Dynamical systems

Tutorial

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Dynamical systems are the universal language of science

- Physics, engineering, chemistry, theoretical biology, economics, quantitative sociology, ...

Modeling processes

- Time-varying measures

- Forces causing/accounting for movement

=> Dynamical systems
time-variation & rate of change

- variable $x(t)$
- rate of change $\frac{dx}{dt}$
time-variation & rate of change

$x(t)$: position

$x(t) = \frac{dx}{dt}$: rate of change (speed)
time-variation & rate of change

\[ \dot{x} = \frac{dx}{dt} = \text{rate of change} = \text{slope of this graph} \]
dynamical system

\[ \frac{dx}{dt} = f(x) \]

relationship between a variable and its rate of change
dynamical system

\[ \frac{dx}{dt} = f(x) \]

What equation is shown here?
dynamics/phase plot

solution/time course

\dot{x}

x(t)

t (time)
dynamical system

- present determines the future
  - given initial condition
  - predict evolution (or predict the past)

\[
dx/dt = f(x)\]
dynamical system

present determines the future

given initial condition

predict evolution (or predict the past)

\[ \frac{dx}{dt} = f(x) \]
dynamical system

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- given initial condition
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\[ \frac{dx}{dt} = f(x) \]
dynamical system

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\[
dx/dt = f(x)
\]
exponential relaxation to attractors

$\tau \frac{dx}{dt} = -x$

$\Rightarrow$ time scale
x: spans the state space (or phase space)

f(x): is the “dynamics” of x (or vector-field)

x(t) is a solution of the dynamical systems to the initial condition x_0

if its rate of change = f(x)

and x(0)=x_0
numerics

- sample time discretely
- compute solution by iterating through time
**_numerics_

\[ x(t) \]

\[ t_0 \quad t_1 \quad t_2 \quad t_3 \]

\[ \Delta t \]

\[ \dot{x} = f(x) \]

\[ t_i = i \times \Delta t; \quad x_i = x(t_i) \]

\[ \dot{x} = \frac{dx}{dt} \approx \frac{\Delta x}{\Delta t} = \frac{x_{i+1} - x_i}{\Delta t} \]

\[ x_{i+1} = x_i + \Delta t \times f(x_i) \]

[forward Euler]
linear dynamics

=> simulation
A fixed point is a constant solution of the dynamical system

\[ \dot{x} = f(x) \]

\[ \dot{x} = 0 \Rightarrow f(x_0) = 0 \]
fixed points
fixed points
fixed points
stability

- mathematically really: asymptotic stability
- defined: a fixed point is asymptotically stable, when solutions of the dynamical system that start nearby converge in time to the fixed point
fixed point, to which neighboring initial conditions converge = attractor
linear approximation near attractor

- non-linearity as a small perturbation/deformation of linear system
- => non-essential non-linearity
if the slope of the linear system is negative, the fixed point is (asymptotically stable)

=> attractor
stability in a linear system

- if the slope of the linear system is positive, then the fixed point is unstable
- $\Rightarrow$ repellor
stability in a linear system

- If the slope of the linear system is zero, then the system is indifferent (marginally stable: stable but not asymptotically stable)
degree of stability

\[ x^* \]

\[ x \]

slope
bifurcations

- Look now at families of dynamical systems, which depend (smoothly) on parameters.
- Ask: as the parameters change (smoothly), how do the solutions change (smoothly?)

  - Smoothly: topological equivalence of the dynamical systems at neighboring parameter values.
  - Bifurcation: dynamical systems NOT topological equivalent as parameter changes infinitesimally.
bifurcation

bifurcation = qualitative change of dynamics (change in number, nature, or stability of fixed points) as the dynamics changes smoothly.
tangent bifurcation

The simplest bifurcation: an attractor collides with a repellor and the two annihilate.
reverse bifurcation

changing the dynamics in the opposite direction
bifurcations are instabilities

- that is, an attractor becomes unstable before disappearing
- (or the attractor appears with reduced stability)
- formally: a zero-real part is a necessary condition for a bifurcation to occur
tangent bifurcation

=> simulation
tangent bifurcation

- normal form of tangent bifurcation
  \[ \dot{x} = \alpha - x^2 \]

- (=simplest polynomial equation whose flow is topologically equivalent to the bifurcation)
Hopf theorem

- when a single (or pair of complex conjugate) eigenvalue crosses the imaginary axis, one of four bifurcations occur

  - tangent bifurcation
  - transcritical bifurcation
  - pitchfork bifurcation
  - Hopf bifurcation
transcritical bifurcation

normal form

\[ \dot{x} = \alpha x - x^2 \]
pitchfork bifurcation

正常形式

\[ \dot{x} = \alpha x - x^3 \]
pitchfork bifurcation

=> simulation
Hopf: need higher dimensions
2D dynamical system: vector-field

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2)
\end{align*}
\]
\[ \dot{x}_1 = f_1(x_1, x_2) \]
\[ \dot{x}_2 = f_2(x_1, x_2) \]
fixed point, stability, attractor

\[
\dot{x}_1 = f_1(x_1, x_2) \\
\dot{x}_2 = f_2(x_1, x_2)
\]
Hopf bifurcation

$\dot{r} = \alpha r - r^3$

$\dot{\phi} = \omega$

normal form
forward dynamics

- given known equation, determined fixed points / limit cycles and their stability
- more generally: determine invariant solutions (stable, unstable and center manifolds)
given solution, find the equation...

this is the problem faced in design of behavioral dynamics...
inverse dynamics: design

- in the design of behavioral dynamics... you may be given:

- attractor solutions/stable states

- and how they change as a function of parameters/conditions

=> identify the class of dynamical systems using the 4 elementary bifurcations

and use normal form to provide an exemplary representative of the equivalence class of dynamics
important concepts

time-variation
rate of change
dynamical system
phase plot vs. time course plot
present determines the future
numerical solutions
fixed points (attractors, repellors)
stability
bifurcations & instabilities