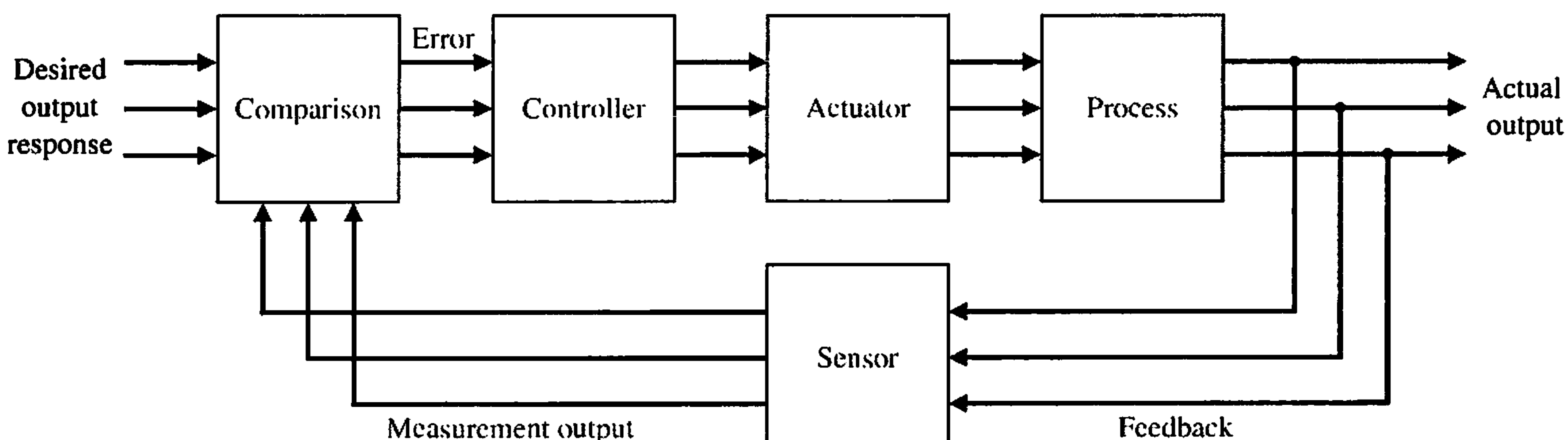
- $$\dot{x} = f(t, x, u) \qquad\qquad y = \eta(t, x, u)$$
- state of process/actuator x
- output, y
- control signal, u




[Dorf, Bischop, 2011]

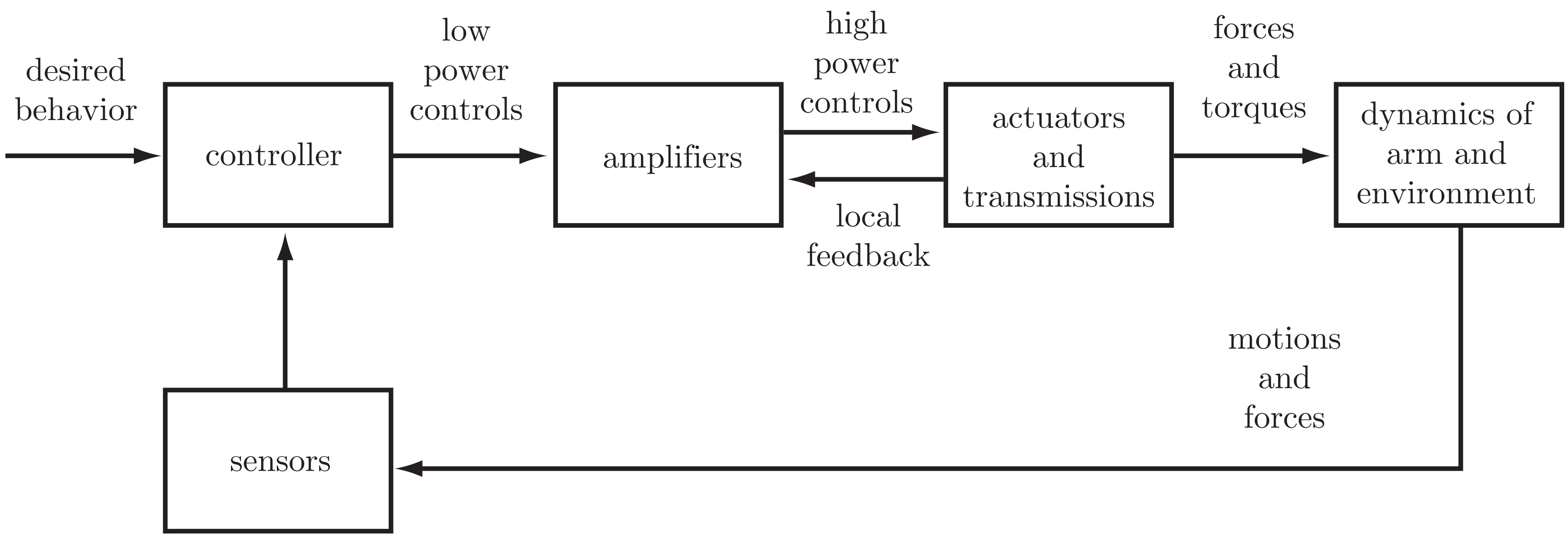
# Control systems

- $$\dot{x} = f(t, x, u) \qquad\qquad y = \eta(t, x, u)$$
- control law: u as a function of y (or ^y), desired response, y_d
- disturbances modeled stochastically




[Dorf, Bischop, 2011]

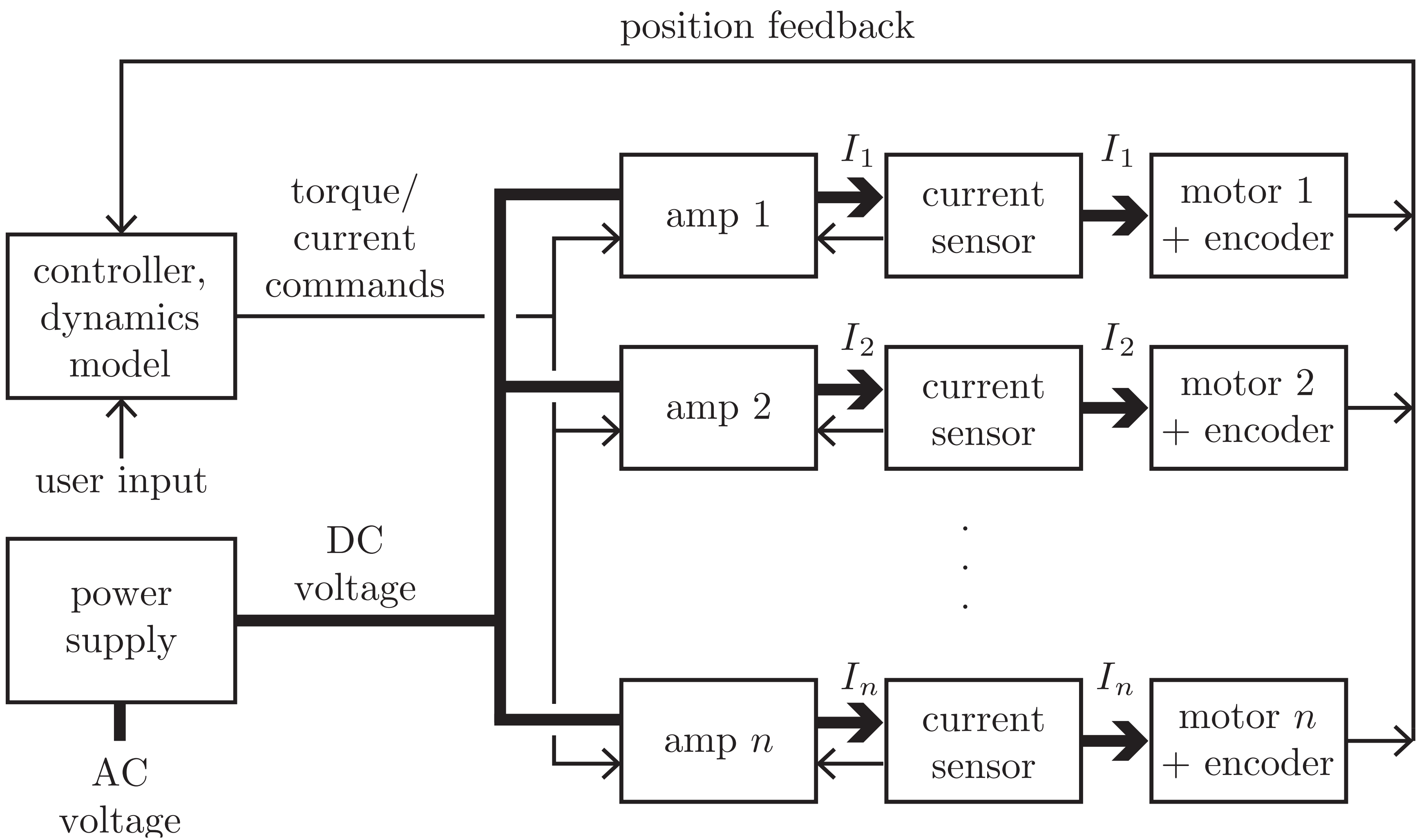
# Robotic control




(a)

[Lunch, Park, 2017]

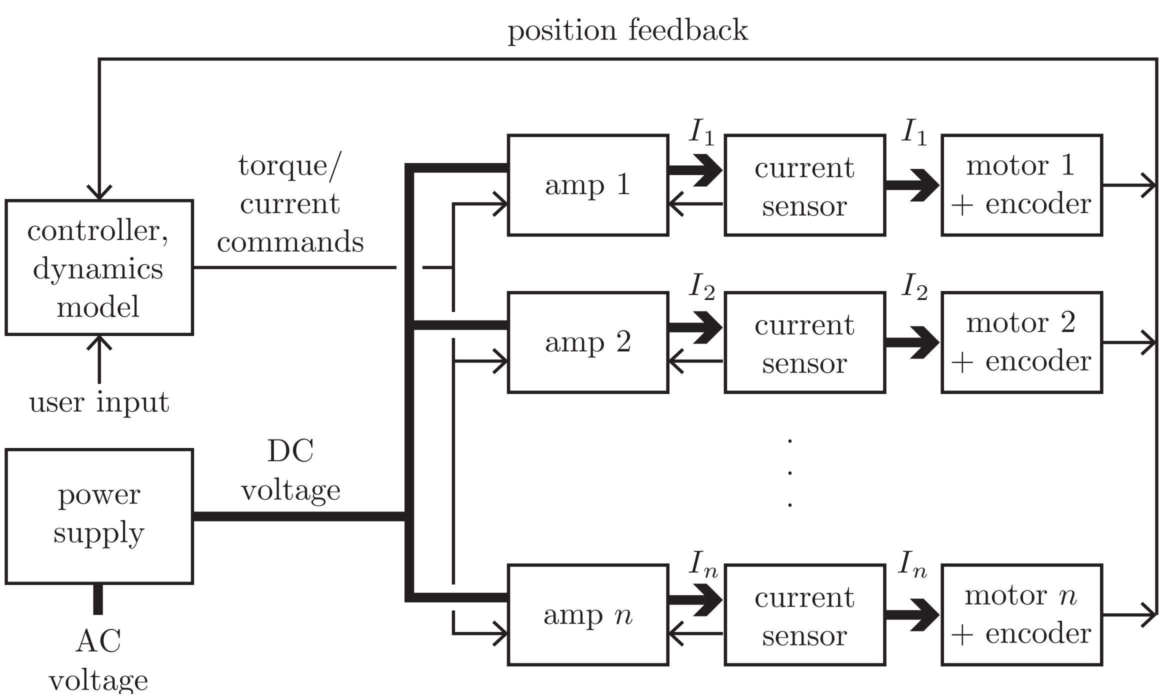
# Robotic control




[Lunch, Park, 2017]

# Robotic control

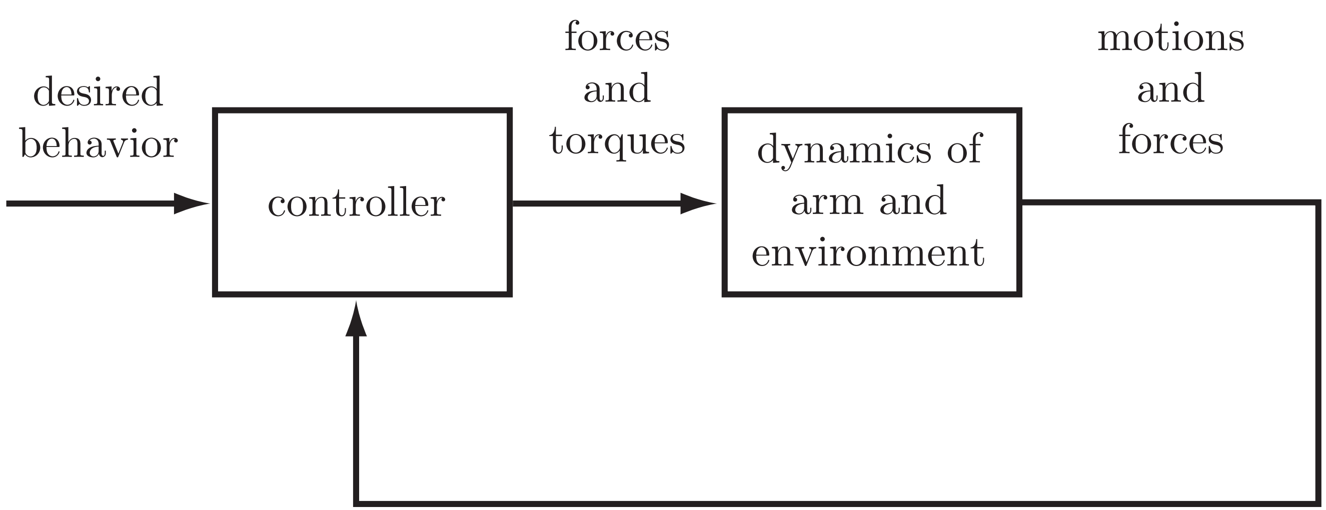
- actuators enable commanding a torque by commanding a current… in good approximation
- => control signal: torque




[Lunch, Park, 2017]

# Robotic control

- $\dot{x} = f(t, x, u)$
- state variable x(t) = output: kinematic state of robot
- desired trajectory: x(t)_d (from motion planning)
- control signal: u = torques




(b)

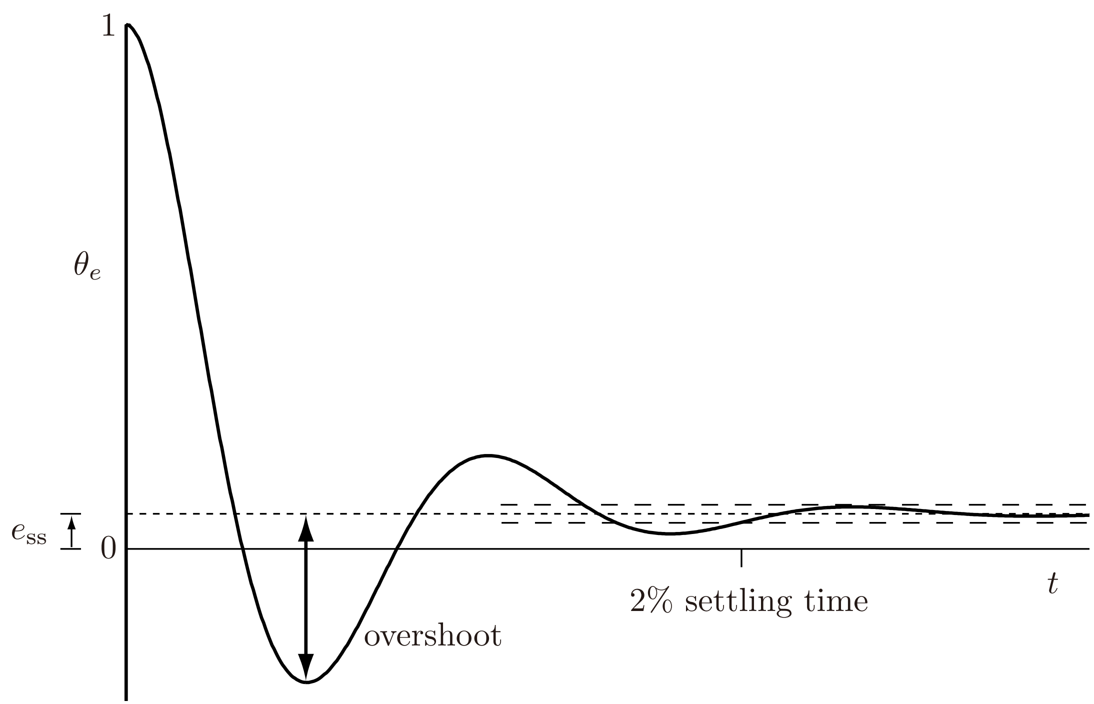
[Lunch, Park, 2017]

# Robotic control

- theoretical core of robotic control theory:
- devise control laws that lead to stable control
- (approximate these numerically on hardware and computers)

# Robotic control

- task: generate joint torques that produce a desired motion… $\theta_d(t)$
- <=> make error: $e(t) = \theta(t) - \theta_d(t)$ small

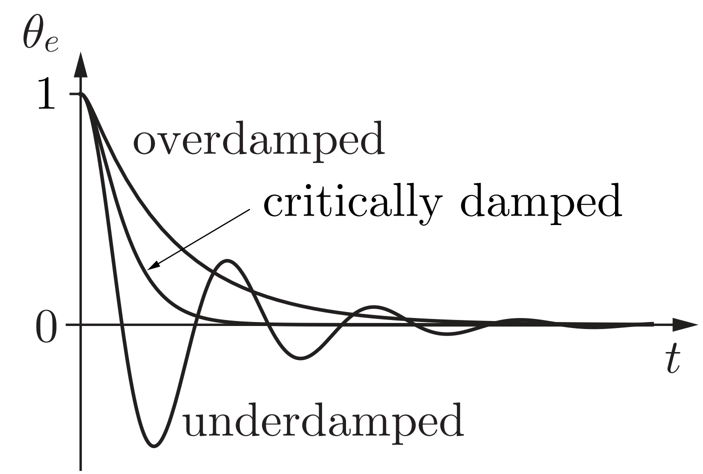


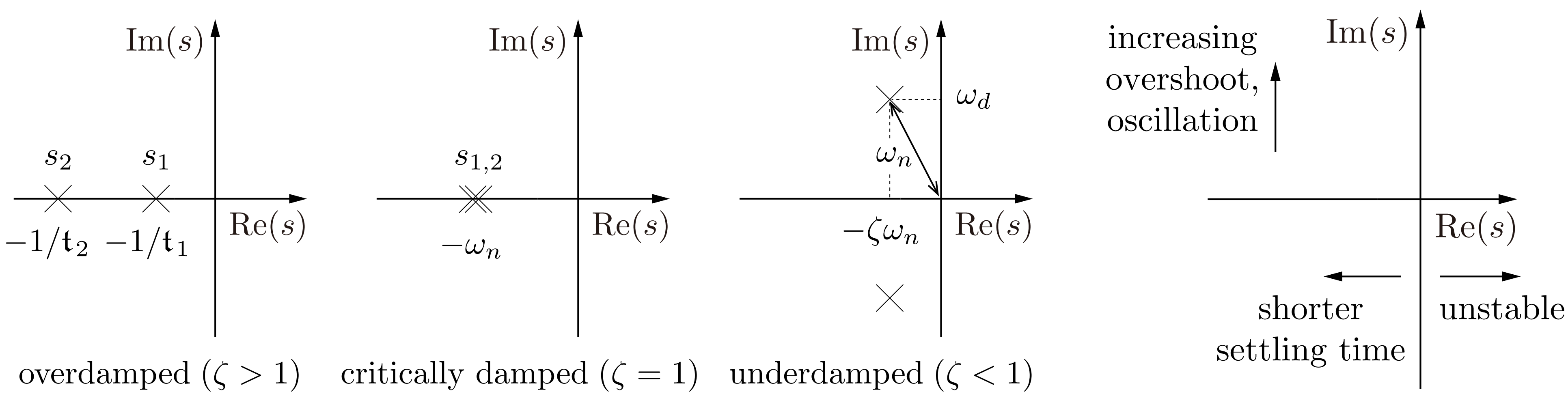

for a constant desired state

[Lunch, Park, 2017]

# Toy example

- analysis by Eigenvalues s

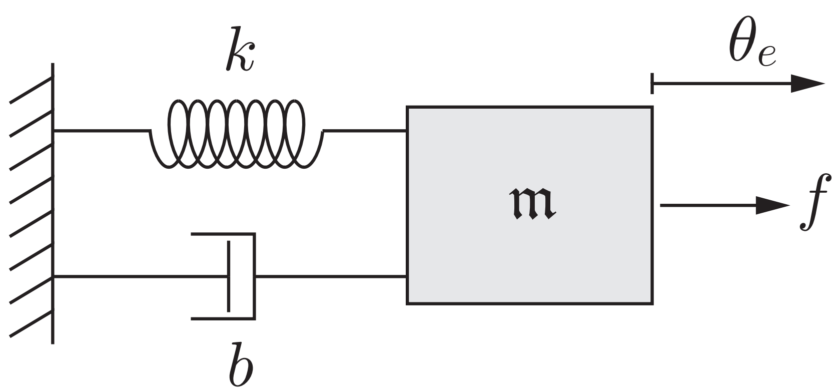


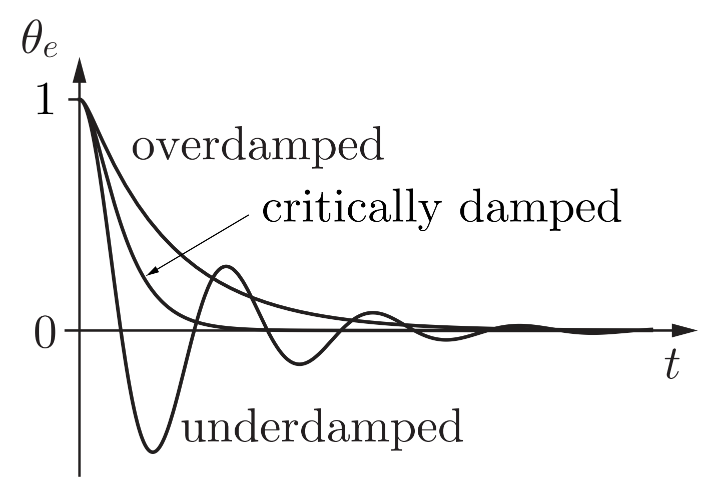




[Lunch, Park, 2017]

# Toy example

- linear mass spring model
  $m\ddot{e}(t) + b\dot{e}(t) + ke(t) = 0$

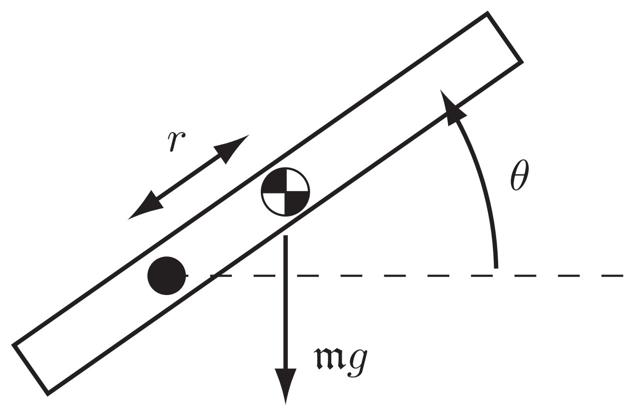



[Lunch, Park, 2017]

# Motion control single joint

- $\tau = M\ddot{\theta} + mgr\cos(\theta) + b\dot{\theta}$




[Lunch, Park, 2017]

# Motion control single joint

- $\tau = M\ddot{\theta} + mgr\cos(\theta) + b\dot{\theta}$
- feedback PID controller
- $\tau = K_p\theta_e + K_d\dot{\theta}_e + K_i \int \theta(t')dt'$

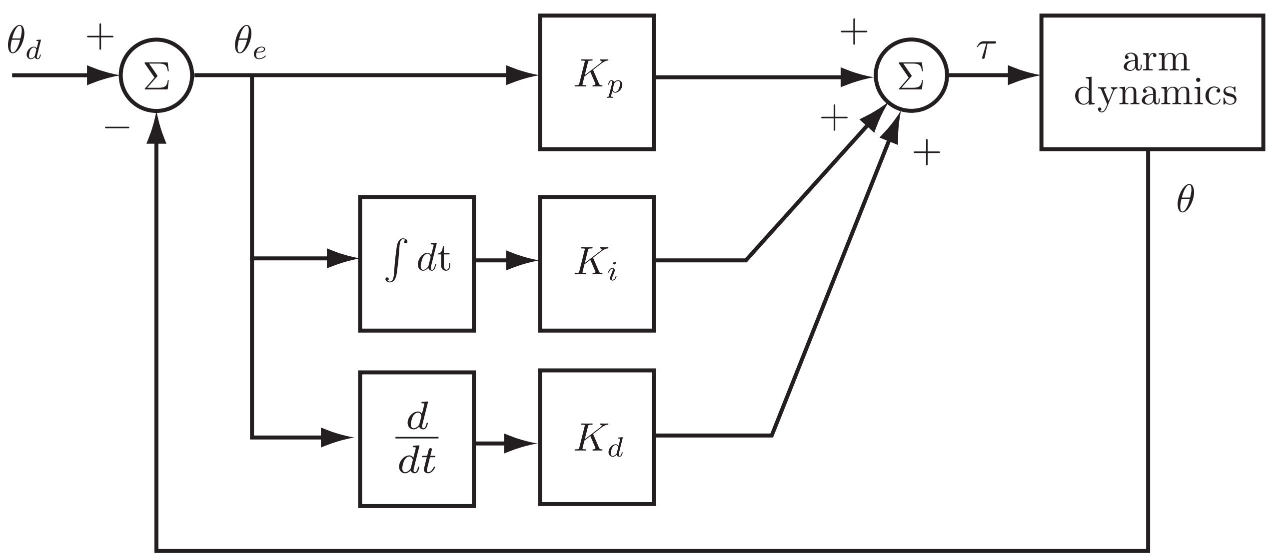
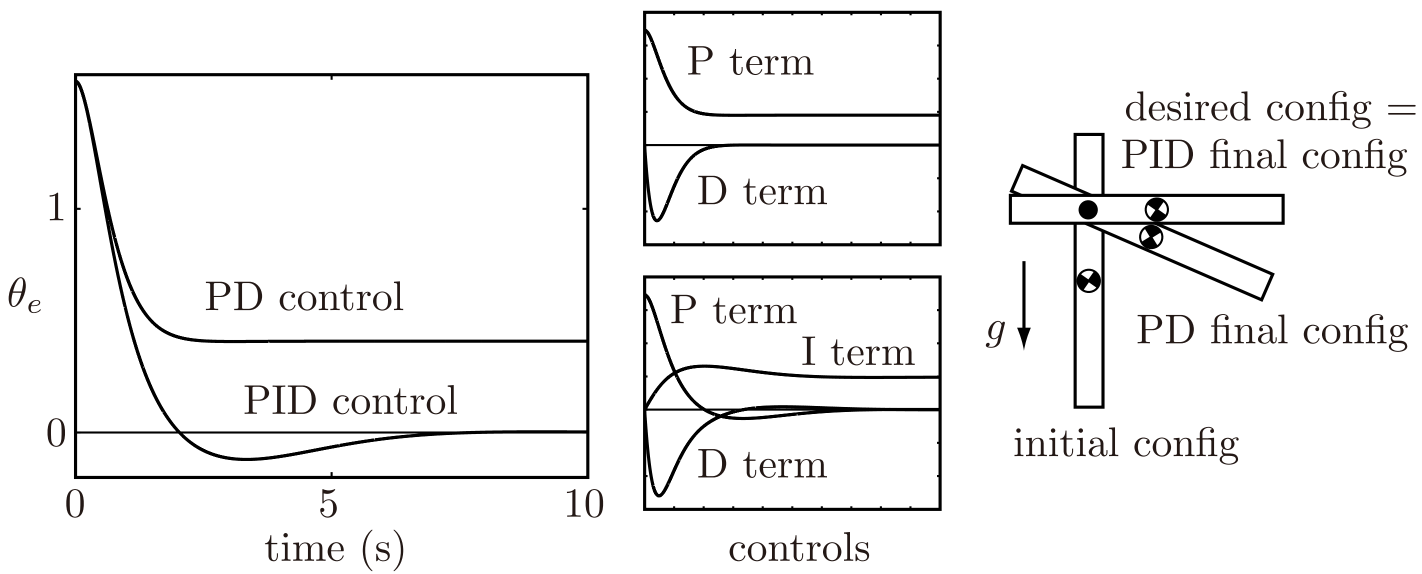



**Figure 11.12:** Block diagram of a PID controller.

[Lunch, Park, 2017]

# Motion control single joint

- $\tau = M\ddot{\theta} + mgr\cos(\theta) + b\dot{\theta}$
- feedback PID controller
- $\tau = K_p\theta_e + K_d\dot{\theta}_e + K_i \int \theta(t')dt'$




[Lunch, Park, 2017]

# Motion control single joint

- $\tau = M\ddot{\theta} + mgr\cos(\theta) + b\dot{\theta} = M\ddot{\theta} + h(\theta, \dot{\theta})$
- feedforward controller
- has model of the dynamics:
- $\tau = \tilde{M}\ddot{\theta} + \tilde{h}(\theta, \dot{\theta})$
- compute forward torque
- $\tau(t) = \tilde{M}(\theta_d(t))\ddot{\theta}_d(t) + \tilde{h}(\theta_d, \dot{\theta}_d)$
- if model exact: $\ddot{\theta} \approx \ddot{\theta}_d$

[Lunch, Park, 2017]

# Motion control single joint

- feedforward controller
- if model wrong..

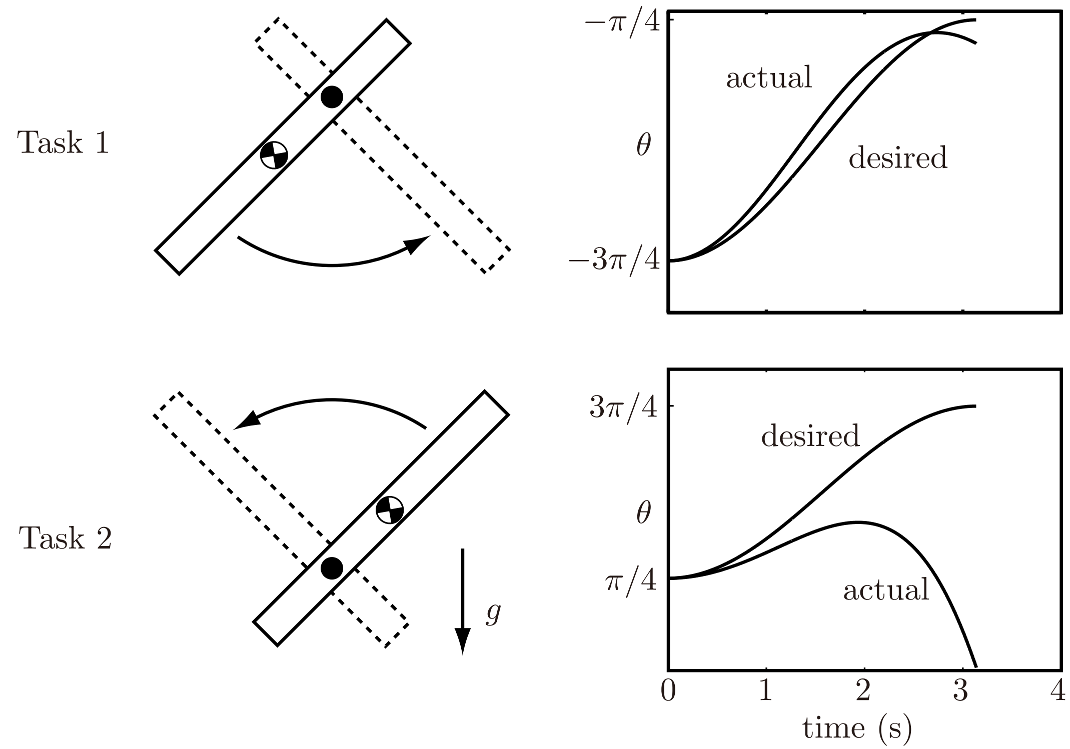



**Figure 11.17:** Results of feedforward control with an incorrect model: $\tilde{r} = 0.08$ m, but $r = 0.1$ m. The desired trajectory in Task 1 is $\theta_d(t) = -\pi/2 - (\pi/4)\cos(t)$ for $0 \le t \le \pi$. The desired trajectory for Task 2 is $\theta_d(t) = \pi/2 - (\pi/4)\cos(t), 0 \le t \le \pi$.

[Lunch, Park, 2017]

# Motion control single joint

- combined feedforward and feedback PID controller …
- $\tau = \tilde{M}(\theta)\left(\ddot{\theta}_d + K_p\theta_e + K_d\dot{\theta}_e + K_i\int\theta(t')dt'\right) + \tilde{h}(\theta, \dot{\theta})$
- = inverse dynamics or computed torque controller

[Lunch, Park, 2017]

# Control of multi-joint arm

- generate joint torques that produce a desired motion… $\theta_d$
- error $\theta_e = \theta - \theta_d$
- PD control $\tau = K_p\theta_e + K_e\dot{\theta}_d + K_i\int\theta_e(t')dt'$
- => controlling joints independently

$$\boxed{M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + N(\theta,\dot{\theta}) = \tau}$$

# Control of multi-joint arm

- there are many more sophisticated models that compensate for interaction torques/ inertial coupling… e.g. computed torque control (inverse dynamics)
- $$\tau = \underbrace{M(\theta)\ddot{\theta}_d + C\dot{\theta} + N}_{\tau_{\mathrm{ff}}} + \underbrace{M(\theta)\left(-K_v\dot{e} - K_p e\right)}_{\tau_{\mathrm{fb}}}.$$

$$\boxed{M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + N(\theta,\dot{\theta}) = \tau}$$

$$\Rightarrow \quad (\ddot{\theta} - \ddot{\theta}_d) = \ddot{e} = -K_v\dot{e} - K_p e$$

# Control of multi-joint arm

- … computed torque control (inverse dynamics)
- but: computational effort can be considerable… simplifcation.. only compensate for gravity…
- $\tau = K_p \theta_e + K_e \dot{\theta}_d + K_i \int \theta_e(t')dt' + \tilde{N}(\theta)$

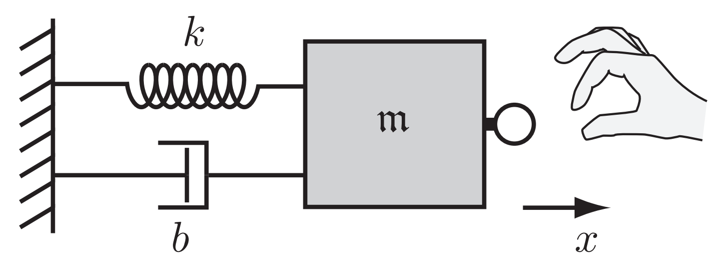
$$\boxed{M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + N(\theta, \dot{\theta}) = \tau}$$

# Problem: contact forces

- as soon as the robot arm makes contact, a host of problems arise from the contact forces and their effect on the arm and controller…
- need compliance… resisting to a well-defined degree
- => impedance control… research frontier

# Impedance

- to control movement well.. need a very stiff arm and "stiff" controller (high gain K_x)
- to control force/limit force (e.g. for interaction with surfaces or humans) you need a relatively soft arm and soft controller
- design system to give hand, x, a desired impedance: m, b, k in
- $m\ddot{x} + b\dot{x} + kx = f$
- where f is force applied..



# Operational space formulation

- Euler-Langrage in end-effector space
- $$\Lambda(\boldsymbol{x})\ddot{\boldsymbol{x}} + \mu(\boldsymbol{x}, \dot{\boldsymbol{x}}) + \boldsymbol{p}(\boldsymbol{x}) = \boldsymbol{F}$$
- with F forces acting on the end-effector
- equivalent dynamics in joint space
- $$A(q)\ddot{q} + \boldsymbol{b}(q, \dot{q}) + \boldsymbol{g}(q) = \Gamma$$
- with joint torques $\Gamma = J^T(q)\boldsymbol{F}$

[Khatib, 1987]

# Impedance control

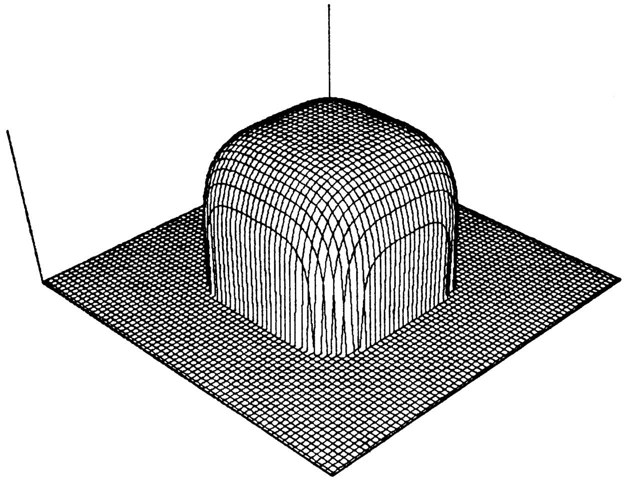
- Hogan 1985…
- $\tau = J^T(\theta)\Big(\tilde{\Lambda}(\theta)\ddot{x} + \tilde{\eta}(\theta, \dot{x}) - (M\ddot{x} + B\dot{x} + Kx)\Big)$

# Link to movement planning

- where does “desired trajectory” come from?
- typically from end-effector level movement planning
    - then add an inverse kinematic…
    - which can be problematic
- alternative: planning and control in end-effector space

# Operational space formulation

- in end-effector space add constraints as contributions to the "virtual forces"
- $$\mathbf{F}^*_{\mathbf{x}_d} = -\operatorname{grad}[U_{\mathbf{x}_d}(\mathbf{x})],$$
  $$\mathbf{F}^*_{O} = -\operatorname{grad}[U_{O}(\mathbf{x})].$$



- $$\Lambda(\boldsymbol{x})\ddot{\boldsymbol{x}} + \mu(\boldsymbol{x},\ \dot{\boldsymbol{x}}) + \boldsymbol{p}(\boldsymbol{x}) = \boldsymbol{F}$$

[Khatib, 1986,1987]

# Optimal control

- given a plant $\dot{x} = f(x, u)$
- find a control signal $u(t)$
- that moves the state from an final position $x_i(0)$ to a terminal position $x_f(t_f)$ within the time $t_f$
- a (difficult) planning problem!
- minimize a cost function to find such a signal

# How does the human (or other animal) movement system generate movement?

- mechanics:… biomechanics
- actuator: muscle
- control?
- optimal control?