Lecture 4
Function Limits and Differentiation

Jan Tekülve
jan.tekuelve@ini.rub.de

Computer Science and Mathematics
Preparatory Course

25.09.2019
Motivation

Estimating Velocity by Differentiation
The Vehicle’s Position

![Diagram of the vehicle's position over time]

- Time: 0, 5, 10, 15, 20, 25, 30, 35, 40
- Position: 0

Graph showing position vs. time with a point at (10, 10)
The Vehicle’s Position
The Vehicle’s Position
The Vehicle’s Position

| 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 |

[Graph showing the vehicle's position over time]
The Vehicle’s Position

- Time: 0, 5, 10, 15, 20, 25, 30, 35, 40
- Position: 0, 10, 20, 30, 40

Graph showing the vehicle's position over time.
The Vehicle’s Position

<table>
<thead>
<tr>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
</tr>
</thead>
</table>

![Graph showing the vehicle's position over time.](image-url)
The Vehicle’s Position

<table>
<thead>
<tr>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
</tr>
</thead>
</table>

![Diagram showing the vehicle's position over time.](image)
The Vehicle’s Position

[Graph showing time and position with points at t=0, 5, 10, 15, 20, 25, 30, 35, 40 with corresponding positions 0, 5, 10, 15, 20, 25, 30, 35, 40]
The Vehicle’s Velocity

Graph showing position and velocity over time.
The Vehicle’s Velocity
The Vehicle’s Velocity

- Diagram showing position over time with a linear increase.
- Diagram showing velocity over time with changes in direction.

Lecture 4 - Sequences

Motivation
The Vehicle’s Velocity
The Vehicle’s Velocity

![Position-time graph](image1)

![Velocity-time graph](image2)
The Vehicle’s Velocity

![Position vs Time Graph](image1)

![Velocity vs Time Graph](image2)
The Vehicle’s Velocity
The Vehicle’s Velocity
Overview

1. Motivation

2. Function Limits
   - Sequences
   - Limit Definition

3. Differentiation
   - Graphical Interpretation
   - Formal Description
   - Rules for Differentiation
   - Numerical Differentiation

4. Tasks
Overview

1. Motivation

2. Function Limits
   - Sequences
   - Limit Definition

3. Differentiation
   - Graphical Interpretation
   - Formal Description
   - Rules for Differentiation
   - Numerical Differentiation

4. Tasks
Sequences

Sequence Definition

Functions with the domain $\mathbb{N}$ are called sequence. A sequence with the codomain $\mathbb{R}$ is called a sequence of real numbers: $f : \mathbb{N} \rightarrow \mathbb{R}, n \rightarrow f(n)$

Examples:

- Constant sequence: $(3)_n \in \mathbb{N} = (3, 3, 3, 3, 3, \ldots)$
- Sequence of natural numbers: $(n)_n \in \mathbb{N} = (1, 2, 3, 4, 5, \ldots)$
- Harmonic sequence: $(\frac{1}{n})_n \in \mathbb{N} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots)$
- Geometric sequence: $(q^n)_n \in \mathbb{N} = (q, q^2, q^3, q^4, q^5, \ldots)$
- Alternating sequence: $((-1)^n)_n \in \mathbb{N} = (-1, 1, -1, 1, -1, \ldots)$
Recursive Sequences

Recursive Sequence Definition

A sequence \((a_n)_{n \in \mathbb{N}}\) may be recursively defined by:

1. The first sequence element: \(a_1\), called **initial value**
2. A recursive rule defining element \(a_{n+1}\) through previous elements \(a_n\)

Example: The Fibonacci Sequence

\[
a_{n+1} = a_n + a_{n-1} = (1, 1, 2, 3, 5, 8, 13, 21, \ldots),
\]

with \(a_1 = 1\) and \(a_2 = 1\)
## Properties of Sequences

### Boundedness

A sequence \((a_n)_{n \in \mathbb{N}}\) has

- an **upper bound**, if there is a \(K \in \mathbb{R}\), such that \(a_n \leq K\) for all \(n \in \mathbb{N}\)

- a **lower bound**, if there is a \(K \in \mathbb{R}\), such that \(a_n \geq K\) for all \(n \in \mathbb{N}\)
Properties of Sequences

Boundedness

A sequence \((a_n)_{n \in \mathbb{N}}\) has

- an upper bound, if there is a \(K \in \mathbb{R}\), such that \(a_n \leq K\) for all \(n \in \mathbb{N}\)

- a lower bound, if there is a \(K \in \mathbb{R}\), such that \(a_n \geq K\) for all \(n \in \mathbb{N}\)

Monotonicity

A sequence \((a_n)_{n \in \mathbb{N}}\) is :

- (strictly) monotonically increasing, if \(a_n(\leq) \leq a_{n+1}\) for all \(n \in \mathbb{N}\)

- (strictly) monotonically decreasing, if \(a_n(\geq) \geq a_{n+1}\) for all \(n \in \mathbb{N}\)
A sequence \((a_n)_{n \in \mathbb{N}}\) of real numbers converges to a real number \(L\), if for all \(\epsilon > 0\), there exists a natural number \(N\):

\[|a_n - L| < \epsilon \text{ for all } n \geq N\]
Convergence and Divergence

Definitions

A sequence \((a_n)_{n \in \mathbb{N}}\) of real numbers converges to a real number \(L\), if for all \(\epsilon > 0\), there exists a natural number \(N\):

\[|a_n - L| < \epsilon \text{ for all } n \geq N\]

Translation: A sequence converges to a real number \(L\), if you get closer to \(L\) with each additional element in the sequence.
Convergence and Divergence

Definitions

- A sequence \((a_n)_{n\in\mathbb{N}}\) of real numbers converges to a real number \(L\), if for all \(\epsilon > 0\), there exists a natural number \(N\):

\[
|a_n - L| < \epsilon \quad \text{for all } n \geq N
\]

**Translation:** A sequence converges to a real number \(L\), if you get closer to \(L\) with each additional element in the sequence.

- \(L\) is called the limit of a sequence

\[
\lim_{n \to \infty} a_n = L
\]
Convergence and Divergence

**Definitions**

▶ A sequence \((a_n)_{n \in \mathbb{N}}\) of real numbers **converges** to a real number \(L\), if for all \(\epsilon > 0\), there exists a natural number \(N\):

\[ |a_n - L| < \epsilon \text{ for all } n \geq N \]

**Translation:** A sequence converges to a real number \(L\), if you get closer to \(L\) with each additional element in the sequence.

▶ \(L\) is called the **limit** of a sequence.

\[ \lim_{{n \to \infty}} a_n = L \]

▶ A sequence that does not converge is called **divergent**.
Convergence Example

The harmonic sequence \((\frac{1}{n})_{n \in \mathbb{N}} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots)\) converges to **Zero**
Convergence Example

The harmonic sequence \( \left( \frac{1}{n} \right)_{n \in \mathbb{N}} = \left( 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \right) \) converges to \textbf{Zero}
Convergence Example

A sequence \((a_n)_{n \in \mathbb{N}}\) of real numbers converges to a real number \(L\), if for all \(\epsilon > 0\), there exists a natural number \(N\) : \(|a_n - L| < \epsilon\) for all \(n \geq N\)
**Convergence Example**

A sequence \((a_n)_{n \in \mathbb{N}}\) of real numbers **converges** to a real number \(L\), if for all \(\varepsilon > 0\), there exists a natural number \(N\) : \(|a_n - L| < \varepsilon\) for all \(n \geq N\)

\[
\varepsilon = 0.28
\]
Convergence Example

A sequence \((a_n)_{n \in \mathbb{N}}\) of real numbers **converges** to a real number \(L\), if for all \(\epsilon > 0\), there exists a natural number \(N\) such that \(|a_n - L| < \epsilon\) for all \(n \geq N\).
Convergence Example

A sequence \((a_n)_{n \in \mathbb{N}}\) of real numbers converges to a real number \(L\), if for all \(\varepsilon > 0\), there exists a natural number \(N\) : \(|a_n - L| < \varepsilon\) for all \(n \geq N\)

\[ \varepsilon = 0.15 \]

\[ N = 6 \]
Convergence Example

A sequence \((a_n)_{n \in \mathbb{N}}\) of real numbers **converges** to a real number \(L\), if for all \(\epsilon > 0\), there exists a natural number \(N\) : \(|a_n - L| < \epsilon\) for all \(n \geq N\).
Convergence Example

A sequence \( (a_n)_{n \in \mathbb{N}} \) of real numbers **converges** to a real number \( L \), if for all \( \epsilon > 0 \), there exists a natural number \( N \) : \( |a_n - L| < \epsilon \) for all \( n \geq N \).

\[ \epsilon = 0.28 \]

\[ N = 3 \]
Properties of Limits

Calculating with Limits

For two converging sequences \((x_n)_{n \in \mathbb{N}}\) and \((y_n)_{n \in \mathbb{N}}\) with limits \(\lim_{n \to \infty} x_n = L_x\) and \(\lim_{n \to \infty} y_n = L_y\) the following holds:

- **Scalar multiplication:** \(\lim_{n \to \infty} (ax_n) = aL_x\) for \(a \in \mathbb{R}\)

- **Addition:** \(\lim_{n \to \infty} (x_n + y_n) = L_x + L_y\)

- **Multiplication:** \(\lim_{n \to \infty} (x_ny_n) = L_xL_y\)

- **Division:** \(\lim_{n \to \infty} \left(\frac{x_n}{y_n}\right) = \frac{L_x}{L_y}\)

- **Norm:** \(\lim_{n \to \infty} (|x_n|) = |L_x|\)
1. **Motivation**

2. **Function Limits**
   - Sequences
   - Limit Definition

3. **Differentiation**
   - Graphical Interpretation
   - Formal Description
   - Rules for Differentiation
   - Numerical Differentiation

4. **Tasks**
A function and its derivative

\[ f(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x \]
A function and its derivative

\[ f(x) = x \quad f'(x) = 1 \]
A function and its derivative

\[ f(x) = 0.5 \quad f'(x) = 0 \]
A function and its derivative

\[ f(x) = \sin(x) \]
\[ f'(x) = \cos(x) \]
Derivative as a Tangent

\[ f(x) = \sin(x) \quad f'(x) = \cos(x) \]
Derivative as a Tangent

\[ f(x) = \sin(x) \quad f'(x) = \cos(x) \]
Derivative as a Tangent

\[ f(x) = \sin(x) \quad f'(x) = \cos(x) \]
Formal Definition

Differentiable Function

- A function \( f \) with domain \( M \) is called differentiable at position \( x_0 \) if, if the limit value

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}
\]

exists.
Formal Definition

Differentiable Function

- A function $f$ with domain $M$ is called differentiable at position $x_0$ if, if the limit value

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists.

- This limit is called $f'$ or **derivative of $f$ at position $x_0$**. If $f'$ is defined for all $x_0 \in M$, then $f'$ becomes a new function called the derivative of $f$. 
Formal Definition

Differentiable Function

- A function \( f \) with domain \( M \) is called differentiable at position \( x_0 \) if, if the limit value
  \[
  \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}
  \]
  exists.

- This limit is called \( f' \) or derivative of \( f \) at position \( x_0 \). If \( f' \) is defined for all \( x_0 \in M \), then \( f' \) becomes a new function called the derivative of \( f \).

- Alternate notations:
  \[
  f'(x_0) = \frac{df}{dx}(x_0) = \lim_{x \to x_0} \frac{f(x + h) - f(x_0)}{h}
  \]
Differentiation as Limit Example

- **Statement:** The derivative of $f(x) = x^2$ is $f'(x) = 2x$
Differentiation as Limit Example

- **Statement:** The derivative of \( f(x) = x^2 \) is \( f'(x) = 2x \)

- Applying the formula

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0}
\]
Differentiation as Limit Example

▶ Statement: The derivative of $f(x) = x^2$ is $f'(x) = 2x$

▶ Applying the formula

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0}$$

▶ Simplifying

$$\lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = \lim_{x \to x_0} (x + x_0)$$
Differentiation as Limit Example

**Statement:** The derivative of $f(x) = x^2$ is $f'(x) = 2x$

**Applying the formula**

$$
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0}
$$

**Simplifying**

$$
\lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = \lim_{x \to x_0} (x + x_0)
$$

**Applying the limit:**

$$
\lim_{x \to x_0} (x + x_0) = 2x
$$
Differentiation is a linear operator

### Rules

- **Constant Factor**
  \[
  \frac{d}{dx} (af) = a \frac{d}{dx} (f)
  \]

- **Sums**
  \[
  \frac{d}{dx} (f + g) = \frac{d}{dx} (f) + \frac{d}{dx} (g)
  \]

### Example:

\[
\frac{d}{dx} (4x^2) = 4 \frac{d}{dx} (x^2) = 4(2x) = 8x
\]
Differentiation is a linear operator

**Rules**

- **Constant Factor**
  \[
  \frac{d}{dx}(af) = a \frac{d}{dx}(f)
  \]

- **Sums**
  \[
  \frac{d}{dx}(f + g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)
  \]

**Example:**

\[
\frac{d}{dx}(4x^2) = 4 \frac{d}{dx}(x^2) = 4(2x) = 8x
\]

\[
\frac{d}{dx}(4x^2 + x^2) = 4 \frac{d}{dx}(x^2) + \frac{d}{dx}(x^2) = 4(2x) + 2x = 10x
\]
Differentiation for Products and Quotients

**Multiplication**

\[
\frac{d}{dx}(fg) = \frac{d}{dx}(f)g + f \frac{d}{dx}(g)
\]

**Exponentiation**

\[
\frac{d}{dx}(f^n) = n \frac{d}{dx}(f)^{n-1}
\]

**Division**

\[
\frac{d}{dx} \left( \frac{f}{g} \right) = \frac{\frac{d}{dx}(f)g - f \frac{d}{dx}(g)}{g^2}
\]
Examples

- **Multiplication**

\[
\frac{d}{dx}(x^2 \sin(x)) = \frac{d}{dx}(x^2) \sin(x) + x^2 \frac{d}{dx}(\sin(x)) = 2x \sin(x) + x^2 \cos(x)
\]
Examples

▶ Multiplication

\[
\frac{d}{dx}(x^2 \sin(x)) = \frac{d}{dx}(x^2)\sin(x) + x^2 \frac{d}{dx}(\sin(x)) = 2x\sin(x) + x^2 \cos(x)
\]

▶ Division

\[
\frac{d}{dx} \left( \frac{1}{x} \right) = \frac{\frac{d}{dx}(1)x - 1 \frac{d}{dx}(x)}{x^2} = \frac{0 - 1}{x^2} = -\frac{1}{x^2}
\]
Exponentiation Rule derives from Multiplication Rule

Example $f'(x^3)$

$$\frac{d}{dx}(x^3) = \frac{d}{dx}(x^2x) = \frac{d}{dx}(x^2)x + x^2 \frac{d}{dx}(x)$$
Exponentiation Rule derives from Multiplication Rule

Example $f'(x^3)$

$$\frac{d}{dx}(x^3) = \frac{d}{dx}(x^2x) = \frac{d}{dx}(x^2)x + x^2 \frac{d}{dx}(x)$$

$$= 2xx + x^2 = 3x^2$$
Exponentiation Rule derives from Multiplication Rule

- Example $f'(x^3)$

\[
\frac{d}{dx}(x^3) = \frac{d}{dx}(x^2 x) = \frac{d}{dx}(x^2) x + x^2 \frac{d}{dx}(x)
\]

\[
= 2xx + x^2 = 3x^2
\]

- Example $f'(x^4)$

\[
\frac{d}{dx}(x^4) = \frac{d}{dx}(x^2 x^2) = \frac{d}{dx}(x^2) x^2 + x^2 \frac{d}{dx}(x^2)
\]
Exponentiation Rule derives from Multiplication Rule

- Example $f'(x^3)$

\[
\frac{d}{dx}(x^3) = \frac{d}{dx}(x^2 \cdot x) = \frac{d}{dx}(x^2)x + x^2 \frac{d}{dx}(x)
\]

\[= 2xx + x^2 = 3x^2\]

- Example $f'(x^4)$

\[
\frac{d}{dx}(x^4) = \frac{d}{dx}(x^2 \cdot x^2) = \frac{d}{dx}(x^2)x^2 + x^2 \frac{d}{dx}(x^2)
\]

\[= 2xx^2 + x^2 \cdot 2x = 2x^3 + 2x^3 = 4x^3\]
Special cases

- The derivative of $f(x) = e^x$ is $f'(x) = e^x$
- The derivative of $f(x) = \ln(x)$ is $f'(x) = \frac{1}{x}$
- The derivative of $f(x) = \sin(x)$ is $f'(x) = \cos(x)$
Composite functions

Chain Rule

- Function $h$ is a composition of functions $g$ and $f$
  \[ h(x) = (g \circ f)(x) = g(f(x)) \]

- If $g$ and $f$ are differentiable, $h$ is also differentiable
  \[ \frac{d}{dx}(h(x)) = \frac{d}{dx}(g(y)) \frac{d}{dx}(f(x)), \text{ with } y = f(x) \]

- Verbal rule: **Inner derivative times outer derivative**
Chain Rule Examples

\[ h(x) = 5(7x + 2)^4 = g(f(x)) \]
Chain Rule Examples

$h(x) = 5(7x + 2)^4 = g(f(x))$

$g(x) = 5x^4 \land f(x) = 7x + 2$
Chain Rule Examples

\[ h(x) = 5(7x + 2)^4 = g(f(x)) \]

\[ g(x) = 5x^4 \land f(x) = 7x + 2 \]
\[ g'(x) = 20x^3 \land f'(x) = 7 \]
Chain Rule Examples

\[ h(x) = 5(7x + 2)^4 = g(f(x)) \]

\[ g(x) = 5x^4 \land f(x) = 7x + 2 \]
\[ g'(x) = 20x^3 \land f'(x) = 7 \]
\[ h'(x) = 20(7x + 2)^3 \cdot 7 = 140(7x + 2)^3 \]
Chain Rule Examples

\[ h(x) = 5(7x + 2)^4 = g(f(x)) \]

\[ g(x) = 5x^4 \land f(x) = 7x + 2 \]
\[ g'(x) = 20x^3 \land f'(x) = 7 \]
\[ h'(x) = 20(7x + 2)^37 = 140(7x + 2)^3 \]

\[ h(x) = e^{5x} = g(f(x)) \]
Chain Rule Examples

1. \( h(x) = 5(7x + 2)^4 = g(f(x)) \)

   \[
   \begin{align*}
   g(x) &= 5x^4 \land f(x) = 7x + 2 \\
   g'(x) &= 20x^3 \land f'(x) = 7 \\
   h'(x) &= 20(7x + 2)^3 \cdot 7 = 140(7x + 2)^3
   \end{align*}
   \]

2. \( h(x) = e^{5x} = g(f(x)) \)

   \[
   \begin{align*}
   g(x) &= e^x \land f(x) = 5x
   \end{align*}
   \]
Chain Rule Examples

1. \[ h(x) = 5(7x + 2)^4 = g(f(x)) \]

   \[ g(x) = 5x^4 \land f(x) = 7x + 2 \]
   \[ g'(x) = 20x^3 \land f'(x) = 7 \]
   \[ h'(x) = 20(7x + 2)^3 \cdot 7 = 140(7x + 2)^3 \]

2. \[ h(x) = e^{5x} = g(f(x)) \]

   \[ g(x) = e^x \land f(x) = 5x \]
   \[ g'(x) = e^x \land f'(x) = 5 \]
Chain Rule Examples

\[ h(x) = 5(7x + 2)^4 = g(f(x)) \]
\[ g(x) = 5x^4 \land f(x) = 7x + 2 \]
\[ g'(x) = 20x^3 \land f'(x) = 7 \]
\[ h'(x) = 20(7x + 2)^3 \cdot 7 = 140(7x + 2)^3 \]

\[ h(x) = e^{5x} = g(f(x)) \]
\[ g(x) = e^x \land f(x) = 5x \]
\[ g'(x) = e^x \land f'(x) = 5 \]
\[ h'(x) = e^{5x} \cdot 5 = 5e^{5x} \]
Finding Local Extrema

\[ f(x) = \sin(x) \]

\[ f'(x) = \cos(x) \]
Finding Local Extrema

\[ f(x) = x^2 \quad f'(x) = 2x \]
Calculation of Local Extrema

\[ f(x) = 4x^2 + 6x \]
Calculation of Local Extrema

\[ f(x) = 4x^2 + 6x \]
\[ f'(x) = 8x + 6 \]
Calculation of Local Extrema

- \( f(x) = 4x^2 + 6x \)
  
  \( f'(x) = 8x + 6 \)
  
  \( f'(x) = 8x + 6 = 0 \)
Calculation of Local Extrema

\[ f(x) = 4x^2 + 6x \]
\[ f'(x) = 8x + 6 \]
\[ f'(x) = 8x + 6 \overset{!}{=} 0 \]
\[ \iff 8x = -6 \]
Calculation of Local Extrema

\[ f(x) = 4x^2 + 6x \]

\[ f'(x) = 8x + 6 \]

\[ f'(x) = 8x + 6 \overset{!}{=} 0 \]

\[ \iff 8x = -6 \]

\[ \iff x = \frac{-6}{8} = \frac{-3}{4} \]
Calculation of Local Extrema

- **$f(x) = 4x^2 + 6x$**
  
  $f'(x) = 8x + 6$
  
  $f''(x) = 8x + 6 \neq 0$
  
  $\Rightarrow 8x = -6$
  
  $\Rightarrow x = \frac{-6}{8} = \frac{-3}{4}$

- **$f(x) = \sin(x)$**
Calculation of Local Extrema

- \( f(x) = 4x^2 + 6x \)
  
  \[
  f'(x) = 8x + 6
  \]
  
  \[
  f'(x) = 8x + 6 = 0
  \]
  
  \[
  \iff 8x = -6
  \]
  
  \[
  \iff x = \frac{-6}{8} = \frac{-3}{4}
  \]

- \( f(x) = \sin(x) \)
  
  \[
  f'(x) = \cos(x)
  \]
Calculation of Local Extrema

**f(x) = 4x^2 + 6x**

\[
f'(x) = 8x + 6
\]

\[
f'(x) = 8x + 6 = 0
\]

\[
\iff 8x = -6
\]

\[
x = \frac{-6}{8} = \frac{-3}{4}
\]

**f(x) = sin(x)**

\[
f'(x) = cos(x)
\]

\[
f'(x) = cos(x) = 0
\]
Calculation of Local Extrema

\[ f(x) = 4x^2 + 6x \]

\[ f'(x) = 8x + 6 \]

\[ f'(x) = 8x + 6 = 0 \]

\[ \iff 8x = -6 \]

\[ x = \frac{-6}{8} = \frac{-3}{4} \]

\[ f(x) = \sin(x) \]

\[ f'(x) = \cos(x) \]

\[ f'(x) = \cos(x) = 0 \]

\[ \iff x = \cos^{-1}(0) \]
Calculation of Local Extrema

\( f(x) = 4x^2 + 6x \)

\( f'(x) = 8x + 6 \)

\( f'(x) = 8x + 6 \overset{!}{=} 0 \)

\( \iff 8x = -6 \)

\( \iff x = \frac{-6}{8} = \frac{-3}{4} \)

\( f(x) = \sin(x) \)

\( f'(x) = \cos(x) \)

\( f'(x) = \cos(x) \overset{!}{=} 0 \)

\( \iff x = \cos^{-1}(0) \)

\( \iff x = 90^\circ = \frac{\pi}{2}, 270^\circ = \frac{3\pi}{2}, \ldots \)
Differentiability is not given

\[ f(x) = \frac{1}{x} \]

\[ f'(x) = \frac{-1}{x^2} \]
Numerical Differentiation

Problem: Only function values \( f(x) \) of \( f(x) \) are known, but not the real function \( f \)

\[
f'(x_i) \approx \frac{f(x_i + \epsilon) - f(x_i)}{\epsilon}
\]
where \( x_i + \epsilon \) is the smallest positive distance from \( x_i \) in \( I \).
Numerical Differentiation

Problem: Only function values $f(x_0)$ of $f(x)$ are known, but not the real function $f$

Instead of calculating the derivative of $f$ analytically, it is possible to approximate $f'(x)$ using numerical differentiation
Numerical Differentiation

- **Problem:** Only function values \( f(x_0) \) of \( f(x) \) are known, but not the real function \( f \)

- Instead of calculating the derivative of \( f \) analytically, it is possible to approximate \( f'(x) \) using **numerical differentiation**

**(Simple) Numerical Differentiation**

The set \( \mathbb{I} \) describes the computable domain of \( f \) in the given context. It is possible to calculate function value \( f(x_i) \), where \( x_i \in \mathbb{I} \).

\[
f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i},
\]

where \( x_{i+1} \) is the smallest positive distance from \( x_i \) in \( \mathbb{I} \).
Numerical Differentiation Example

- From a sensor we receive the following values:

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x_i)$</td>
<td>3.1</td>
<td>2.9</td>
<td>2.4</td>
<td>1.4</td>
<td>1.6</td>
<td>3</td>
<td>3.1</td>
<td>3.3</td>
<td>3.5</td>
<td>4.2</td>
</tr>
</tbody>
</table>

- The derivative at $x_3$ equals:

$$f'(x_3) = \frac{f(x_{3+1}) - f(x_3)}{x_{3+1} - x_3} \Rightarrow \frac{f(x_4) - f(x_3)}{4 - 3} = \frac{1.6 - 1.4}{1} = 0.2$$

- The change at position $x_3$ is 0.2
Task Template Braitenberg

- Download the archive `task_template_4.zip` from the course homepage. Extract it into a folder of your choice.

- The archive contains `task_4_1.py`, `task_4_1_student_code.py` and `braitenberg.png`.

- Use `task_4_1.py` to run the program, but edit code only in `task_4_1_student_code.py`.

Explain Task Template!
Tasks

1. Calculate the vehicle’s velocity through numerical differentiation.
   - Open `task_4_1_student_code.py` and implement the function `calc_velocity_from_position`.
   - Use the given list of positions to estimate the vehicle’s velocity using numerical differentiation.
   - Append the resulting velocity values to the `player_velocities_x` list.
   - **Tip**: Use a for-loop that runs through the position values and compares the current list-entry to the preceding one.

2. Write a script that calculates the Fibonacci sequence for an arbitrary number $N$ of elements. Print the numbers to the console.
   - The first two elements of $a_1$ and $a_2$ are always 1
   - Write a loop that runs $N$ times and calculates the Fibonacci number $a_{n+1} = a_n + a_{n-1}$
   - **Tip**: Use variables to store the values for the current value $a_n$ and the previous value $a_{n-1}$ and update them in each loop.