Dynamical systems tutorial

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Dynamical systems: Tutorial

- the word “dynamics”
  - time-varying measures
  - range of a quantity
  - forces causing/accounting for movement => dynamical systems

- dynamical systems are the universal language of science
  - physics, engineering, chemistry, theoretical biology, economics, quantitative sociology, ...
time-variation and rate of change

- variable $x(t)$;
- rate of change $\frac{dx}{dt}$
dynamical system: relationship between a variable and its rate of change
(linear) dynamical system

\[ \frac{dx}{dt} = f(x) \]
exponential relaxation to attractors

\[ \tau \, \frac{dx}{dt} = -x \]

\[ x(t) \]

\[ x(0) \]

\[ x(0)/e \]

\[ x(\tau) \]

\[ x(2\tau) \]

\[ \Rightarrow \text{time scale} \]
(nonlinear) dynamical system

dx/dt = f(x)
dynamical system

- present determines the future
- given initial condition
- predict evolution (or predict the past)

\[
dx/dt = f(x)\]
dynamical systems

- \( x \): spans the state space (or phase space)
- \( f(x) \): is the "dynamics" of \( x \) (or vector-field)
- \( x(t) \) is a solution of the dynamical systems to the initial condition \( x(0) = x_0 \)
  - if its rate of change = \( f(x) \)
  - and \( x(0) = x_0 \)
Dynamical systems

as differential equations: initial state determines the future

\[ \dot{x} = f(x) \]
Dynamical systems

A vector of initial states determines the future: systems of differential equations:

\[ \dot{x} = f(x) \quad \text{where} \quad x = (x_1, x_2, \ldots, x_n) \]
Dynamical systems

Continuously many variables \( x(y) \) determine the future = an initial function \( x(y) \) determines the future

Partial differential equations

\[
\dot{x}(y, t) = f \left( x(y, y), \frac{\partial x(y, t)}{\partial y}, \ldots \right)
\]

Functional differential equations

\[
\dot{x}(y, t) = \int dy' g \left( x(y, t), x(y', t) \right)
\]
Dynamical systems

- a piece of past trajectory determines the future
- delay differential equations
- functional differential equations

\[
\dot{x}(t) = f(x(t - \tau)) \\
\dot{x}(t) = \int_{t'}^{t} dt'f(x(t'))
\]
sample time discretely

compute solution by iterating through time

\[ \dot{x} = f(x) \]

\[ t_i = i \cdot \Delta t; \quad x_i = x(t_i) \]

\[ \dot{x} = \frac{dx}{dt} \approx \frac{\Delta x}{\Delta t} = \frac{x_{i+1} - x_i}{\Delta t} \]

\[ x_{i+1} = x_i + \Delta t \cdot f(x_i) \]

[forward Euler]
linear dynamics

=> simulation
fixed point, to which neighboring initial conditions converge = attractor

dx/dt = f(x)
A fixed point is a constant solution of the dynamical system

\[ \dot{x} = f(x) \]

\[ \dot{x} = 0 \Rightarrow f(x_0) = 0 \]
stability

Mathematically, really: asymptotic stability

Defined: a fixed point is asymptotically stable, when solutions of the dynamical system that start nearby converge in time to the fixed point.
the mathematical concept of stability (which we do not use) requires only that nearby solutions stay nearby

Definition: a fixed point is **unstable** if it is not stable in that more general sense,

that is: if nearby solutions do not necessarily stay nearby (may diverge)
linear approximation near attractor

- non-linearity as a small perturbation/deformation of linear system

=> non-essential non-linearity
stability in a linear system

if the slope of the linear system is negative, the fixed point is (asymptotically stable)

d\lambda/dt = f(\lambda)
if the slope of the linear system is positive, then the fixed point is unstable

\[ d\lambda/dt = f(\lambda) \]
stability in a linear system

If the slope of the linear system is zero, then the system is indifferent (marginally stable: stable but not asymptotically stable)

\[ \frac{d\lambda}{dt} = f(\lambda) \]
stability in linear systems

generalization to multiple dimensions

- if the real-parts of all Eigenvalues are negative: stable
- if the real-part of any Eigenvalue is positive: unstable
- if the real-part of any Eigenvalue is zero: marginally stable in that direction (stability depends on other eigenvalues)
stability in nonlinear systems

- stability is a local property of the fixed point

- => linear stability theory

  - the eigenvalues of the linearization around the fixed point determine stability

  - all real-parts negative: stable

  - any real-part positive: unstable

  - any real-part zero: undecided: now nonlinearity decides (nonhyberpolic fixed point)
stability in nonlinear systems

- all real-parts negative: stable
- any real-part positive: unstable
stability in nonlinear systems

- any real-part zero: undecided; now nonlinearity decides (non-hyperpolic fixed point)
look now at families of dynamical systems, which depend (smoothly) on parameters

ask: as the parameters change (smoothly), how do the solutions change (smoothly?)

smoothly: topological equivalence of the dynamical systems at neighboring parameter values

bifurcation: dynamical systems NOT topological equivalent as parameter changes infinitesimally
bifurcation

\[ \frac{dx}{dt} = f(x) \]

attractor 1

attractor 2
bifurcation

A bifurcation is a qualitative change of dynamics (change in number, nature, or stability of fixed points) as the dynamics changes smoothly.

dx/dt = f(x)
tangent bifurcation

the simplest bifurcation (co-dimension 0): an attractor collides with a repellor and the two annihilate

\[ \frac{dx}{dt} = f(x) \]

Diagram: Two attractors labeled attractor 1 and attractor 2, with the function \( f(x) \) depicted as a system of curves in the phase space.
local bifurcation

dx/dt = f(x)

attractor 1

attractor 2
reverse bifurcation

changing the dynamics in the opposite direction

dx/dt = f(x)

attractor 1

attractor 2
bifurcations are instabilities

that is, an attractor becomes unstable before disappearing

(or the attractor appears with reduced stability)

formally: a zero-real part is a necessary condition for a bifurcation to occur
tangent bifurcation

normal form of tangent bifurcation

\[ \dot{x} = \alpha - x^2 \]

(=simplest polynomial equation whose flow is topologically equivalent to the bifurcation)

\[
\begin{align*}
\text{fixed point} & \quad x_0 = \sqrt{\alpha} \\
\text{dx/dt} & \\
\alpha \text{ positive} & \quad x \\
\alpha = 0 & \quad x \\
\alpha \text{ negative} & \\
\alpha & \\
\text{stable} & \\
\text{unstable} &
\end{align*}
\]
Hopf theorem

- When a single (or pair of complex conjugate) eigenvalue crosses the imaginary axis, one of four bifurcations occur:
  - tangent bifurcation
  - transcritical bifurcation
  - pitchfork bifurcation
  - Hopf bifurcation
transcritical bifurcation

\[ \dot{x} = \alpha x - x^2 \]

- normal form

\[ x = \alpha x - x^2 \]

- fixed point

- positive

- negative

- \( \alpha = 0 \)

- stable

- unstable

- \( \alpha \)
pitchfork bifurcation

normal form

\[ \dot{x} = \alpha x - x^3 \]

\( \alpha \) positive

\( \alpha \) negative

\( \alpha = 0 \)

fixed point

\[ \dot{x} = -2x_0 x = -2\sqrt{\alpha} x \]

stable

unstable
Hopf: need higher dimensions
2D dynamical system: vector-field

\[ \dot{x}_1 = f_1(x_1, x_2) \]
\[ \dot{x}_2 = f_2(x_1, x_2) \]
\[ \dot{x}_1 = f_1(x_1, x_2) \]
\[ \dot{x}_2 = f_2(x_1, x_2) \]
fixed point, stability, attractor

\[ \dot{x}_1 = f_1(x_1, x_2) \]
\[ \dot{x}_2 = f_2(x_1, x_2) \]
**Hopf bifurcation**

\[ \dot{r} = \alpha r - r^3 \]

\[ \dot{\phi} = \omega \]

**normal form**
forward dynamics

given known equation, determined fixed points / limit cycles and their stability

more generally: determine invariant solutions (stable, unstable and center manifolds)
inverse dynamics

given solution, find the equation…

d this is the problem faced in design of behavioral dynamics…
inverse dynamics: design

- in the design of behavioral dynamics… you may be given:
  - attractor solutions/stable states
  - and how they change as a function of parameters/conditions
  => identify the class of dynamical systems using the 4 elementary bifurcations

- and use normal form to provide an exemplary representative of the equivalence class of dynamics