Lecture 4
Function Limits and Differentiation

Jan Tekülve

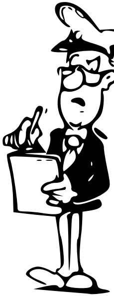
jan.tekuelve@ini.rub.de

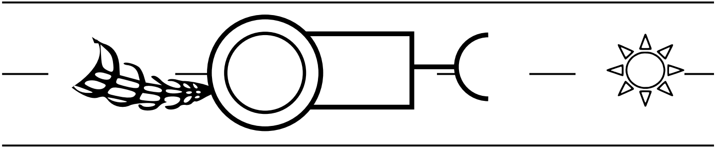
Computer Science and Mathematics
Preparatory Course

25.09.2019

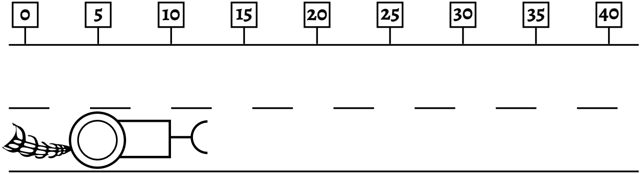
Motivation

Estimating Velocity by Differentiation

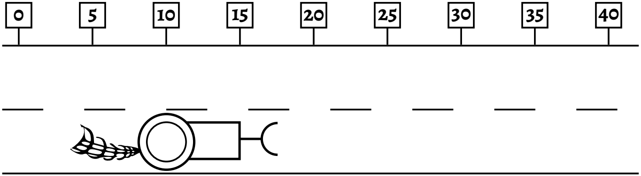


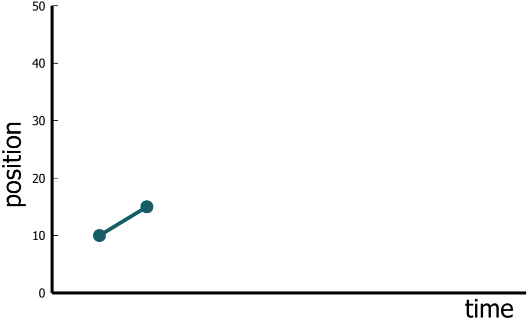


The Vehicle's Position

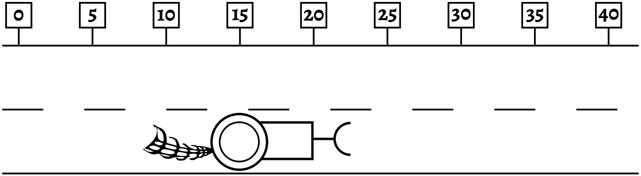


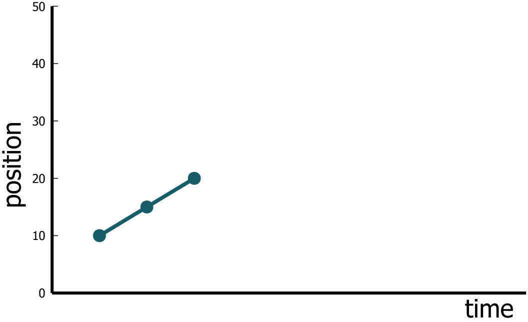
The Vehicle’s Position



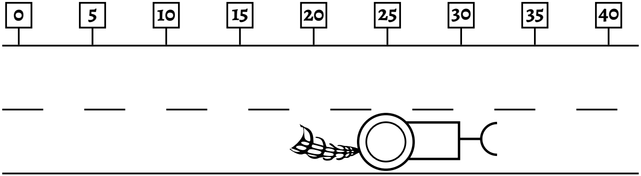


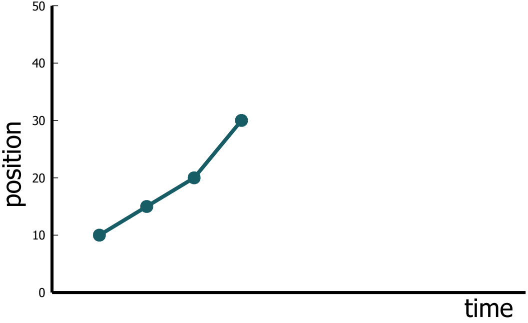
The Vehicle's Position



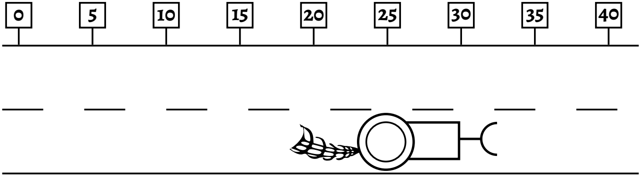


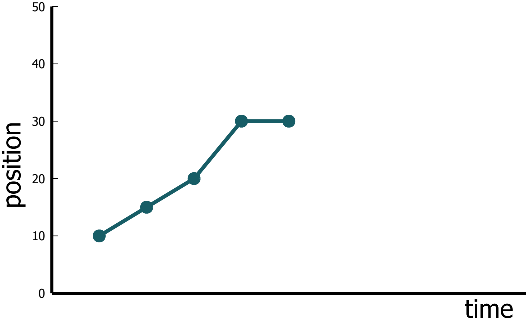
The Vehicle's Position



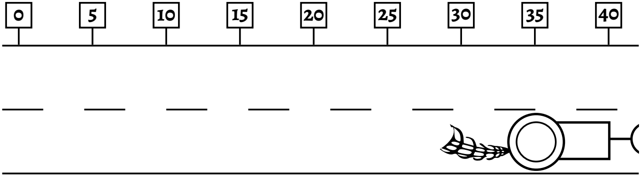


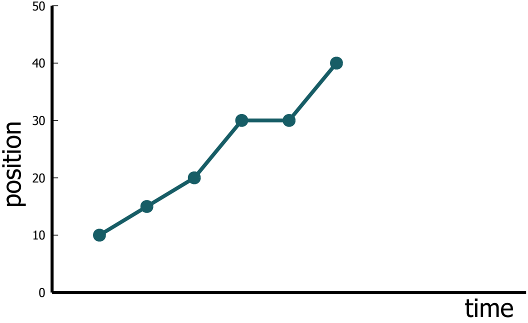
The Vehicle's Position



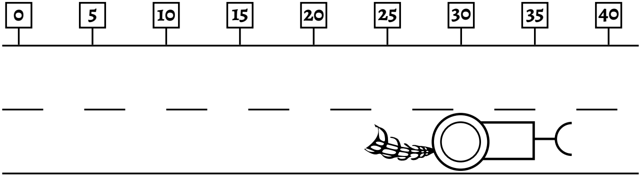


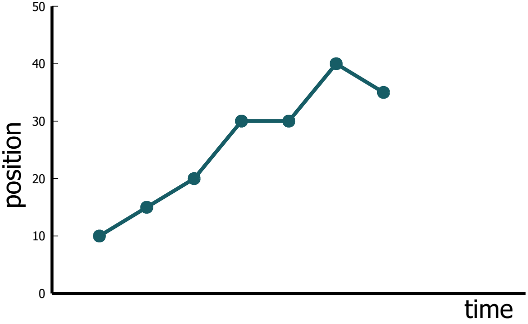
The Vehicle's Position



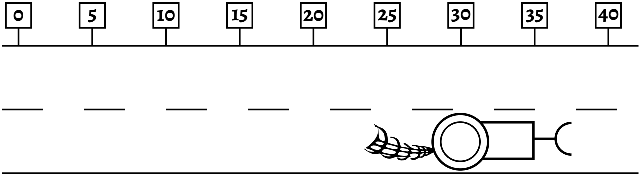


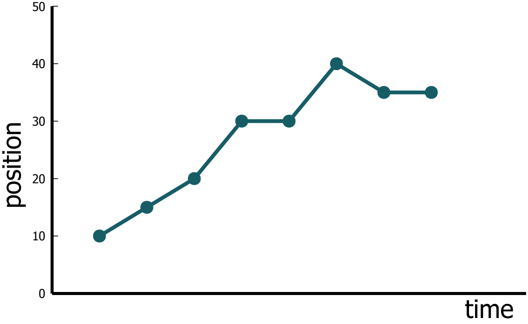
The Vehicle's Position



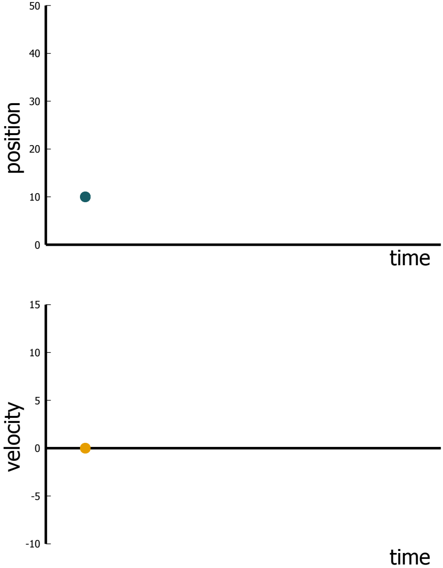


The Vehicle's Position

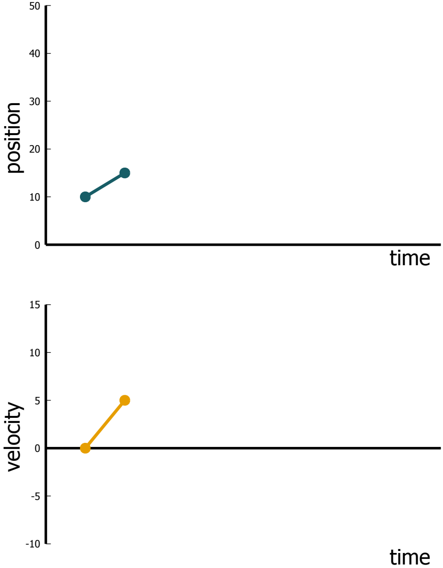




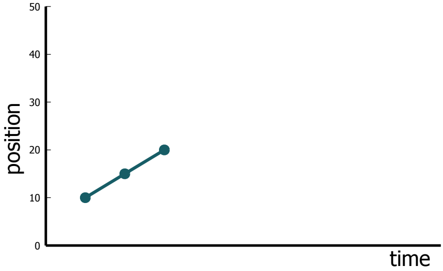
The Vehicle's Velocity

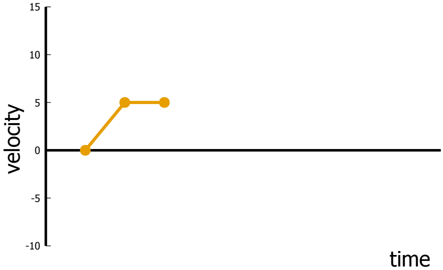


The Vehicle's Velocity

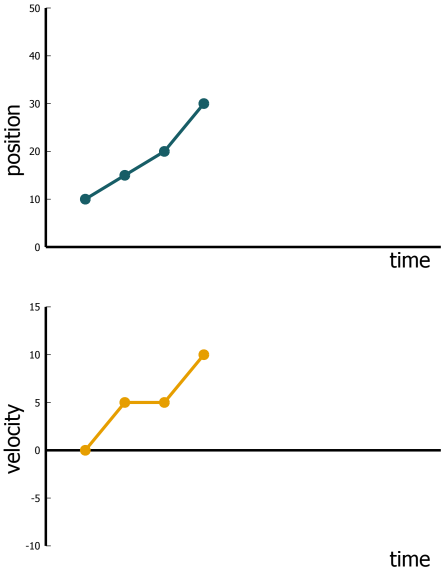


The Vehicle's Velocity

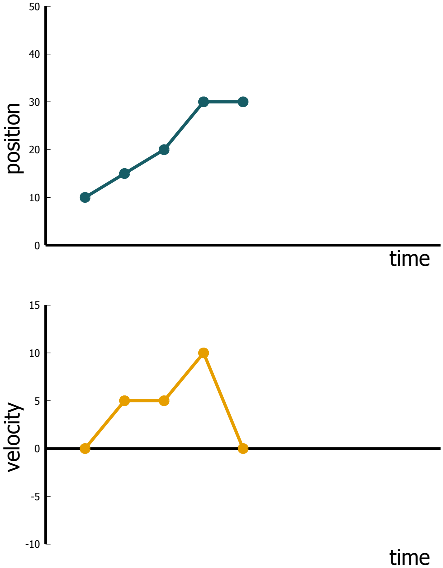




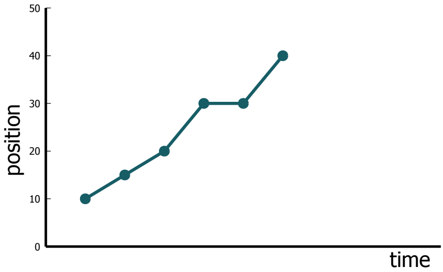
The Vehicle's Velocity

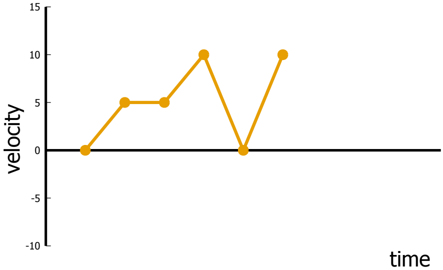


The Vehicle's Velocity

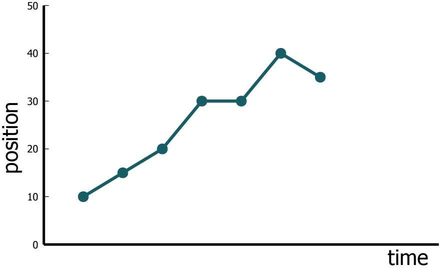


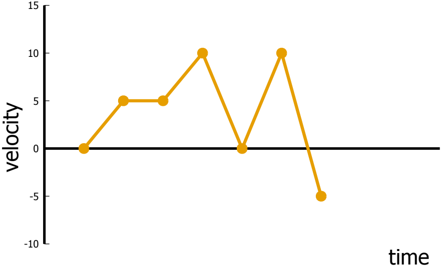
The Vehicle's Velocity



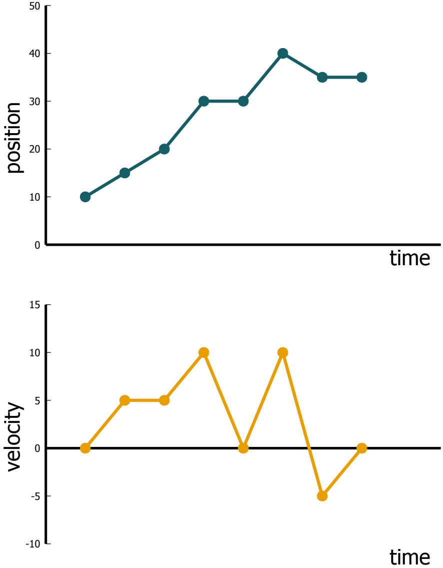


The Vehicle's Velocity





The Vehicle's Velocity



Overview

Overview

Sequences

Sequence Definition

Functions with the domain $\mathbb{N}$ are called **sequence**. A sequence with the codomain $\mathbb{R}$ is called a sequence of real numbers: $f : \mathbb{N} \to \mathbb{R}, n \to f(n)$

Examples:

- Constant sequence: $(3)_{n\in\mathbb{N}} = (3, 3, 3, 3, 3, \dots)$
- Sequence of natural numbers: $(n)_{n\in\mathbb{N}} = (1, 2, 3, 4, 5, \dots)$
- Harmonic sequence: $(\frac{1}{n})_{n\in\mathbb{N}} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots)$
- Geometric sequence: $(q^n)_{n\in\mathbb{N}} = (q, q^2, q^3, q^4, q^5, \dots)$
- Alternating sequence: $((-1)^n)_{n\in\mathbb{N}} = (-1, 1, -1, 1, -1, \dots)$

Recursive Sequences

Recursive Sequence Definition

A sequence $(a_n)_{n\in\mathbb{N}}$ may be recursively defined by:

1. The first sequence element : a_1, called **initial value**
2. A recursive rule defining element a_{n+1} through previous elements a_n

Example: The Fibonacci Sequence

$$a_{n+1} = a_n + a_{n-1} = (1, 1, 2, 3, 5, 8, 13, 21, \dots),$$
$$\text{with } a_1 = 1 \text{ and } a_2 = 1$$

Properties of Sequences

Boundedness

A sequence $(a_n)_{n\in\mathbb{N}}$ has

- an **upper bound**, if there is a $K \in \mathbb{R}$, such that $a_n \leq K$ for all $n \in \mathbb{N}$
- a **lower bound**, if there is a $K \in \mathbb{R}$, such that $a_n \geq K$ for all $n \in \mathbb{N}$

Properties of Sequences

Boundedness

A sequence $(a_n)_{n\in\mathbb{N}}$ has

- an **upper bound**, if there is a $K \in \mathbb{R}$, such that $a_n \leq K$ for all $n \in \mathbb{N}$
- a **lower bound**, if there is a $K \in \mathbb{R}$, such that $a_n \geq K$ for all $n \in \mathbb{N}$

Monotonicity

A sequence $(a_n)_{n\in\mathbb{N}}$ is :

- **(strictly) monotonically increasing**, if $a_n (<) \leq a_{n+1}$ for all $n \in \mathbb{N}$
- **(strictly) monotonically decreasing**, if $a_n (>) \geq a_{n+1}$ for all $n \in \mathbb{N}$

Convergence and Divergence

Definitions

- A sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers **converges** to a real number L, if for all $\epsilon > 0$, there exists a natural number N:

$$|a_n - L| < \epsilon \text{ for all } n \geq N$$

Convergence and Divergence

Definitions

- A sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers **converges** to a real number L, if for all $\epsilon > 0$, there exists a natural number N:

$$|a_n - L| < \epsilon \text{ for all } n \geq N$$

Translation: A sequence converges to a real number L, if you get closer to L with each additional element in the sequence

Convergence and Divergence

Definitions

- A sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers **converges** to a real number L, if for all $\epsilon > 0$, there exists a natural number N:

$$|a_n - L| < \epsilon \text{ for all } n \geq N$$

Translation: A sequence converges to a real number L, if you get closer to L with each additional element in the sequence

- L is called the **limit** of a sequence

$$\lim_{n\to\infty} a_n = L$$

Convergence and Divergence

Definitions

- A sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers **converges** to a real number L, if for all $\epsilon > 0$, there exists a natural number N:

$$|a_n - L| < \epsilon \text{ for all } n \geq N$$

Translation: A sequence converges to a real number L, if you get closer to L with each additional element in the sequence

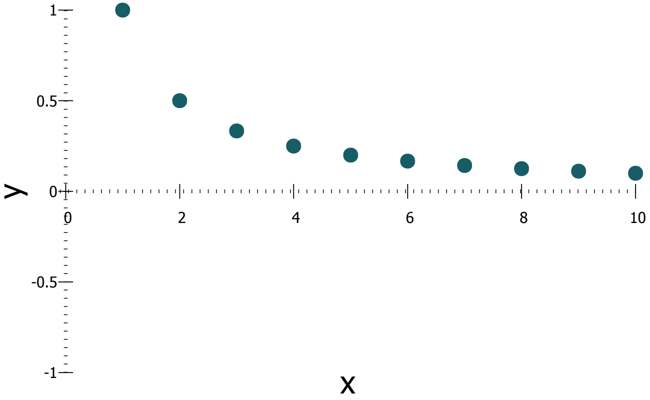
- L is called the **limit** of a sequence

$$\lim_{n\to\infty} a_n = L$$

- A sequence that does not converge is called **divergent**

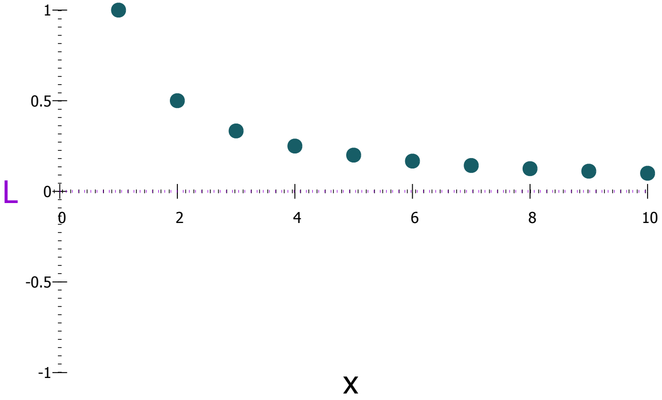
Convergence Example

The harmonic sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right)$ converges to **Zero**



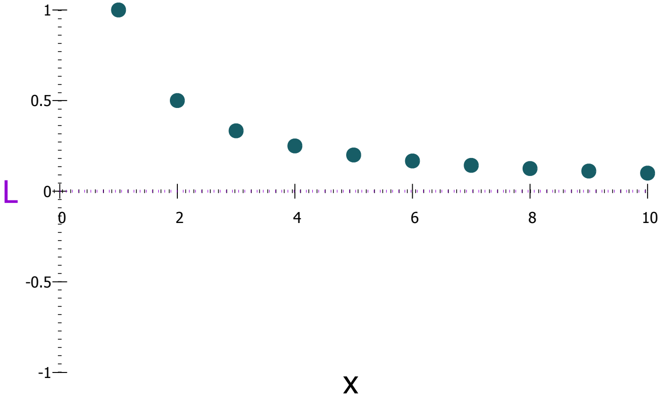
Convergence Example

The harmonic sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right)$ converges to **Zero**



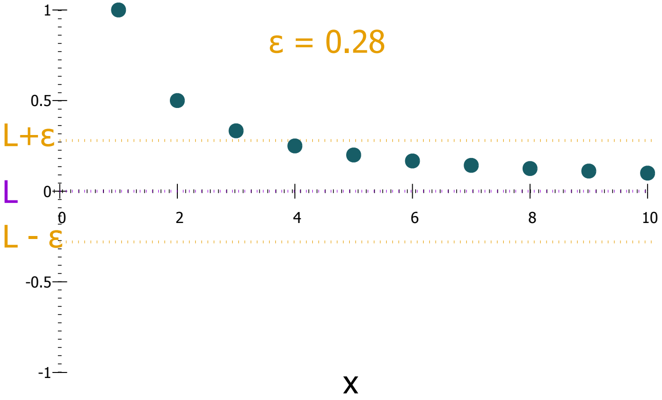
Convergence Example

A sequence $(a_n)_{n \in \mathbb{N}}$ of real numbers **converges** to a real number L, if for all $\epsilon > 0$, there exists a natural number $N : |a_n - L| < \epsilon$ for all $n \geq N$



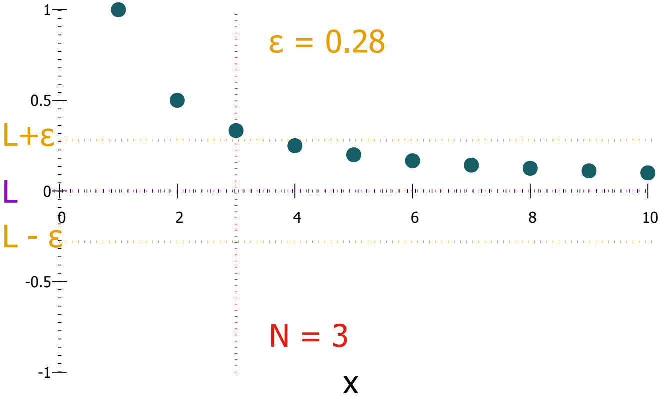
Convergence Example

A sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers **converges** to a real number L, if for all $\epsilon > 0$, there exists a natural number $N : |a_n - L| < \epsilon$ for all $n \geq N$



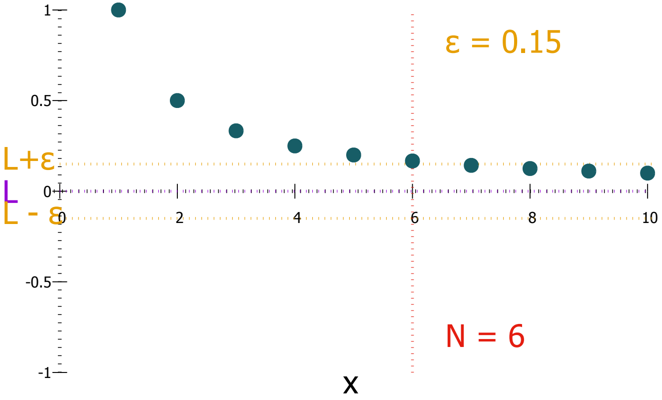
Convergence Example

A sequence $(a_n)_{n \in \mathbb{N}}$ of real numbers **converges** to a real number L, if for all $\epsilon > 0$, there exists a natural number $N : |a_n - L| < \epsilon$ for all $n \geq N$



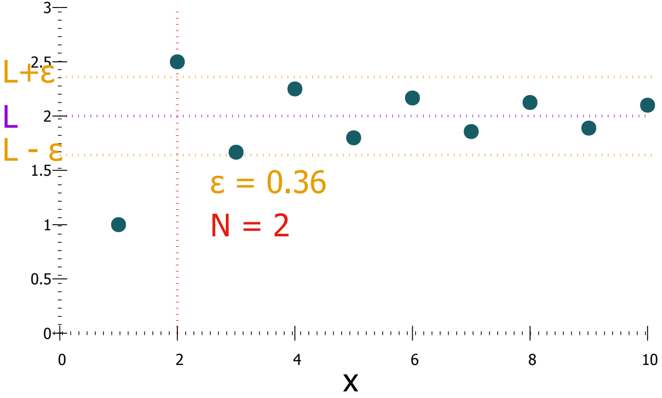
Convergence Example

A sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers **converges** to a real number L, if for all $\epsilon > 0$, there exists a natural number $N : |a_n - L| < \epsilon$ for all $n \geq N$



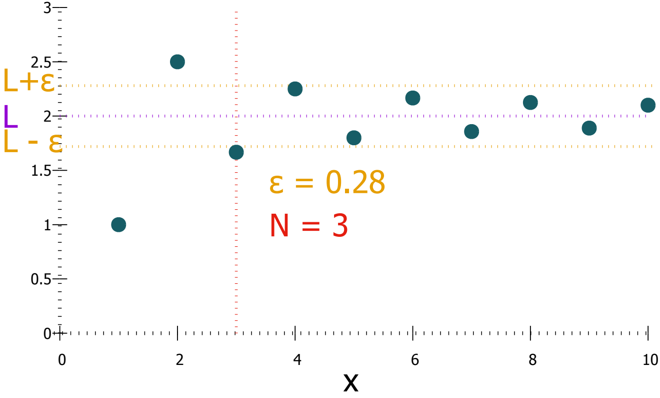
Convergence Example

A sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers **converges** to a real number L, if for all $\epsilon > 0$, there exists a natural number $N : |a_n - L| < \epsilon$ for all $n \geq N$



Convergence Example

A sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers **converges** to a real number L, if for all $\epsilon > 0$, there exists a natural number $N : |a_n - L| < \epsilon$ for all $n \geq N$



Properties of Limits

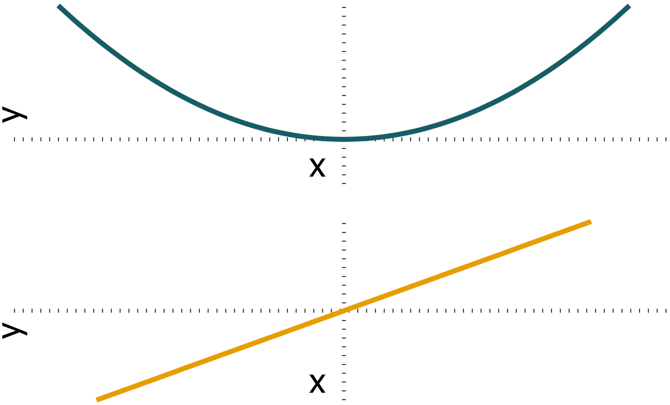
Calculating with Limits

For two converging sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ with limits $\lim_{n\to\infty} x_n = L_x$ and $\lim_{n\to\infty} y_n = L_y$ the following holds:

- **Scalar multiplication:** $\lim_{n\to\infty}(ax_n) = aL_x$ for $a \in \mathbb{R}$
- **Addition:** $\lim_{n\to\infty}(x_n + y_n) = L_x + L_y$
- **Multiplication:** $\lim_{n\to\infty}(x_n y_n) = L_x L_y$
- **Division:** $\lim_{n\to\infty}(\frac{x_n}{y_n}) = \frac{L_x}{L_y}$
- **Norm:** $\lim_{n\to\infty}(|x_n|) = |L_x|$

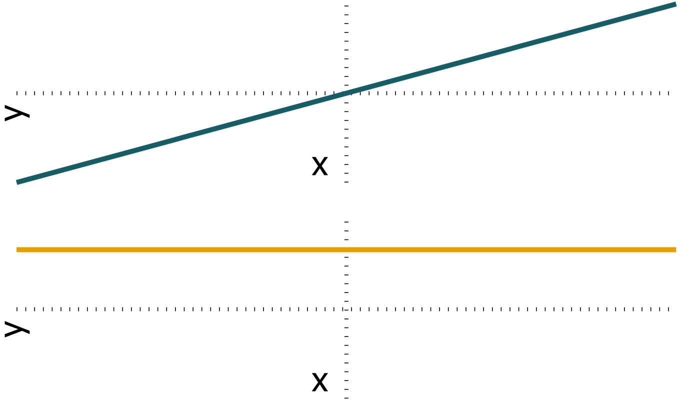
A function and its derivative

$f(x) = x^2$ $\quad$ $f'(x) = 2x$



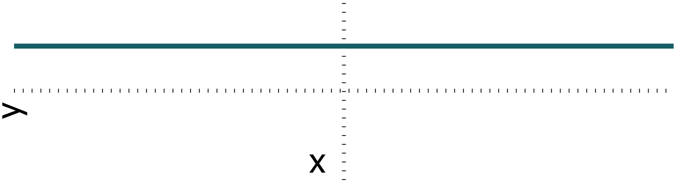
A function and its derivative

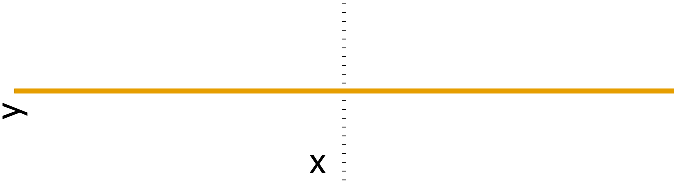
$f(x) = x$ $\quad f'(x) = 1$



A function and its derivative

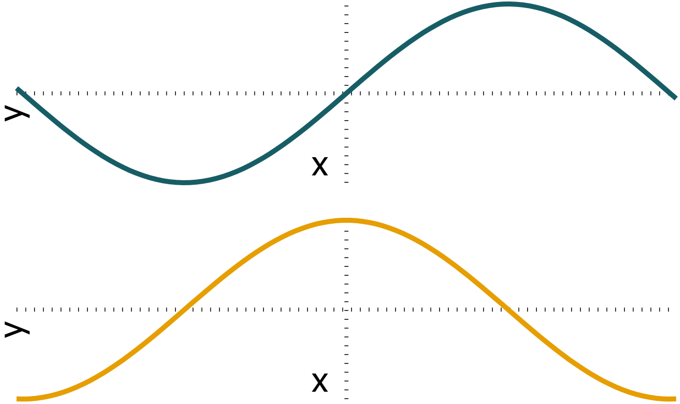
$f(x) = 0.5$ $\qquad$ $f'(x) = 0$





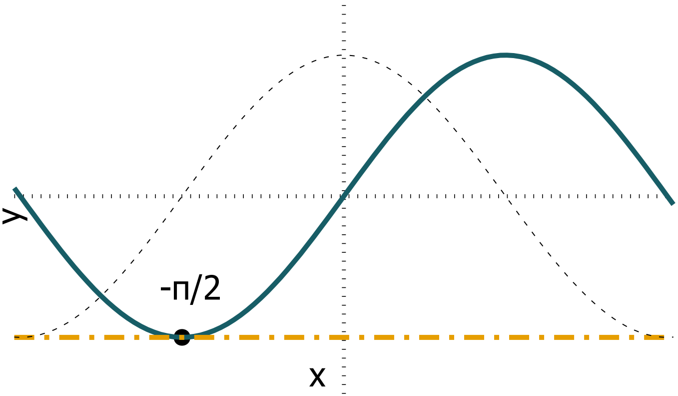
A function and its derivative

$f(x) = sin(x)$ $f'(x) = cos(x)$



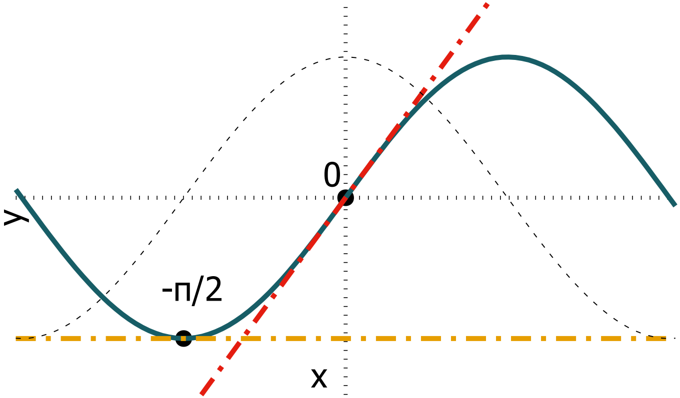
Derivative as a Tangent

$f(x) = sin(x)$ $\qquad$ $f'(x) = cos(x)$



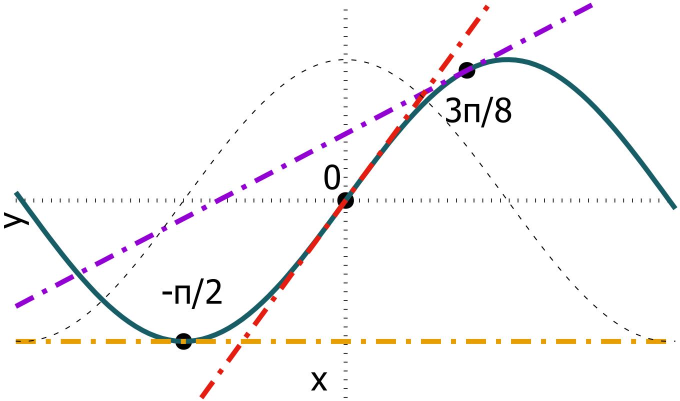
Derivative as a Tangent

$f(x) = sin(x)$ $\quad$ $f'(x) = cos(x)$



Derivative as a Tangent

$f(x) = sin(x)$ $\quad$ $f'(x) = cos(x)$



Formal Definition

Differentiable Function

- A function f with domain M is called differentiable at position x_0 if, if the limit value

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists.

Formal Definition

Differentiable Function

- A function f with domain M is called differentiable at position x_0 if, if the limit value

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

  exists.

- This limit is called f' or **derivative of f at position x_0**. If f' is defined for all $x_0 \in M$, then f' becomes a new function called the derivative of f

Formal Definition

Differentiable Function

- A function f with domain M is called differentiable at position x_0 if, if the limit value
$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$
exists.

- This limit is called f' or **derivative of f at position x_0**. If f' is defined for all $x_0 \in M$, then f' becomes a new function called the derivative of f

- Alternate notations:
$$f'(x_0) = \frac{df}{dx}(x_0) = \lim_{x \to 0} \frac{f(x+h) - f(x_0)}{h}$$

Differentiation as Limit Example

- **Statement:** The derivative of $f(x) = x^2$ is $f'(x) = 2x$

Differentiation as Limit Example

- **Statement:** The derivative of $f(x) = x^2$ is $f'(x) = 2x$
- Applying the formula

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - {x_0}^2}{x - x_0}$$

Differentiation as Limit Example

- **Statement:** The derivative of $f(x) = x^2$ is $f'(x) = 2x$
- Applying the formula

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - {x_0}^2}{x - x_0}$$

- Simplifying

$$\lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\cancel{(x - x_0)}(x + x_0)}{\cancel{x - x_0}} = \lim_{x \to x_0} (x + x_0)$$

Differentiation as Limit Example

- **Statement:** The derivative of $f(x) = x^2$ is $f'(x) = 2x$
- Applying the formula

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - {x_0}^2}{x - x_0}$$

- Simplifying

$$\lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\cancel{(x - x_0)}(x + x_0)}{\cancel{x - x_0}} = \lim_{x \to x_0} (x + x_0)$$

- Applying the limit:

$$\lim_{x \to x_0} (x + x_0) = 2x$$

Differentiation is a linear operator

Rules

- **Constant Factor**

$$\frac{d}{dx}(af) = a\frac{d}{dx}(f)$$

- **Sums**

$$\frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)$$

Example:

$$\frac{d}{dx}(4x^2) = 4\frac{d}{dx}(x^2) = 4(2x) = 8x$$

Differentiation is a linear operator

Rules

- **Constant Factor**

$$\frac{d}{dx}(af) = a\frac{d}{dx}(f)$$

- **Sums**

$$\frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)$$

Example:

$$\frac{d}{dx}(4x^2) = 4\frac{d}{dx}(x^2) = 4(2x) = 8x$$

$$\frac{d}{dx}(4x^2 + x^2) = 4\frac{d}{dx}(x^2) + \frac{d}{dx}(x^2) = 4(2x) + 2x = 10x$$

Differentiation for Products and Quotients

Rules

- **Multiplication**

$$\frac{d}{dx}(fg) = \frac{d}{dx}(f)g + f\frac{d}{dx}(g)$$

- **Exponentiation**

$$\frac{d}{dx}(f^n) = n\frac{d}{dx}(f)^{n-1}$$

- **Division**

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{d}{dx}(f)g - f\frac{d}{dx}(g)}{g^2}$$

Examples

- **Multiplication**

$$\frac{d}{dx}(x^2 sin(x)) = \frac{d}{dx}(x^2)sin(x) + x^2\frac{d}{dx}(sin(x)) = 2xsin(x) + x^2cos(x)$$

Examples

- **Multiplication**

$$\frac{d}{dx}(x^2 sin(x)) = \frac{d}{dx}(x^2)sin(x) + x^2\frac{d}{dx}(sin(x)) = 2xsin(x) + x^2 cos(x)$$

- **Division**

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{\frac{d}{dx}(1)x - 1\frac{d}{dx}(x)}{x^2} = \frac{0-1}{x^2} = \frac{-1}{x^2}$$

Exponentiation Rule derives from Multiplication Rule

- Example $f'(x^3)$

$$\frac{d}{dx}(x^3) = \frac{d}{dx}(x^2x) = \frac{d}{dx}(x^2)x + x^2\frac{d}{dx}(x)$$

Exponentiation Rule derives from Multiplication Rule

- Example $f'(x^3)$

$$\frac{d}{dx}(x^3) = \frac{d}{dx}(x^2x) = \frac{d}{dx}(x^2)x + x^2\frac{d}{dx}(x)$$

$$= 2xx + x^2 = 3x^2$$

Exponentiation Rule derives from Multiplication Rule

- Example $f'(x^3)$

$$\frac{d}{dx}(x^3) = \frac{d}{dx}(x^2x) = \frac{d}{dx}(x^2)x + x^2\frac{d}{dx}(x)$$

$$= 2xx + x^2 = 3x^2$$

- Example $f'(x^4)$

$$\frac{d}{dx}(x^4) = \frac{d}{dx}(x^2x^2) = \frac{d}{dx}(x^2)x^2 + x^2\frac{d}{dx}(x^2)$$

Exponentiation Rule derives from Multiplication Rule

- Example $f'(x^3)$

$$\frac{d}{dx}(x^3) = \frac{d}{dx}(x^2x) = \frac{d}{dx}(x^2)x + x^2\frac{d}{dx}(x)$$

$$= 2xx + x^2 = 3x^2$$

- Example $f'(x^4)$

$$\frac{d}{dx}(x^4) = \frac{d}{dx}(x^2x^2) = \frac{d}{dx}(x^2)x^2 + x^2\frac{d}{dx}(x^2)$$

$$= 2xx^2 + x^2 2x = 2x^3 + 2x^3 = 4x^3$$

Special cases

- The derivative of $f(x) = e^x$ is $f'(x) = e^x$
- The derivative of $f(x) = ln(x)$ is $f'(x) = \frac{1}{x}$
- The derivative of $f(x) = sin(x)$ is $f'(x) = cos(x)$

Composite functions

Chain Rule

- Function h is a composition of functions g and f

$$h(x) = (g \circ f)(x) = g(f(x))$$

- If g and f are differentiable, h is also differentiable

$$\frac{d}{dx}(h(x)) = \frac{d}{dx}(g(y))\frac{d}{dx}(f(x)), \text{ with } y = f(x)$$

- Verbal rule: **Inner derivative times outer derivative**

Chain Rule Examples

- $h(x) = 5(7x+2)^4 = g(f(x))$

Chain Rule Examples

- $\boldsymbol{h(x) = 5(7x+2)^4 = g(f(x))}$

  $g(x) = 5x^4 \wedge f(x) = 7x + 2$

Chain Rule Examples

- $h(x) = 5(7x+2)^4 = g(f(x))$

$$g(x) = 5x^4 \wedge f(x) = 7x + 2$$
$$g'(x) = 20x^3 \wedge f'(x) = 7$$

Chain Rule Examples

- $\boldsymbol{h(x) = 5(7x+2)^4 = g(f(x))}$

$$g(x) = 5x^4 \wedge f(x) = 7x + 2$$
$$g'(x) = 20x^3 \wedge f'(x) = 7$$
$$h'(x) = 20(7x+2)^3 7 = 140(7x+2)^3$$

Chain Rule Examples

- $h(x) = 5(7x+2)^4 = g(f(x))$

$$\begin{aligned} g(x) &= 5x^4 \wedge f(x) = 7x + 2 \\ g'(x) &= 20x^3 \wedge f'(x) = 7 \\ h'(x) &= 20(7x+2)^3 7 = 140(7x+2)^3 \end{aligned}$$

- $h(x) = e^{5x} = g(f(x))$

Chain Rule Examples

- $\boldsymbol{h(x) = 5(7x+2)^4 = g(f(x))}$

$$
\begin{aligned}
g(x) &= 5x^4 \wedge f(x) = 7x + 2 \\
g'(x) &= 20x^3 \wedge f'(x) = 7 \\
h'(x) &= 20(7x+2)^3 7 = 140(7x+2)^3
\end{aligned}
$$

- $\boldsymbol{h(x) = e^{5x} = g(f(x))}$

$$
g(x) = e^x \wedge f(x) = 5x
$$

Chain Rule Examples

- $\boldsymbol{h(x) = 5(7x+2)^4 = g(f(x))}$

$$g(x) = 5x^4 \wedge f(x) = 7x + 2$$
$$g'(x) = 20x^3 \wedge f'(x) = 7$$
$$h'(x) = 20(7x+2)^3 7 = 140(7x+2)^3$$

- $\boldsymbol{h(x) = e^{5x} = g(f(x))}$

$$g(x) = e^x \wedge f(x) = 5x$$
$$g'(x) = e^x \wedge f'(x) = 5$$

Chain Rule Examples

- $\boldsymbol{h(x) = 5(7x+2)^4 = g(f(x))}$

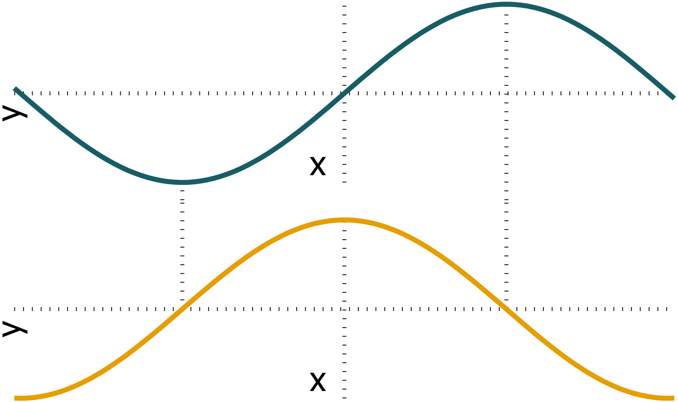
$$g(x) = 5x^4 \wedge f(x) = 7x + 2$$
$$g'(x) = 20x^3 \wedge f'(x) = 7$$
$$h'(x) = 20(7x+2)^3 7 = 140(7x+2)^3$$

- $\boldsymbol{h(x) = e^{5x} = g(f(x))}$

$$g(x) = e^x \wedge f(x) = 5x$$
$$g'(x) = e^x \wedge f'(x) = 5$$
$$h'(x) = e^{5x} 5 = 5e^{5x}$$

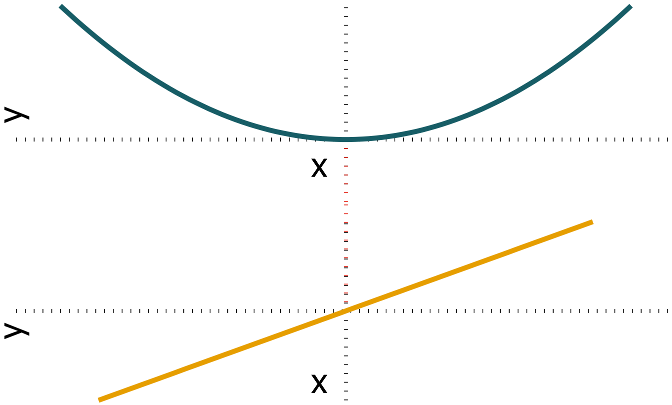
Finding Local Extrema

$f(x) = sin(x)$ $f'(x) = cos(x)$



Finding Local Extrema

$f(x) = x^2$ $f'(x) = 2x$



Calculation of Local Extrema

- $f(x) = 4x^2 + 6x$

Calculation of Local Extrema

- $\boldsymbol{f(x) = 4x^2 + 6x}$

  $f'(x) = 8x + 6$

Calculation of Local Extrema

- $\boldsymbol{f(x) = 4x^2 + 6x}$

  $$f'(x) = 8x + 6$$

  $$f'(x) = 8x + 6 \stackrel{!}{=} 0$$

Calculation of Local Extrema

- $f(x) = 4x^2 + 6x$

  $f'(x) = 8x + 6$

  $f'(x) = 8x + 6 \overset{!}{=} 0$

  $\iff 8x = -6$

Calculation of Local Extrema

- $\boldsymbol{f(x) = 4x^2 + 6x}$

  $$f'(x) = 8x + 6$$

  $$f'(x) = 8x + 6 \stackrel{!}{=} 0$$

  $$\iff 8x = -6$$

  $$\iff x = \frac{-6}{8} = \frac{-3}{4}$$

Calculation of Local Extrema

- $\boldsymbol{f(x) = 4x^2 + 6x}$

  $$f'(x) = 8x + 6$$

  $$f'(x) = 8x + 6 \overset{!}{=} 0$$

  $$\iff 8x = -6$$

  $$\iff x = \frac{-6}{8} = \frac{-3}{4}$$

- $\boldsymbol{f(x) = sin(x)}$

Calculation of Local Extrema

- $\boldsymbol{f(x) = 4x^2 + 6x}$

  $$f'(x) = 8x + 6$$

  $$f'(x) = 8x + 6 \overset{!}{=} 0$$

  $$\iff 8x = -6$$

  $$\iff x = \frac{-6}{8} = \frac{-3}{4}$$

- $\boldsymbol{f(x) = sin(x)}$

  $$f'(x) = cos(x)$$

Calculation of Local Extrema

- $f(x) = 4x^2 + 6x$

  $$f'(x) = 8x + 6$$

  $$f'(x) = 8x + 6 \stackrel{!}{=} 0$$

  $$\iff 8x = -6$$

  $$\iff x = \frac{-6}{8} = \frac{-3}{4}$$

- $f(x) = sin(x)$

  $$f'(x) = cos(x)$$

  $$f'(x) = cos(x) \stackrel{!}{=} 0$$

Calculation of Local Extrema

- $\boldsymbol{f(x) = 4x^2 + 6x}$

$$f'(x) = 8x + 6$$

$$f'(x) = 8x + 6 \overset{!}{=} 0$$

$$\iff 8x = -6$$

$$\iff x = \frac{-6}{8} = \frac{-3}{4}$$

- $\boldsymbol{f(x) = sin(x)}$

$$f'(x) = cos(x)$$

$$f'(x) = cos(x) \overset{!}{=} 0$$

$$\iff x = cos^{-1}(0)$$

Calculation of Local Extrema

- $\boldsymbol{f(x) = 4x^2 + 6x}$

$$f'(x) = 8x + 6$$

$$f'(x) = 8x + 6 \stackrel{!}{=} 0$$

$$\iff 8x = -6$$

$$\iff x = \frac{-6}{8} = \frac{-3}{4}$$

- $\boldsymbol{f(x) = sin(x)}$

$$f'(x) = cos(x)$$

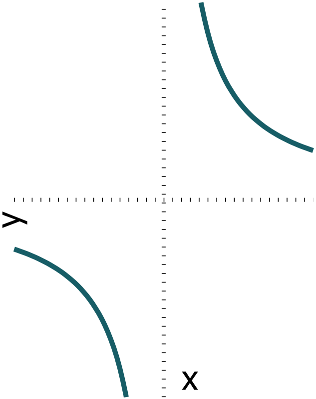
$$f'(x) = cos(x) \stackrel{!}{=} 0$$

$$\iff x = cos^{-1}(0)$$

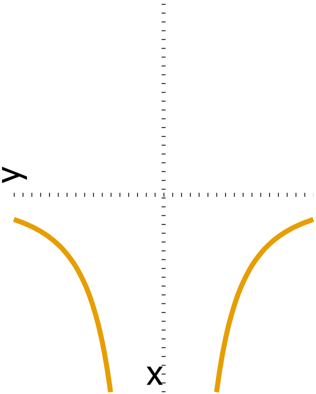
$$\iff x = 90^\circ = \frac{\pi}{2}, 270^\circ = \frac{3\pi}{2}, \dots$$

Differentiability is not given

$f(x) = \frac{1}{x}$



$f'(x) = \frac{-1}{x^2}$



Numerical Differentiation

- **Problem:** Only function values $f(x_0)$ of $f(x)$ are known, but not the real function f

Numerical Differentiation

- **Problem:** Only function values $f(x_0)$ of $f(x)$ are known, but not the real function f
- Instead of calculating the derivative of f analytically, it is possible to approximate $f'(x)$ using **numerical differentiation**

Numerical Differentiation

- **Problem:** Only function values $f(x_0)$ of $f(x)$ are known, but not the real function f
- Instead of calculating the derivative of f analytically, it is possible to approximate $f'(x)$ using **numerical differentiation**

(Simple) Numerical Differentiation

The set $\mathbb{I}$ describes the computable domain of f in the given context. It is possible to calculate function value $f(x_i)$, where $x_i \in \mathbb{I}$.

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i},$$

where x_{i+1} is the smallest positive distance from x_i in $\mathbb{I}$.

Numerical Differentiation Example

- From a sensor we receive the following values:

x_i	0	1	2	3	4	5	6	7	8	9
$f(x_i)$	3.1	2.9	2.4	1.4	1.6	3	3.1	3.3	3.5	4.2

- The derivative at x_3 equals:

$$f'(x_3) = \frac{f(x_{3+1}) - f(x_3)}{x_{3+1} - x_3} \Rightarrow \frac{f(x_4) - f(x_3)}{4 - 3} = \frac{1.6 - 1.4}{1} = 0.2$$

- The change at position x_3 is 0.2

Task Template Braitenberg

- Download the archive *task_template_4.zip* from the course homepage. Extract it into a folder of your choice.
- The archive contains *task_4_1.py*, *task_4_1_student_code.py* and *braitenberg.png*.
- Use *task_4_1.py* to run the program, but edit code only in *task_4_1_student_code.py*.

Explain Task Template!

Tasks

1. Calculate the vehicle's velocity through numerical differentiation.
   - Open *task_4_1_student_code.py* and implement the function *calc_velocity_from_position*.
   - Use the given list of positions to estimate the vehicles velocity using numerical differentiation.
   - Append the resulting velocity values to the *player_velocities_x* list.
   - **Tip**: Use a for-loop that runs through the position values and compares the current list-entry to the preceding one.

2. Write a script the calculates the Fibonacci sequence for an arbitrary number N of elements. Print the numbers to the console.
   - The first two elements of a_1 and a_2 are always 1
   - Write a loop that runs N times and calculates the Fibonacci number $a_{n+1} = a_n + a_{n-1}$
   - **Tip:** Use variables to store the values for the current value a_n and the previous value a_{n-1} and update them in each loop.