Lecture 6
Differential Equations

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Computer Science and Mathematics
Preparatory Course

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Overview

1. Motivation

2. Mathematics
   - Solving Differential Equations
   - Qualitative Analysis
   - Numerical Approximation

3. Tasks
The Vehicle’s Behavior as Function of Angle Change
The Vehicle’s Behavior as Function of Angle Change
The Vehicle’s Behavior as Function of Angle Change
**Motivation**

The Vehicle’s Behavior as Function of Angle Change

![Diagram showing the vehicle's behavior change due to angle change](image)
The Vehicle’s Behavior as Function of Angle Change

The vehicle’s change in angle depends on its current sensor input.

\[
\frac{d\beta}{dt} = -S_L + S_R,
\]

where \( t \) describes time and \( S_L \), \( S_R \) left and right sensor values.
The Vehicle’s Behavior as Function of Angle Change

- The vehicle’s change in angle depends on its current sensor input.
- The following equation may describe its behavior:

\[
\frac{d\beta}{dt} = -S_L + S_R,
\]

where \(t\) describes time and \(S_L, S_R\) left and right sensor values.
Differential Equation as Rule System

A differential equation describes how the rate of change of a system depends on its current state. For example:

\[ f'(x) = 4f(x) + 5 = g(f(x)) \quad \text{with} \quad g(x) = 4x + 5 \]
Differential Equation as Rule System

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- A differential equation describes how a system should change in a given state.

- Brief oversimplification:

  A differential equation describes rules for the future
Solving Differential Equations

Given a differential equation of the form \( f'(x) = g(f(x)) \) ... the original function \( f(x) \) is usually not known.

Solving a differential equation describes the process of finding an \( f(x) \) that follows the above rule for all \( x \)

Differential equations entail two equations

1. The function \( g(f(x)) \) governing the rate of change
2. The function \( f(x) \) describing the overall behavior
Derivative vs. Differential equation

- \( f'(x) = cx \)

- \( f''(x) = cf(x) \)
Derivative vs. Differential equation

- \( f'(x) = cx \)
  - The rate of change depends on a fixed rule depending on \( x \)

- \( f''(x) = cf(x) \)
Derivative vs. Differential equation

- \( f'(x) = cx \)
  - The rate of change depends follows a fixed rule depending on \( x \)
  - The solution can be described by the antiderivative \( f(x) = \frac{1}{2} cx^2 \)

- \( f'(x) = cf(x) \)

- The rate of change is a scaled version of the function itself:
  - \( g(f(x)) = cf(x) \)
- The only function that stays the same when differentiated is the exponential function \( e^x \)

- Considering the chain rule the derivative of \( e^{cx} \) is exactly \( c e^{cx} \) therefore \( f(x) = c e^{cx} \)

- Usually a differential equation is not that easily solvable
Derivative vs. Differential equation

- $f'(x) = cx$
  - The rate of change depends follows a fixed rule depending on $x$
  - The solution can be described by the antiderivative $f(x) = \frac{1}{2}cx^2$
  - This is not a differential equation as no $f(x)$ is on the right side

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    \[ f(x) = ce^{cx} \]
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Dynamical Systems Theory

- Mathematicians want to find solutions to particular differential equations

- **Dynamical Systems Theory** is concerned with analyzing the qualitative behavior of the system
Qualitative Behavior of Differential Equations

\[ f'(x) = y' = \frac{dy}{dx} = -f(x) \]
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Initial Condition Matters

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Back to the Braitenberg Vehicle

We govern the vehicles behavior with a differential equation:

\[
\frac{d\beta}{dt} = -\beta - S_L + S_R,
\]
Back to the Braitenberg Vehicle

▶ We govern the vehicles behavior with a differential equation

\[
\frac{d\beta}{dt} = -\beta - S_L + S_R,
\]

▶ Adding an attractor gives the vehicle a preferred orientation
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2. Mathematics
   - Solving Differential Equations
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   - Numerical Approximation

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Euler Approximation

\[
\frac{dy}{dx} = y \quad y(0) = 1
\]

\[y(x) = e^x\]
Euler Approximation

\[ \frac{dy}{dx} = y \quad y(0) = 1 \]

\[ \Delta x = 1 \]

\[ \begin{array}{c|c|c}
  x & y & \frac{dy}{dx} \\
  0 & 1 & 1 \\
\end{array} \]

\[ y(x) = e^x \]
Euler Approximation

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Varying the stepsize

\[ \frac{dy}{dx} = y \quad y(0) = 1 \]

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Varying the stepsize

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\frac{dy}{dx} = y \quad y(0) = 1 \\
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$y(x) = e^x$
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\[y(x) = e^x\]
Euler Approximation in Words

1. Start with a certain value for $x$ and $y$ and the differential equation $\frac{dy}{dx} = \ldots$ you want to approximate

2. Decide for a step size that determines the accuracy of your approximation

3. Repeat as long as you like:
   3.1 Use the current $y$-value to calculate the current rate of change $\frac{dy}{dx}$
   3.2 Calculate the next $y$-value by taking the current $y$-value and adding to it the rate of change times the step size
   3.3 Increase $x$ by the step size
Task Template

- Download the archive `task_template_6.zip` from the course homepage. Extract it into a folder of your choice.

- The archive contains `task_61.py`, `student_code_61.py`, and `braitenberg.png`.

- You only need to edit code in the `student_code` file.

**Explain Task Template!**
Tasks

1. Change the behavior of the vehicle by implementing the function `calc_angle_change`.
   - `current_angle` is the current orientation of the vehicle in degree.
   - `left_sensor_values` and `right_sensor_values` are the measured values of the sensors. They increase the closer they are to an obstacle.
   - First make the angle change dependent on the current sensor values. How can you make the vehicle avoid obstacles?
   - Let your change in the angle depend on the current angle itself. Set an attractor at 45°, such that the vehicle will turn towards 45° degrees in the absence of obstacles.
   - What do you need to change to make the vehicle go towards obstacles?
2. Imagine the differential equation $\frac{dy}{dx} = -y + 20$, where $y$ describes the heading of your vehicle. You know that your initial orientation is $y(0) = 40^\circ$.

- Use the euler approximation method to calculate the $y$-values up to an $x$-value of 4. Use a step size of 0.5.
- Implement the euler approximation method in a python script, which can go to a certain $x$-value with a certain step size.
- Hint: You can reuse a lot of the code from yesterday.
- Calculate how long your for-loop has to run depending on the desired $x$-value and your step size.
- Save your results in three different lists. One for the $x$-values, one for the $y$-values and one for the $\frac{dy}{dx}$ term.
- Plot your $x$-values against your $y$-values and your $y$-values against your $\frac{dy}{dx}$-values. (See the next slide for plotting commands.)
Matplotlib.pyplot

The pyplot submodule

# A submodule can be imported with the . operator
import matplotlib.pyplot as plt
# The as operator allows renaming for convenience
xValues = [1,1,2,3,5,8,13]
yValues = [3,4,7,6,9,10,12]
plt.plot(xValues,yValues) #plots lines
# This generates the plot and .show() displays it
plt.show()

#plots points and lines
plt.plot(xValues,yValues,linestyle = "-", marker="o")
plt.show()