Lecture 5
Integration

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Computer Science and Mathematics
Preparatory Course

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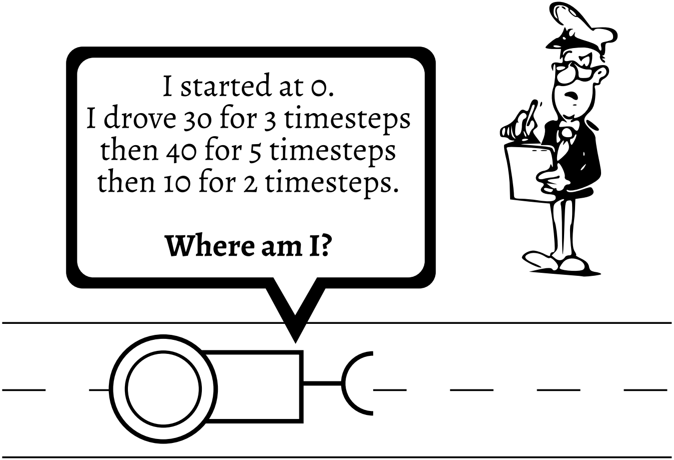
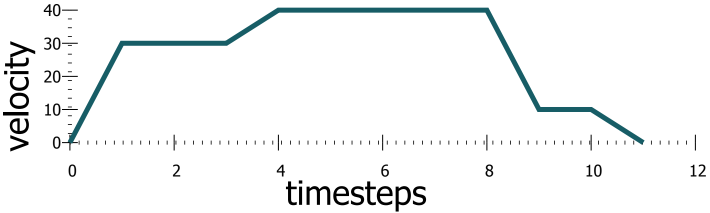
Reverting Differentiation



From Velocity to position



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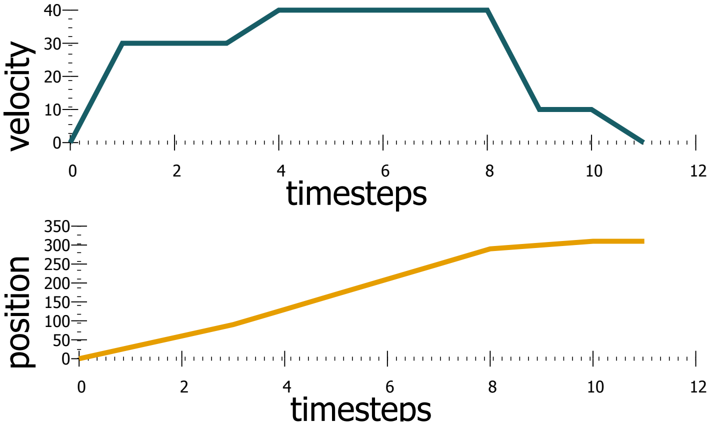
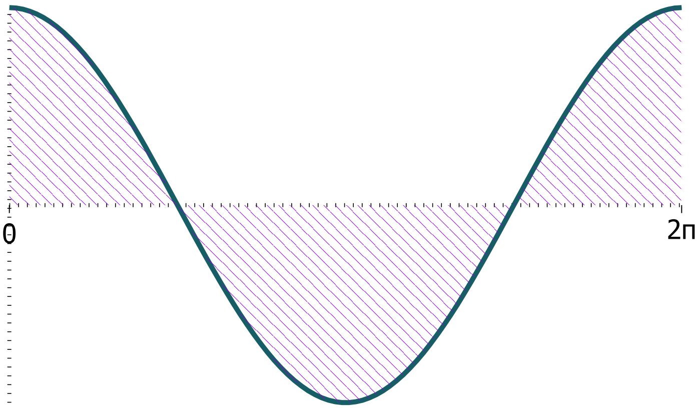


Overview

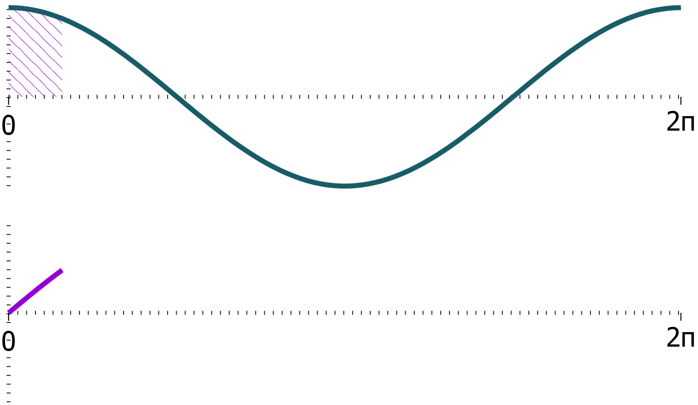
Integral as Area

$f(x) = cos(x)$ $\int_0^{2\pi} cos(x)$



Integral as Function

$f(x) = cos(x)$ $\int_0^{2\pi} cos(x)$



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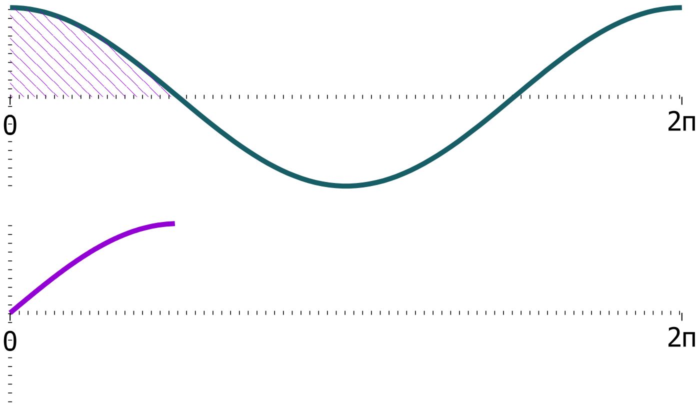

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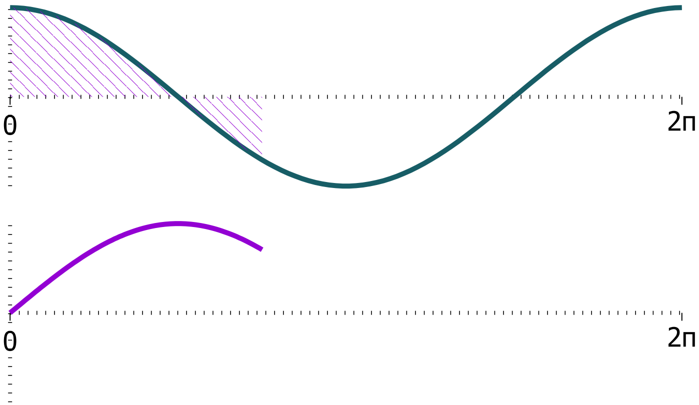

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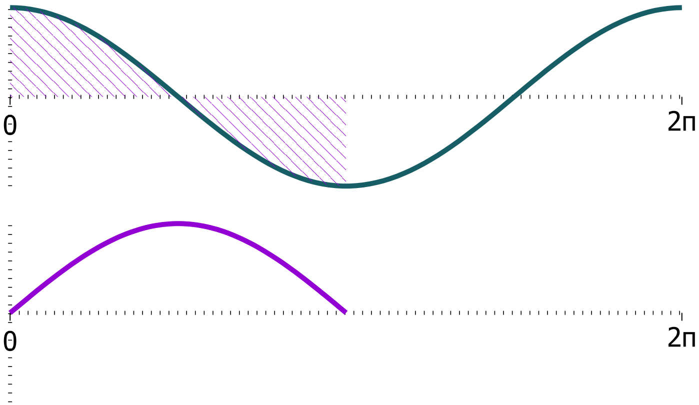

Integral as Function

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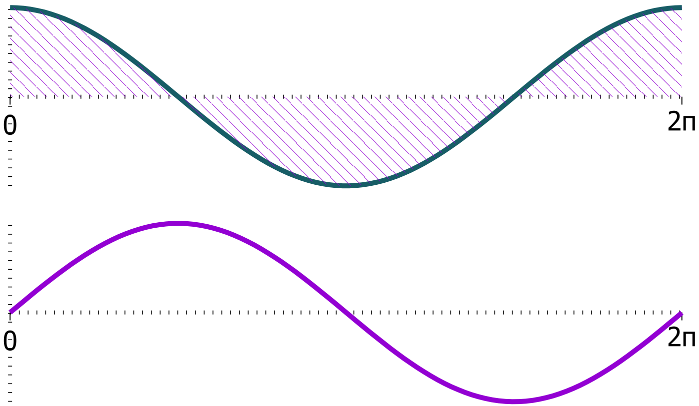
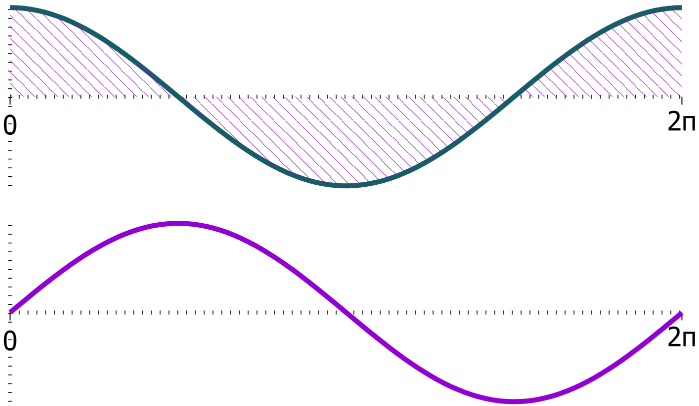


Integral as Function

$f(x) = cos(x)$ $\int_0^{2\pi} cos(x) = sin(2\pi)$



Geometric Definition

Definite Integral

The **definite integral** of a function $f(x)$ between the **lower boundary** a and the **upper boundary** b

$$\int_a^b f(x)$$

is defined as the size of the area between f and the x-axis inside the boundaries. Areas above the x-Axis are considered positive and areas below negative.

The Antiderivative

Definition

If f is a function with domain $[a, b] \to \mathbb{R}$ and there is a function F, which is differentiable in the interval $[a, b]$ with the property that

$$F'(x) = f(x),$$

then F is considered the **antiderivative** of f

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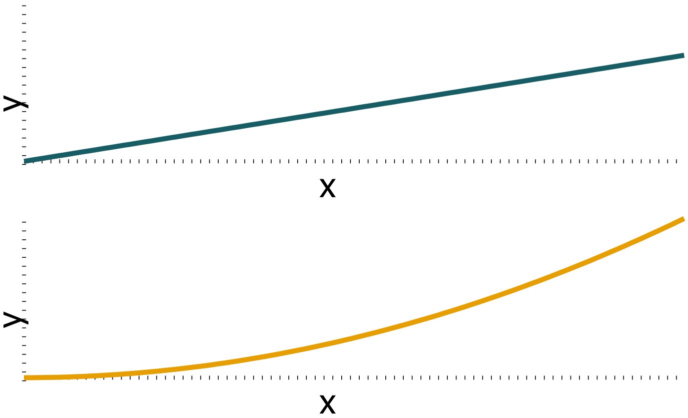
Properties of the antiderivative

- Differentiation removes constants, because of that an antiderivative is described by a family of functions $F(x) + c$
- Unlike with differentiation there are no fixed rules to compute an antiderivative from a given f

A function and its antiderivative

$f(x) = x$ $\qquad$ $F(x) = \frac{1}{2}x^2$



Calculating the Integral

Calculating the area in an interval

If f is integrable and continuous in $[a, b]$. Then the following holds for each antiderivative F of f

$$\int_a^b f(x)dx = \int_a^b F'(x)dx = [F(x)]_a^b = F(b) - F(a)$$

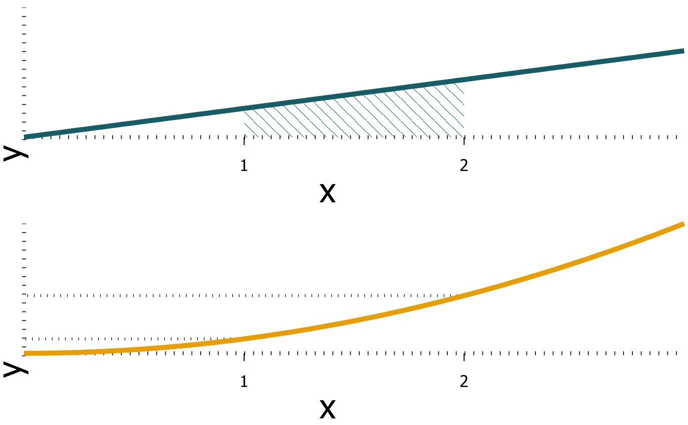
Example:

- Area under $f(x)$ between values 1 and 2

$$\int_1^2 xdx = \left[\frac{1}{2}x^2\right]_1^2 = \frac{1}{2}2^2 - \frac{1}{2}1^2 = 1.5$$

Integral as area underneath a function

$f(x) = x$ $F(x) = \frac{1}{2}x^2$ $\int_1^2 f(x)dx = F(2) - F(1)$



The Integral as Linear Operator

Integration Rules

- **Summation**

$$\int_a^b f(x) + g(x) = \int_a^b f(x) + \int_a^b g(x)$$

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$$\int_a^b f(x) + g(x) = \int_a^b f(x) + \int_a^b g(x)$$

- **Scalar Multiplication**

$$\int_a^b cf(x) = c\int_a^b f(x)$$

The Integral as Linear Operator

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- **Summation**

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$$\int_a^b cf(x) = c \int_a^b f(x)$$

- **Boundary Transformations**

$$\int_a^b f(x) + \int_b^c f(x) = \int_a^c f(x) \quad \wedge \quad \int_a^b f(x) = -\int_b^a f(x)$$

Improper Integrals

Infinite Intervals

It is possible to calculate the area in infinitely large intervals. Intervals with an infinite boundary are called **Improper Integrals**

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

Example:

- Convergent improper integral

$$\int_1^{\infty} x^{-2}dx = \lim_{b\to\infty} \int_1^b x^{-2}dx = \lim_{b\to\infty} \left[-x^{-1}\right]_1^b = \lim_{b\to\infty} \left(-b^{-1} + 1\right) = 1$$

Numerical Approximation

- It is not trivial to find the antiderivative to a given function or a given dataset

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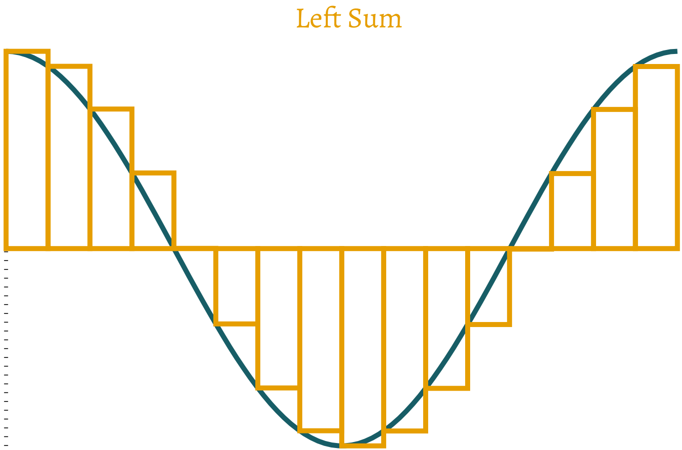
Partioning an Interval

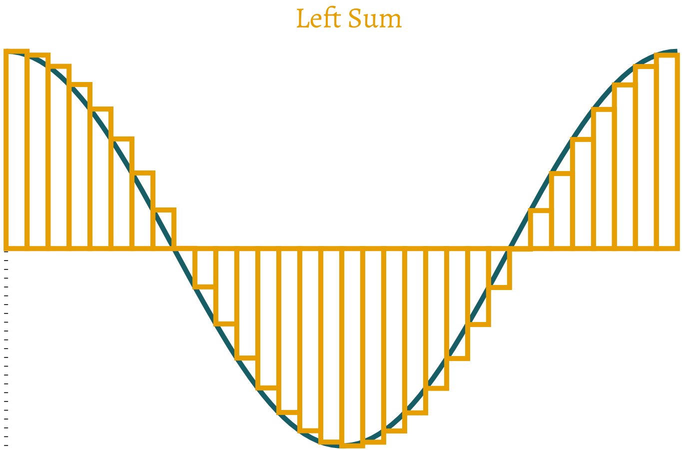
Let $(x_i)_{i \in [a,b]}$ be a sequence of n increasing numbers in $[a, b]$ with fixed distance h between x_i and $x_i + 1$ for all x_i.

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

Riemann Sums



Riemann Sums



Riemann Sums

Left and Right Sum

- For an interval $[x_i, x_{i+1}]$ and a function f the functions

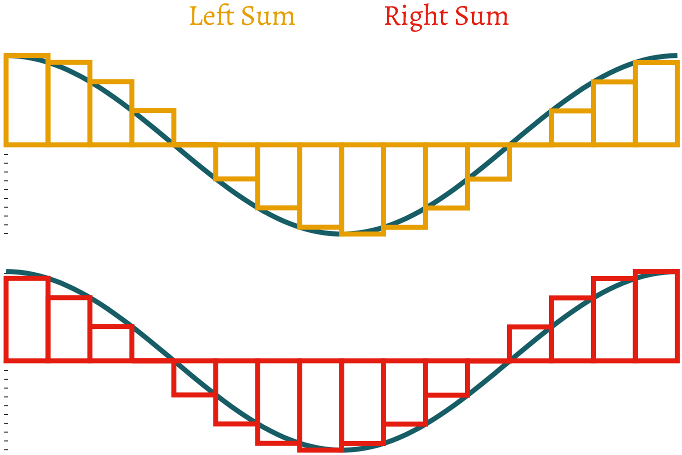
$$\text{Left}(f, [x_i, x_{i+1}[) = f(x_i) \text{ and } \text{Right}(f, [x_i, x_{i+1}]) = f(x_{i+1})$$

  are defined to return the leftmost or rightmost value of the function in the interval.

- **Left and Right Sum** are defined as the Sums of Left and Right across whole partitioned interval $(x_i)_{i \in [a,b]}$

$$I_L = \sum_i^n \text{Left}(f, x_i, x_{i+1}) \text{ and } I_R = \sum_i^n \text{Right}(f, x_i, x_{i+1})$$

Left and Right Sum



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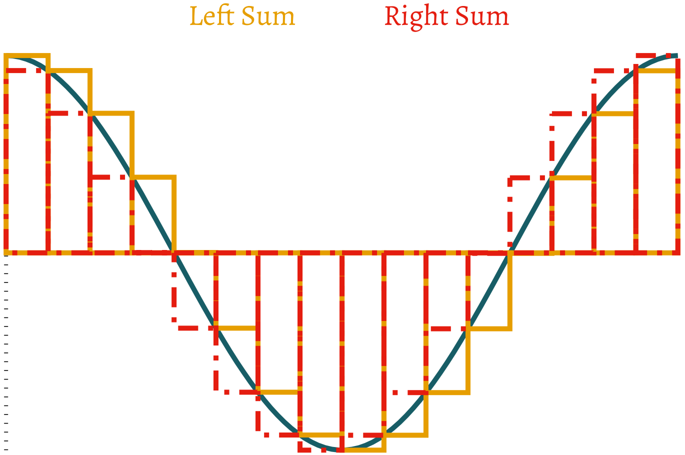

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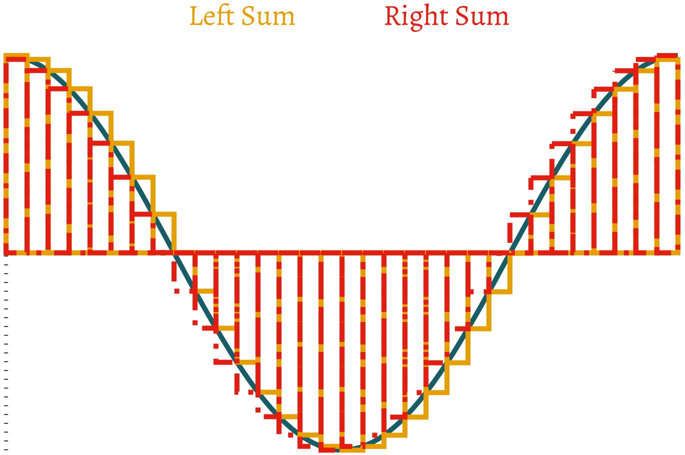


Estimation of the True Integral

- Left and Right Sums for a partition $(x_i)_{i\in[a,b]}$ give us an estimate of the integral
$$I_L \leq \int_a^b f(x)dx \leq I_R,$$
if the function in the interval is increasing and
$$I_R \leq \int_a^b f(x)dx \leq I_L,$$
if the function in the interval is decreasing.

Midpoint Method

Calculating Midpoints

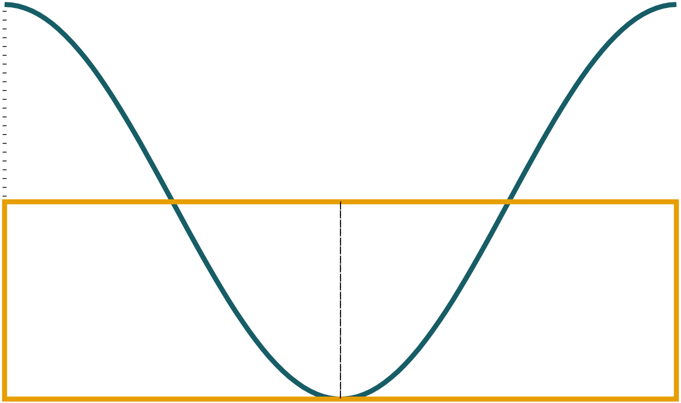
Another way of approximating an integral with finite sums is the **Midpoint Method**, which uses the function value in the middle of a given interval $[x_i, x_{i+1}]$

$$\text{Mid}(f, [x_i, x_{i+1}]) = f(\frac{x_i + x_{i+1}}{2})$$

The sum of Midpoints also yields an estimation of the area under the curve

$$I_M = \int_i^n Mid(f, [x_i, x_{i+1}])$$

Midpoint Sums



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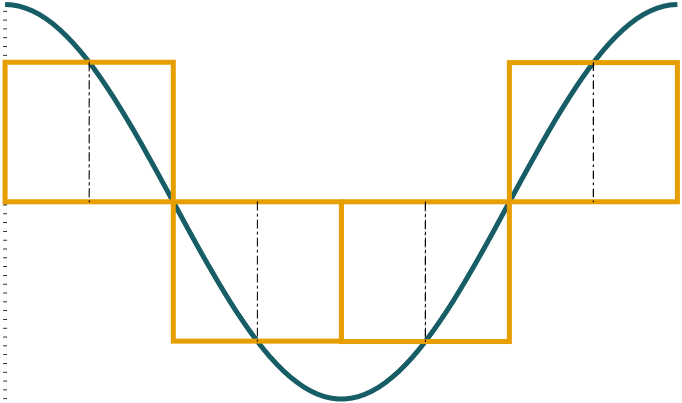

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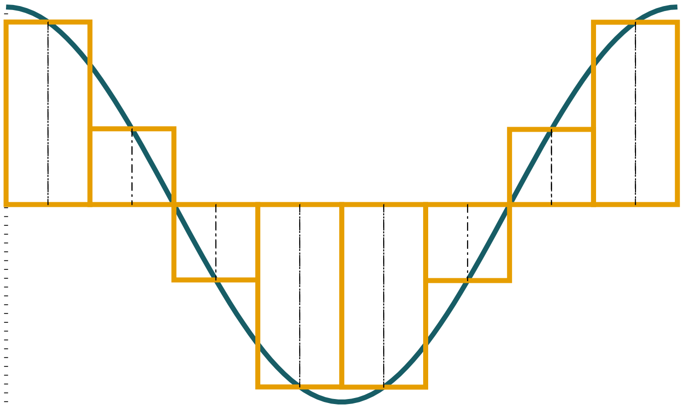

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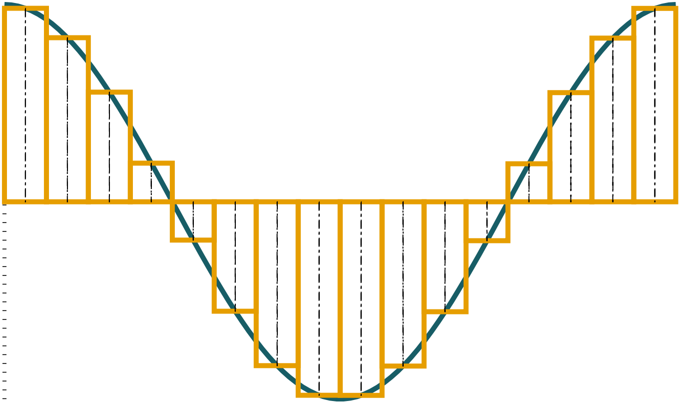


(Simple) Numerical Integration

- We can understand Numerical Integration as the opposite of Numerical Differentiation
- Thus instead of subtracting following elements, we will add to the previous element
- This is equivalent to the left Riemann Sums

(Simple) Numerical Differentiation

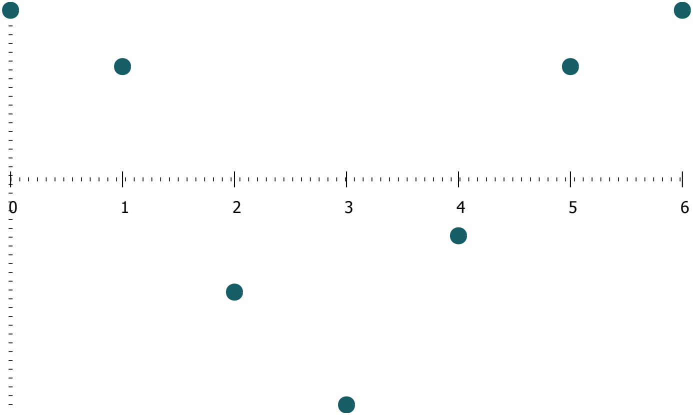
The set $\mathbb{I}$ describes the computable domain of f in the given context. It is possible to calculate function value $f(x_i)$, where $x_i \in \mathbb{I}$.

$$F(x_i) \approx F(x_{i-1}) + (f(x_i) \cdot (x_i - x_{i-1})),$$

where x_{i-1} is the smallest negative distance from x_i in $\mathbb{I}$.

Graphical Example

A List of Datapoints



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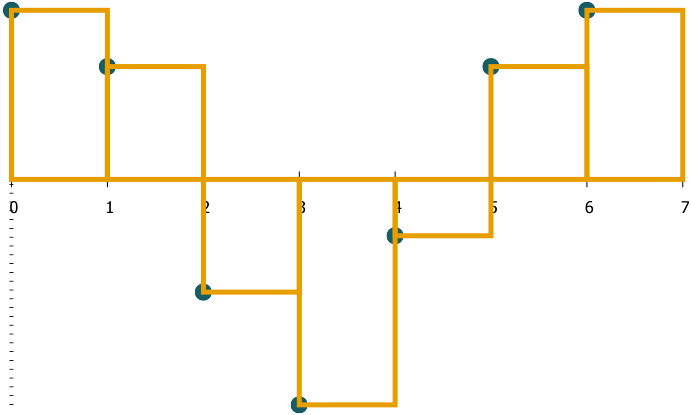

Integrating a List of Datapoints

- From a sensor we receive the following velocity values $f(x_i)$:

i	0	1	2	3	4	5	6
x_i	0	1	2	3	4	5	6
$f(x_i)$	3	2	-2	-4	-1	2	3

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- The integrated position for $x_3 = 3$ and startpoint $s = 2$ equals:

$$F(x_3) = s + 1f(x_0) + 1f(x_1) + 1f(x_2) + 1f(x_3) = 2 + 3 + 2 + (-2) + (-4) = 1$$

- The position at time-step $x_3 = 3$ is 1

Changing the Precision

- From a sensor we receive the following velocity values $f(x_i)$:

i	0	1	2	3	4	5	6
x_i	0	0.5	1	1.5	2	2.5	3
$f(x_i)$	3	2.5	2	0	-2	-2.5	-4

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- The distance between each point is 0.5.
  The area underneath each point is therefore $0.5 * f(x_i)$

- The integrated position for $x_6 = 3$ and startpoint $s = 2$ equals:

$$\begin{aligned} F(x_6) &= s + 0.5f(x_0) + 0.5f(x_1) + 0.5f(x_2) \\ &\quad + 0.5f(x_3) + 0.5f(x_4) + 0.5f(x_5) + 0.5f(x_6) \\ &= 2 + 1.5 + 1.25 + 1 + 0 + (-1) + (-1.25) + (-2) = 1.5 \end{aligned}$$

- The position at time-step $x_6 = 3$ is 1.5

Reading Files

- Opening a file

```
fileObject = open("file.txt", "r")
#The option r stands for read
```

- Reading the file contents

```
#readlines creates a list containing each line
lines = fileObject.readlines()
for line in lines:
    print(line)
```

- Close the file after usage:

```
fileObj.close()#This can be done right after readlines()
```

Details on Strings

- Useful string operations

```
#Strip removes the new-line character '\n'
line = line.strip()
#Split tokenizes the string at the given character
line = line.split(" ")# 'Hello you' to ['Hello','you']
line = line.split("o")# 'Hello you' to ['Hell',' y','u']
line = line.replace("l","b")# 'Hello you' to 'Hebbo you'
```

Task Template

- Download the archive *task_template_5.zip* from the course homepage. Extract it into a folder of your choice.
- The archive contains *task_51.py*, *student_code_51.py*, *task_52.py*, *student_code_52.py*, and *velocity_series.txt*.
- You only need to edit code in the *student_code* files.

Explain Task Template!

Tasks

1. Implement the function *area(a,b)*, which returns the area between x-axis and function curve $f(x)$ in the interval a, b.
   - The functions $f(x)$ and $F(x)$ are implemented already and you can call them in your code
   - Run *task51.py* to verify your result and plot an example area.
   - Given $F(x) = 4x^3 + 10$ calculate $f(x)$ on a piece of paper.
   - Implement both functions $F(x)$ and $f(x)$ instead of the given ones. Test using *task51.py*.
2. Calculate the derivatives of $F(x) = sin(x)$ and verify the result via numerical integration. Do the same for: $F(x) = 3(5x^2)^2$ and $F(x) = (3x^3 + 3) \cdot (2x^2 + 2)$
   - Implement the function *numerical_integration* using the formula for (simple) numerical integration
   - Use a for loop through all y-values of function $f(x)$ and use them to calculate the values $F(x)$.
   - Append them to the list *y_values_F*

File Reading Task (optional)

1. Write a script that opens *velocity_series.txt* from the course page, reads its contents and stores them as a list of floating values.
   - Use *file.readlines*() to receive a list of strings containing each line
   - Extract the velocity in each line by applying the *split*() method in a for-loop
   - In the loop typecast the velocity into a float and append it to a second list
2. Copy your function *numerical_integration* from the previous task and calculate the resulting series of positions.
   - You can receive the step size and the starting point from the file *velocity_series.txt*