Dynamical systems

Tutorial

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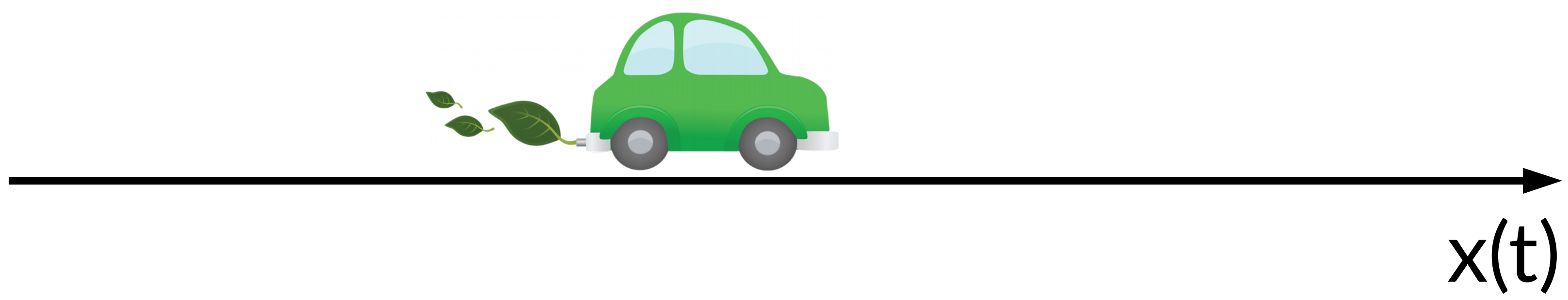
- dynamical systems are the universal language of science
  - physics, engineering, chemistry, theoretical biology, economics, quantitative sociology, ...
- modeling processes
  - time-varying measures
  - forces causing/accounting for movement

    => dynamical systems

time-variation & rate of change

- variable x(t)
- rate of change dx/dt

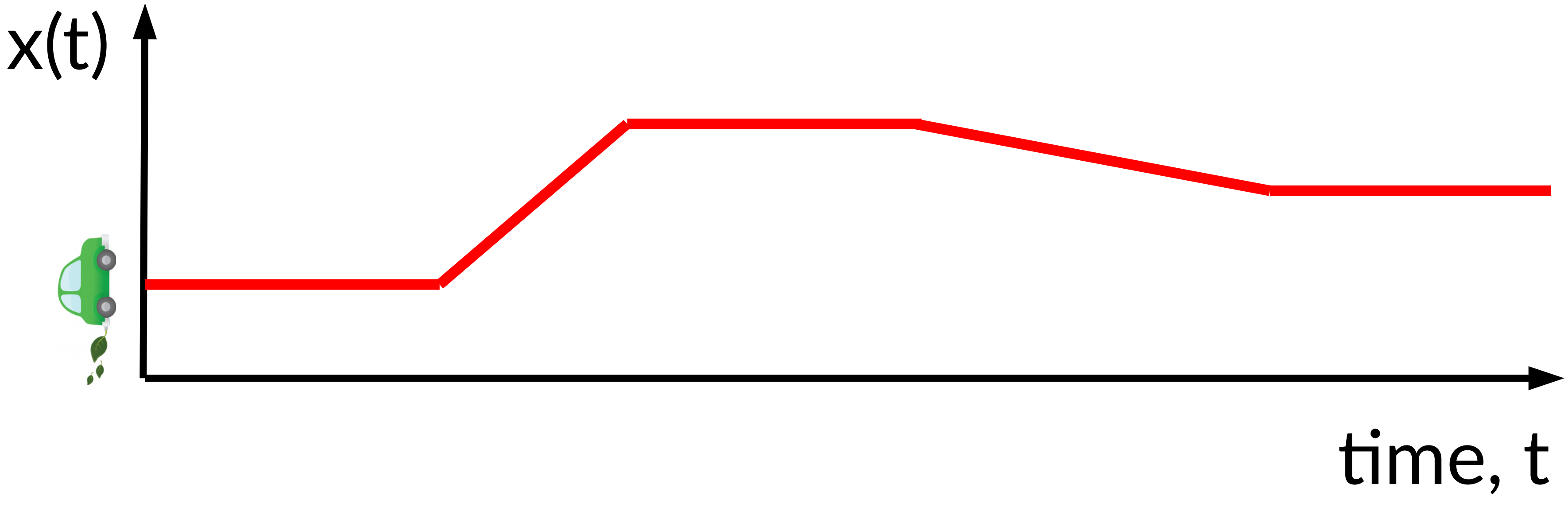
time-variation & rate of change



$x(t)$: position

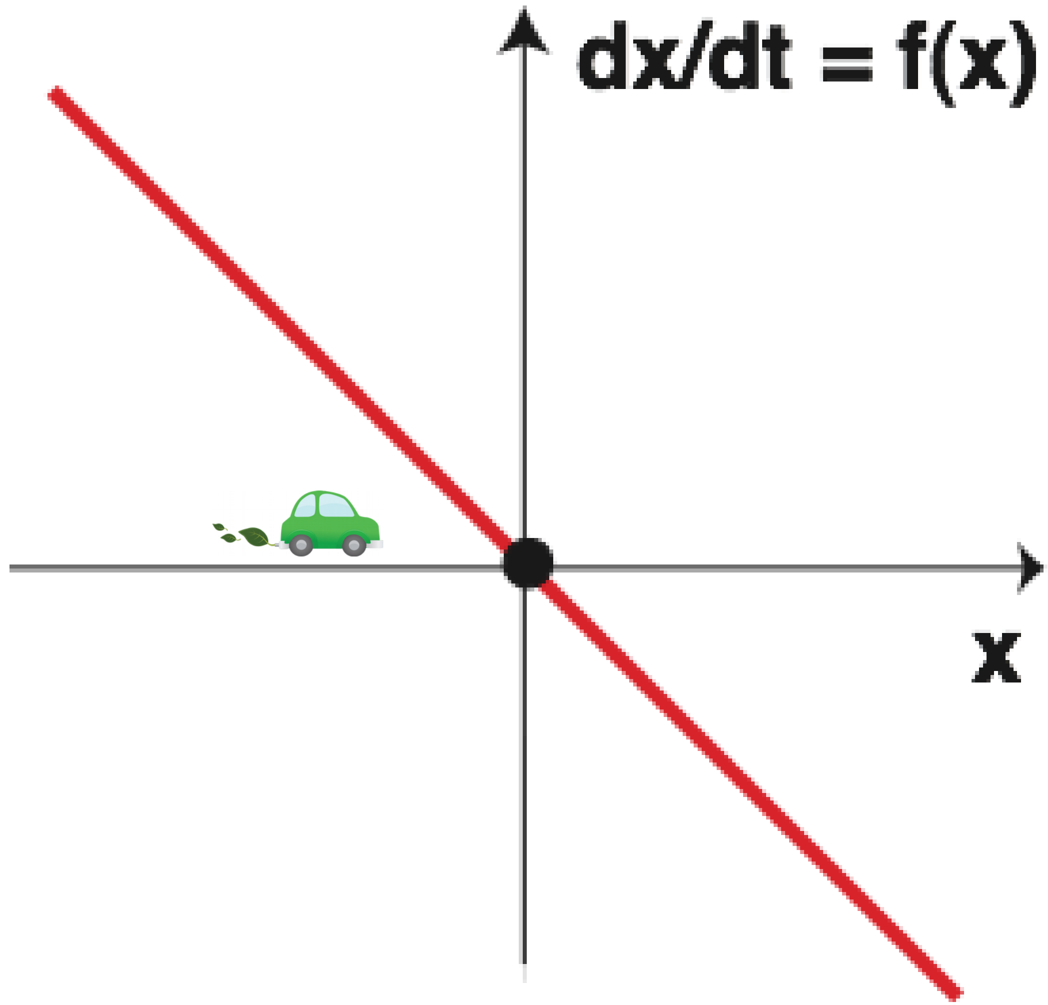
$\dot{x}(t) = dx/dt$: rate of change (speed)

time-variation & rate of change



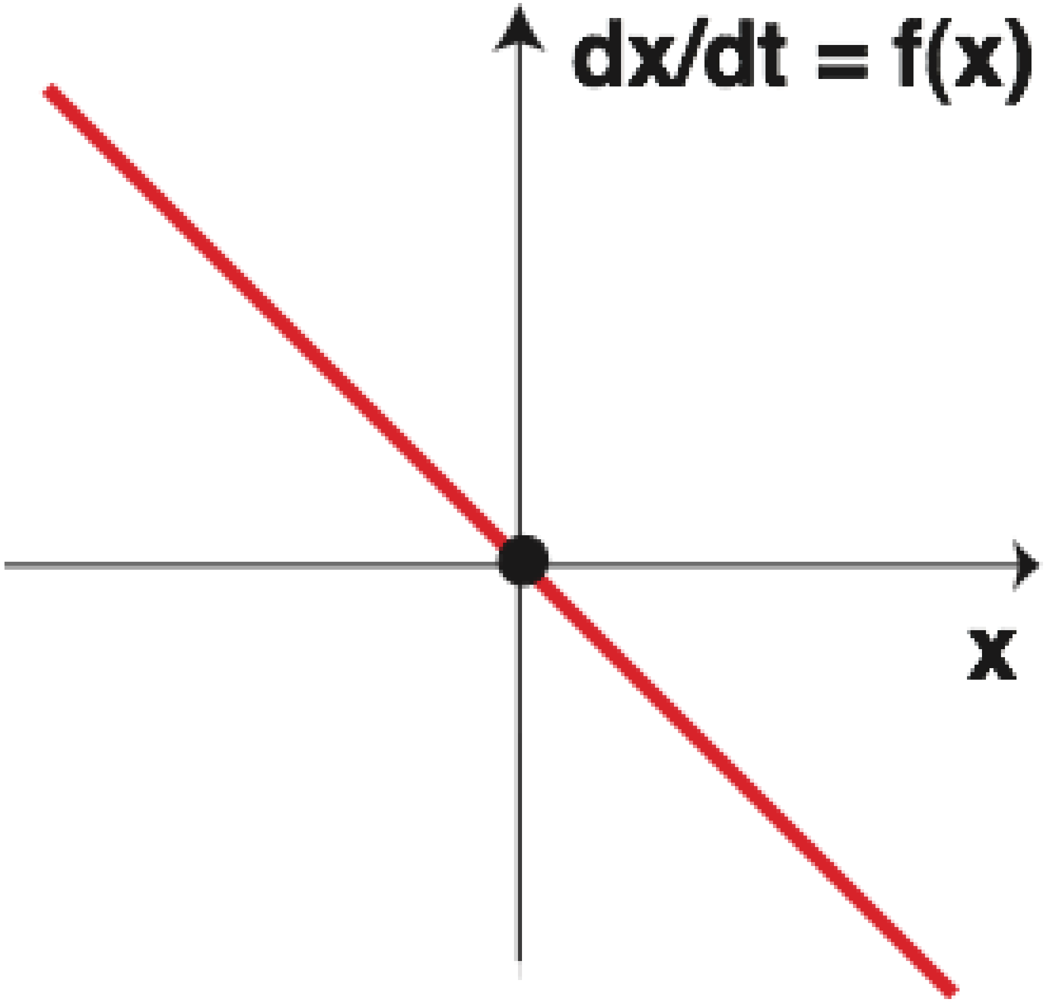
$\dot{x}$ = dx/dt = rate of change = slope of this graph

dynamical system



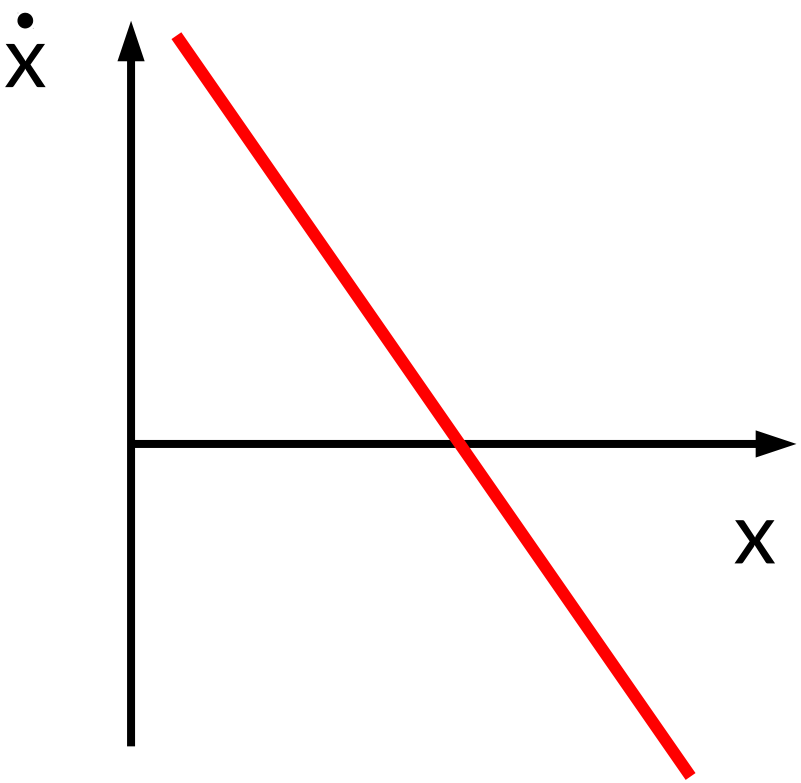
relationship between a variable and its rate of change

dynamical system

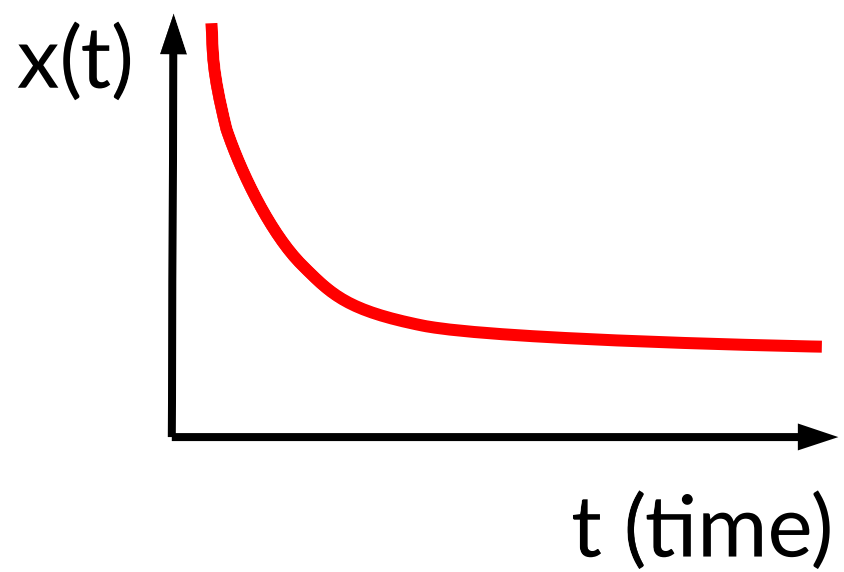


What equation is shown here?

dynamics/
phase plot

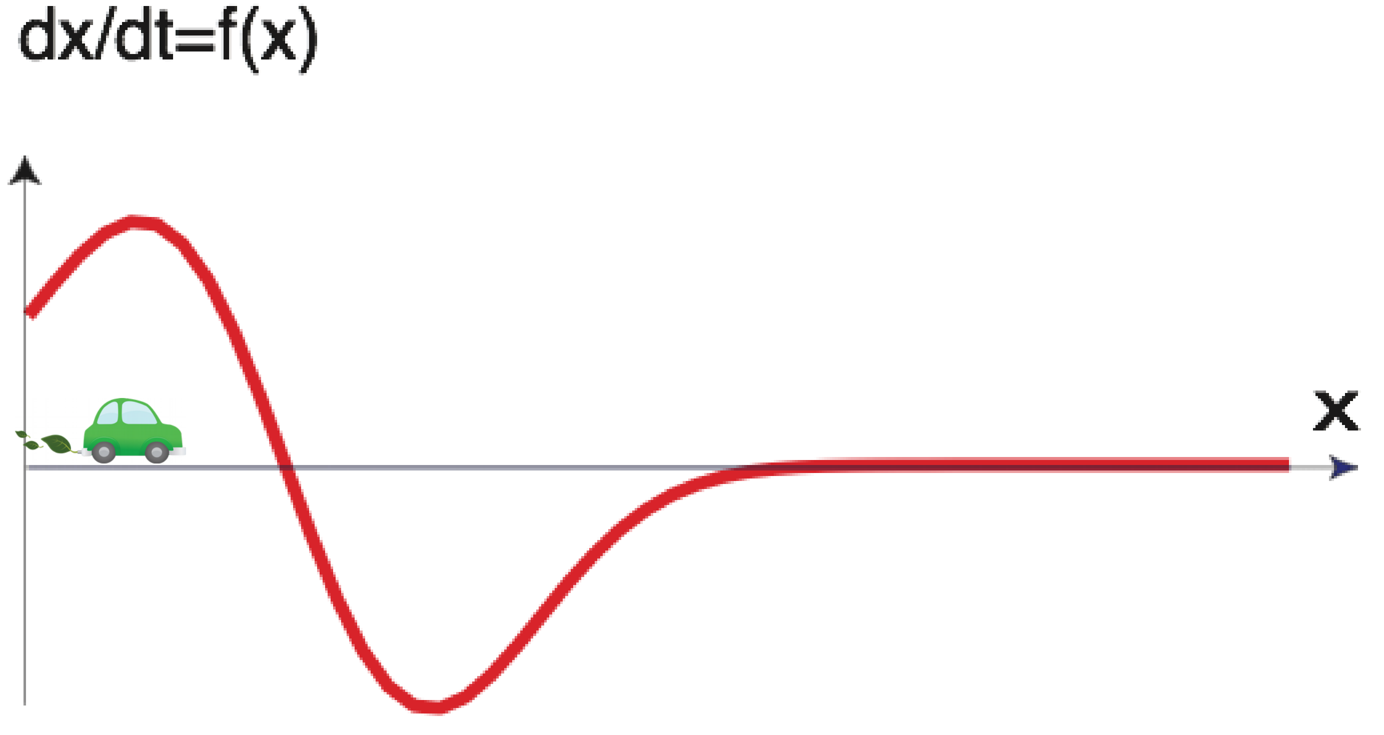


solution/
time course



dynamical system

- present determines the future
  - given initial condition
  - predict evolution (or predict the past)



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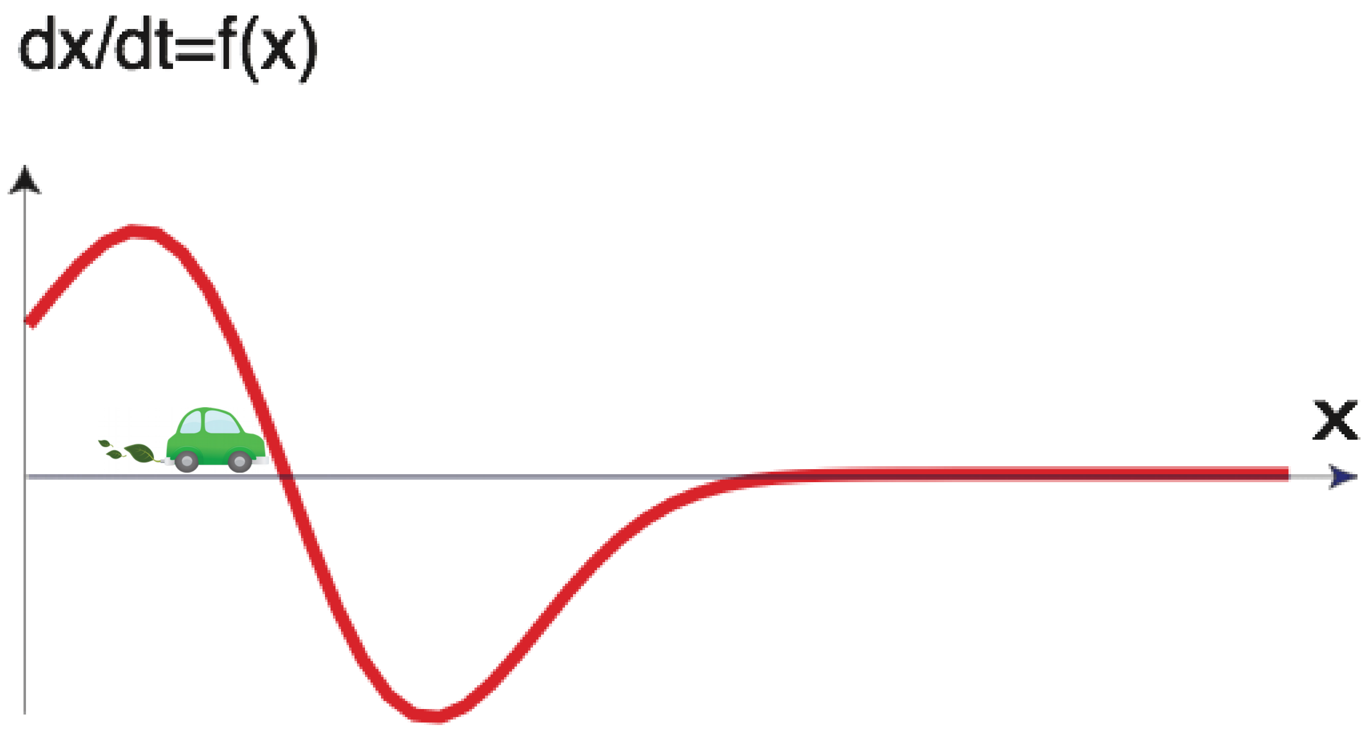

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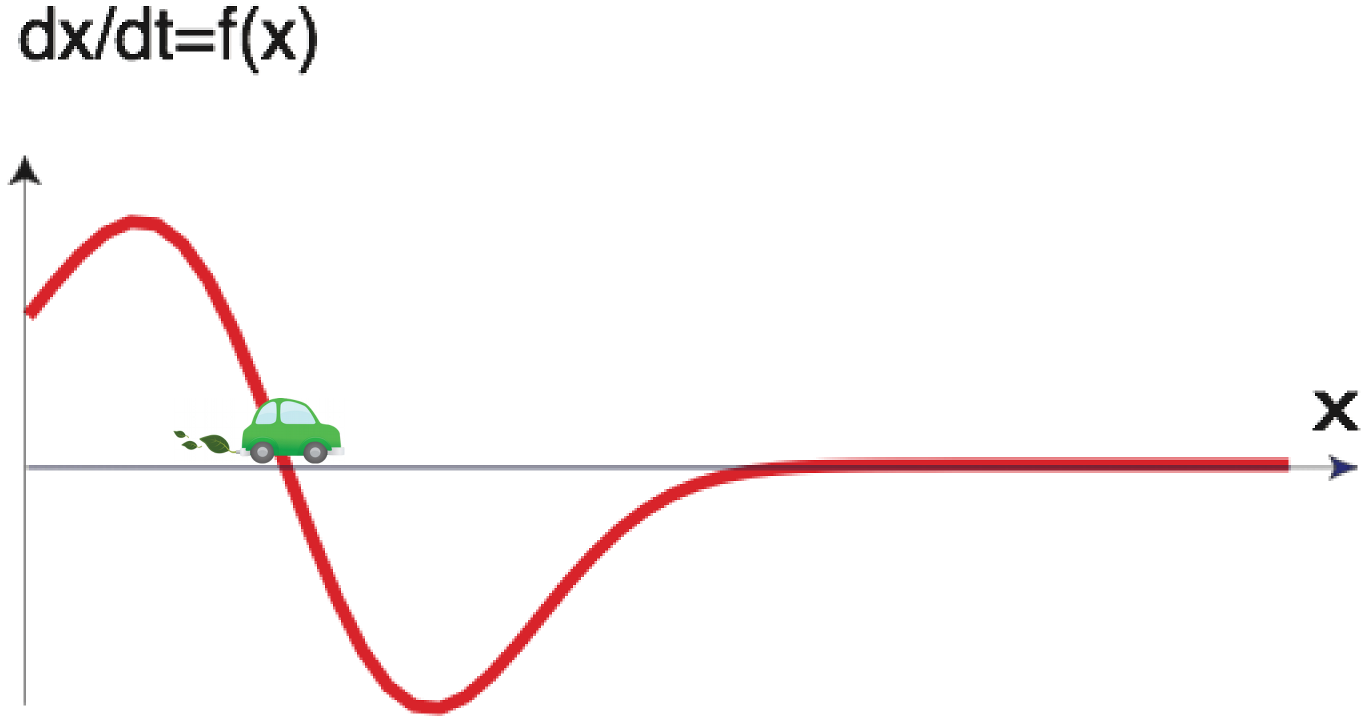

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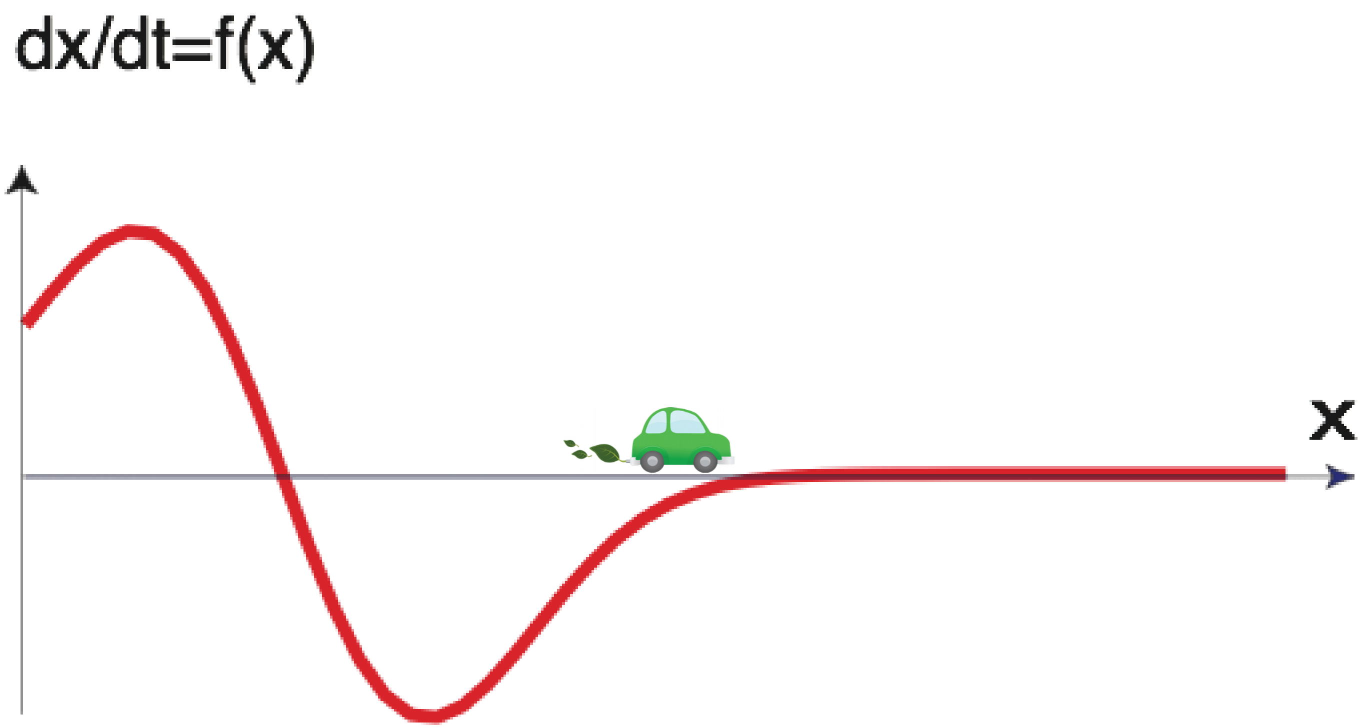

dynamical system

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  - given initial condition
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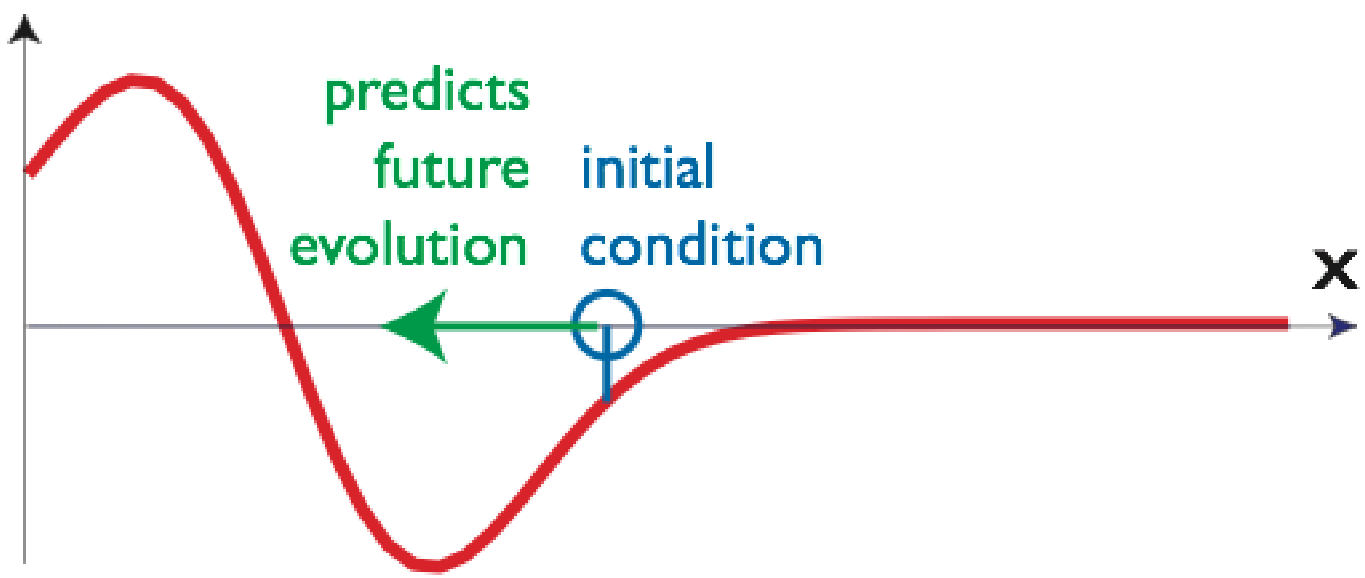
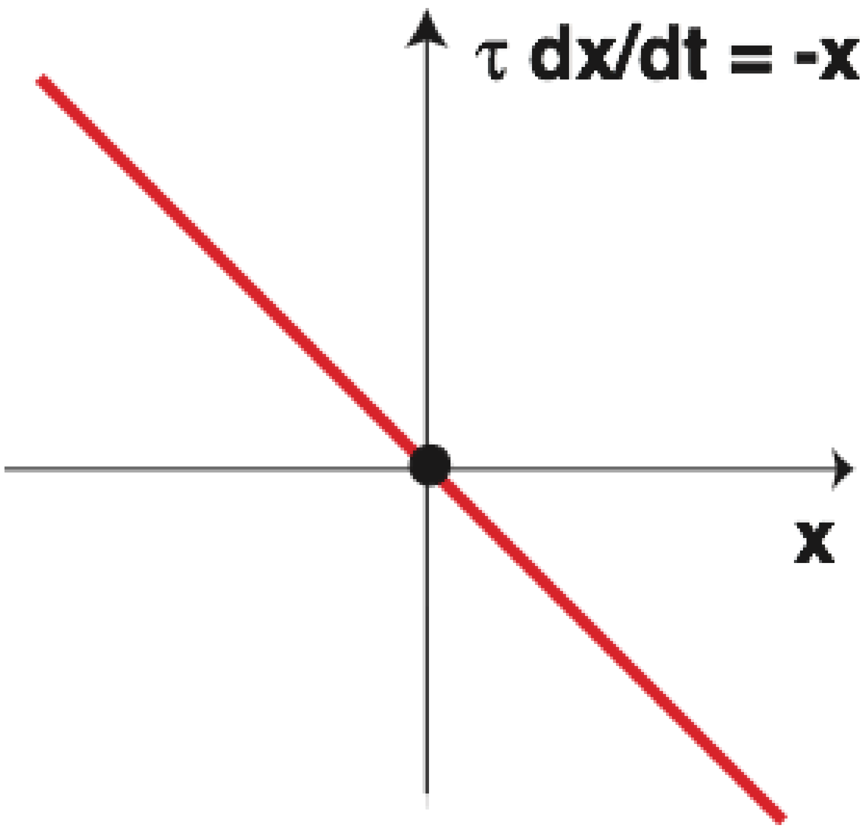
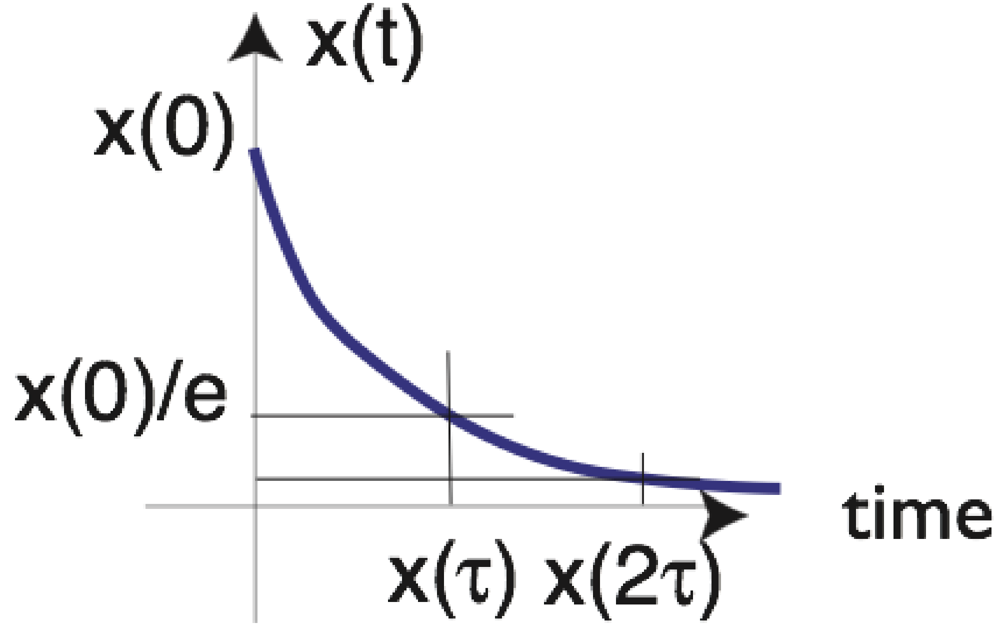
$dx/dt=f(x)$



exponential relaxation to attractors

- => time scale





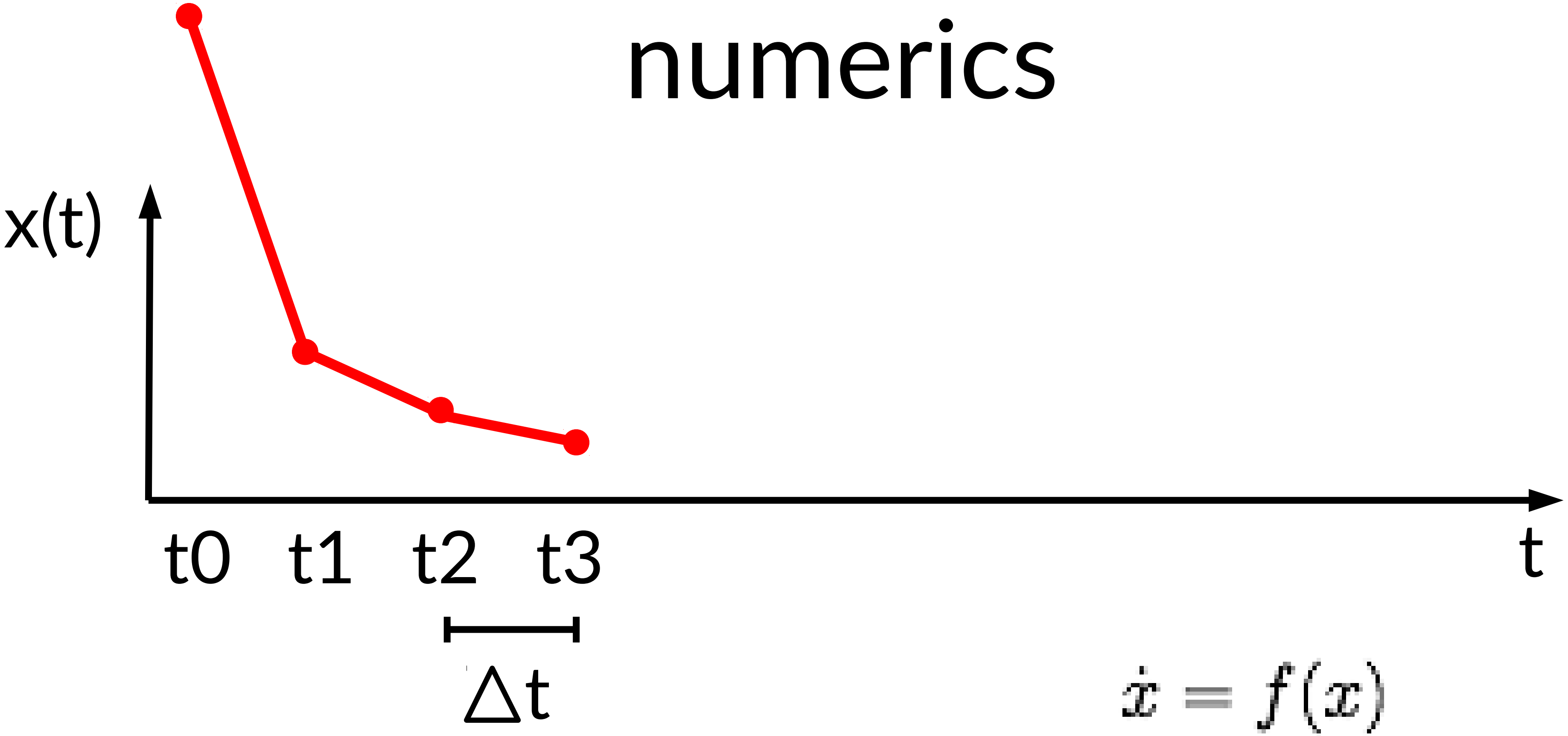
dynamical systems

- x: spans the state space (or phase space)
- f(x): is the “dynamics” of x (or vector-field)
- x(t) is a solution of the dynamical systems to the initial condition x_0
  - if its rate of change = f(x)
  - and x(0)=x_0

numerics

- sample time discretely
- compute solution by iterating through time

numerics



$$\dot{x} = f(x)$$

$$t_i = i * \Delta t; \qquad x_i = x(t_i)$$

$$\dot{x} = \frac{dx}{dt} \approx \frac{\Delta x}{\Delta t} = \frac{x_{i+1} - x_i}{\Delta t}$$

[forward Euler]

$$x_{i+1} = x_i + \Delta t * f(x_i)$$

linear dynamics

=> simulation

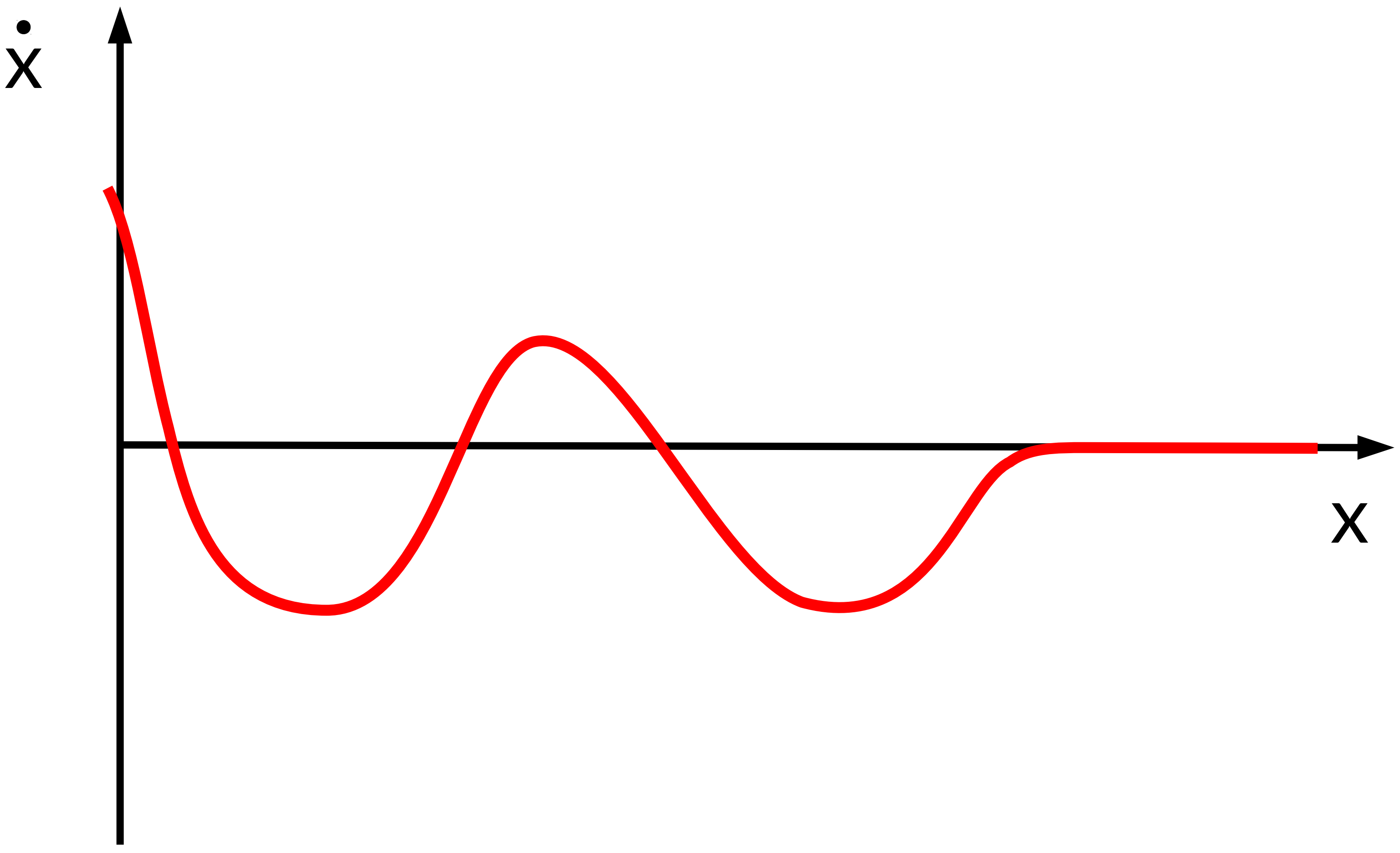
fixed point

- is a constant solution of the dynamical system

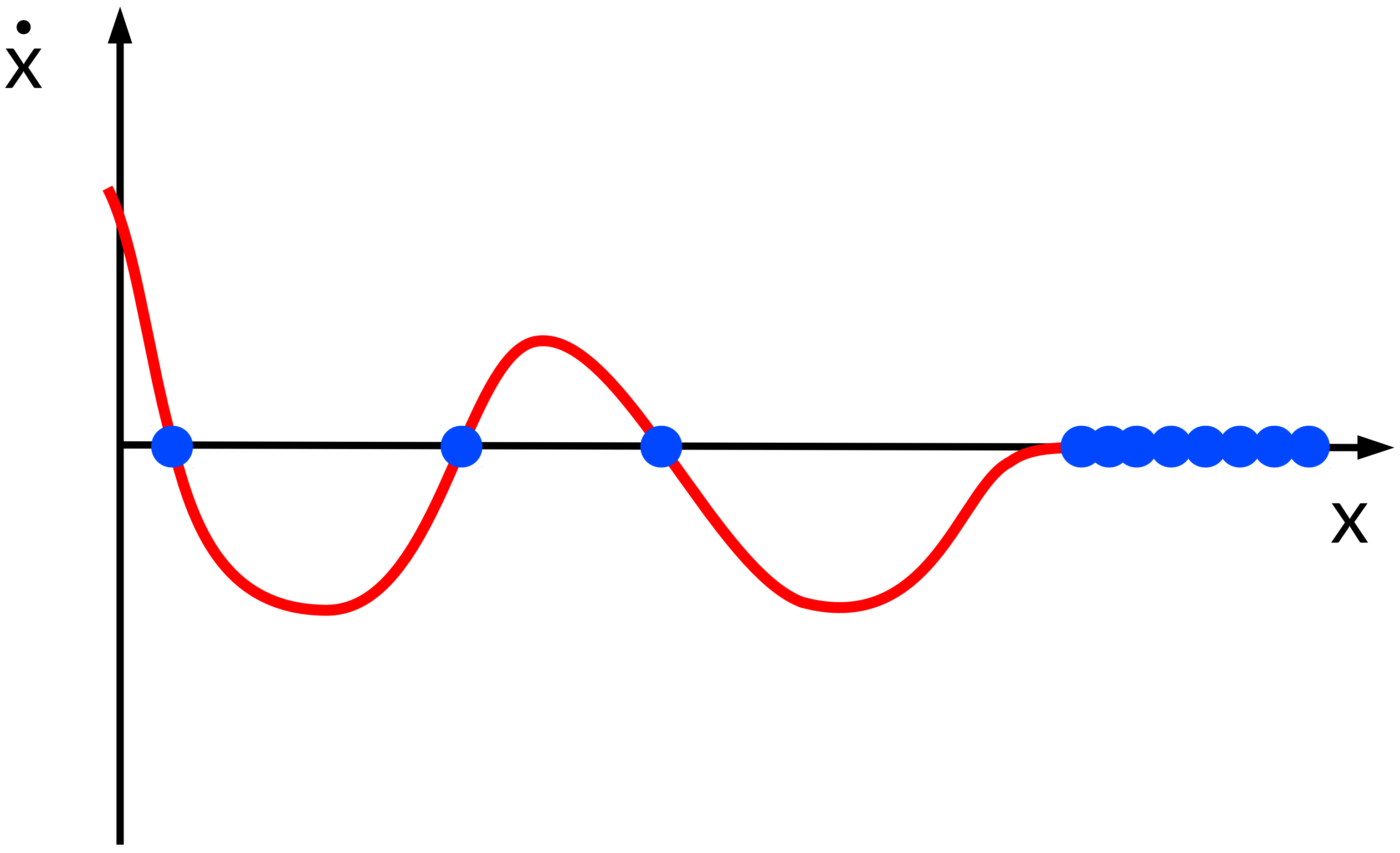
$$\dot{x} = f(x)$$

$$\dot{x} = 0 \Rightarrow f(x_0) = 0$$

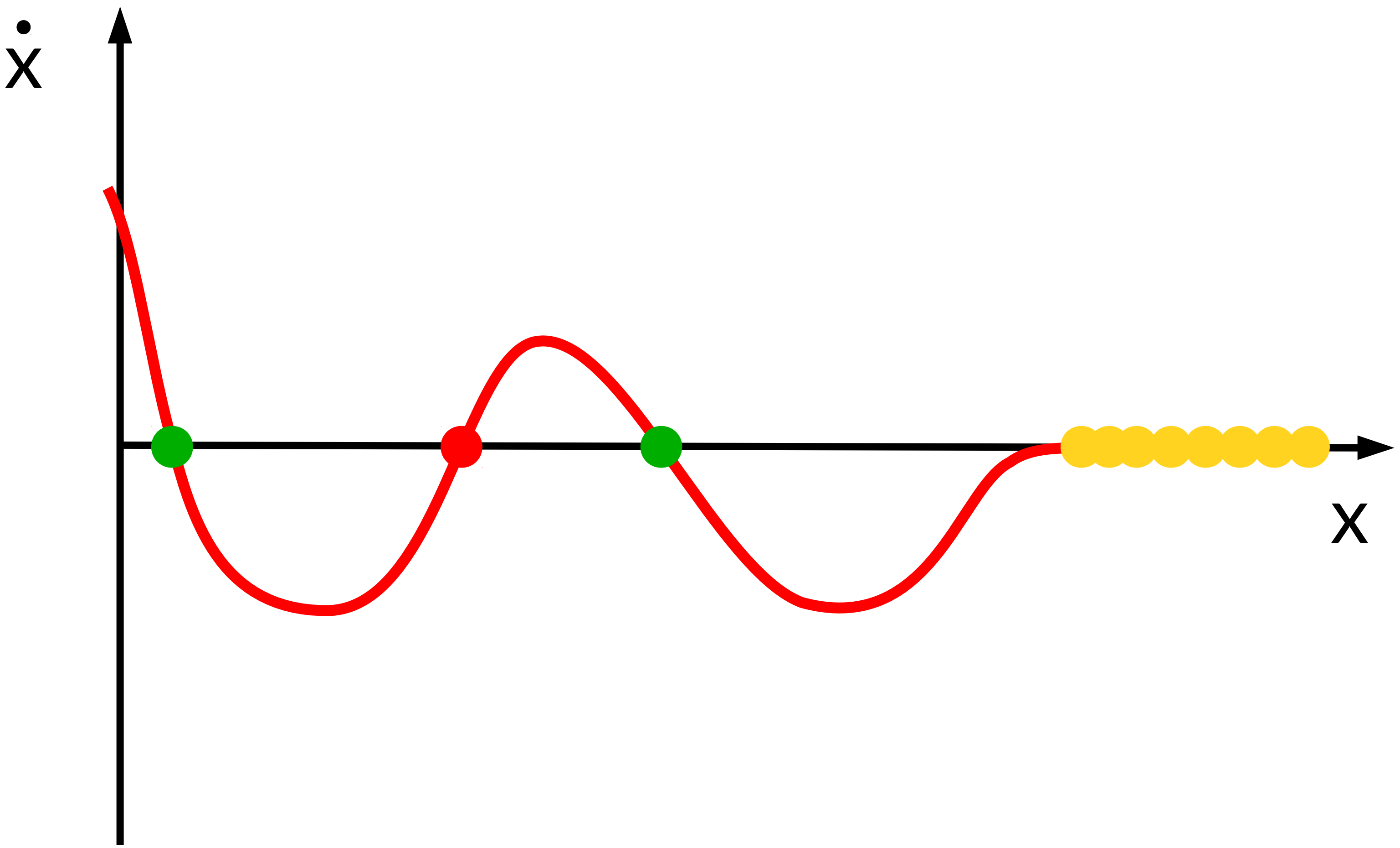
fixed points



fixed points



fixed points

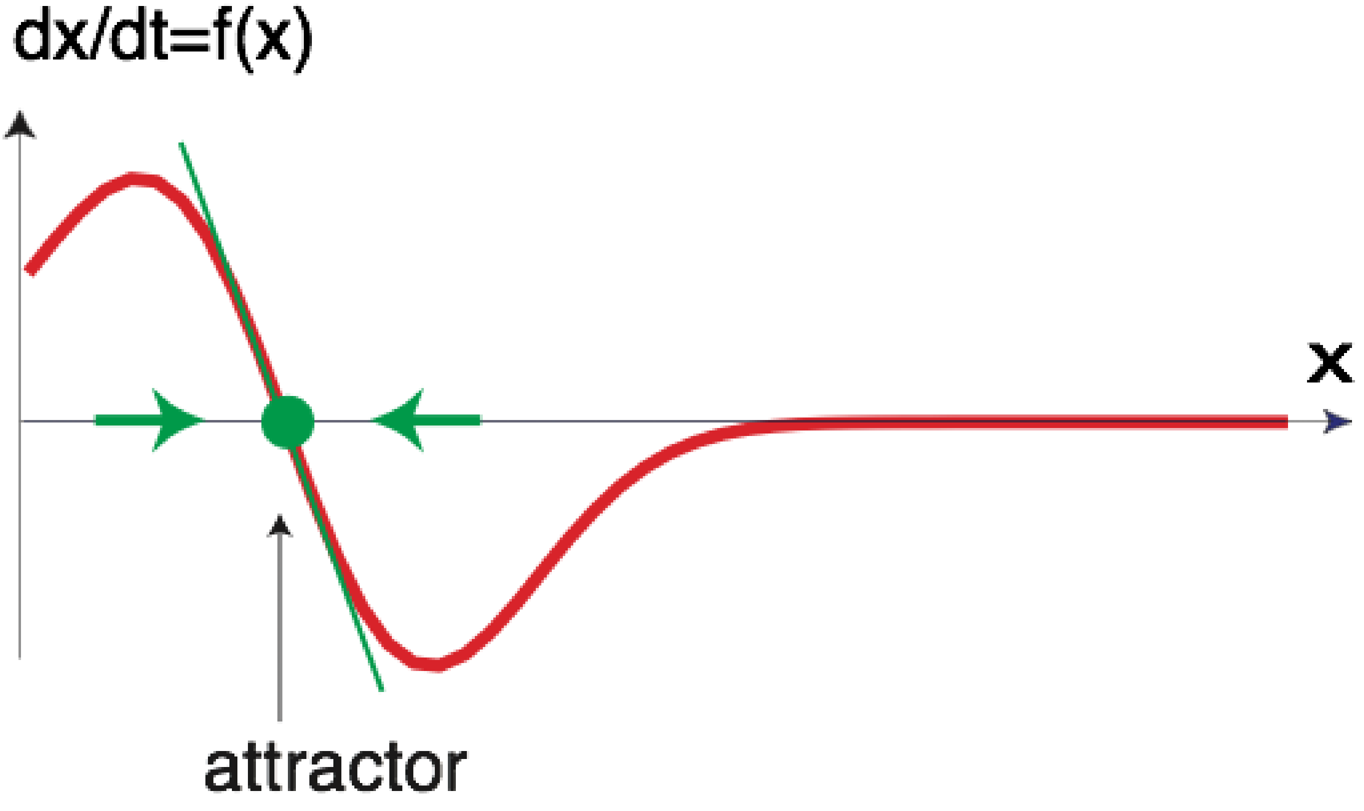


stability

- mathematically really: asymptotic stability
- defined: a fixed point is asymptotically stable, when solutions of the dynamical system that start nearby converge in time to the fixed point

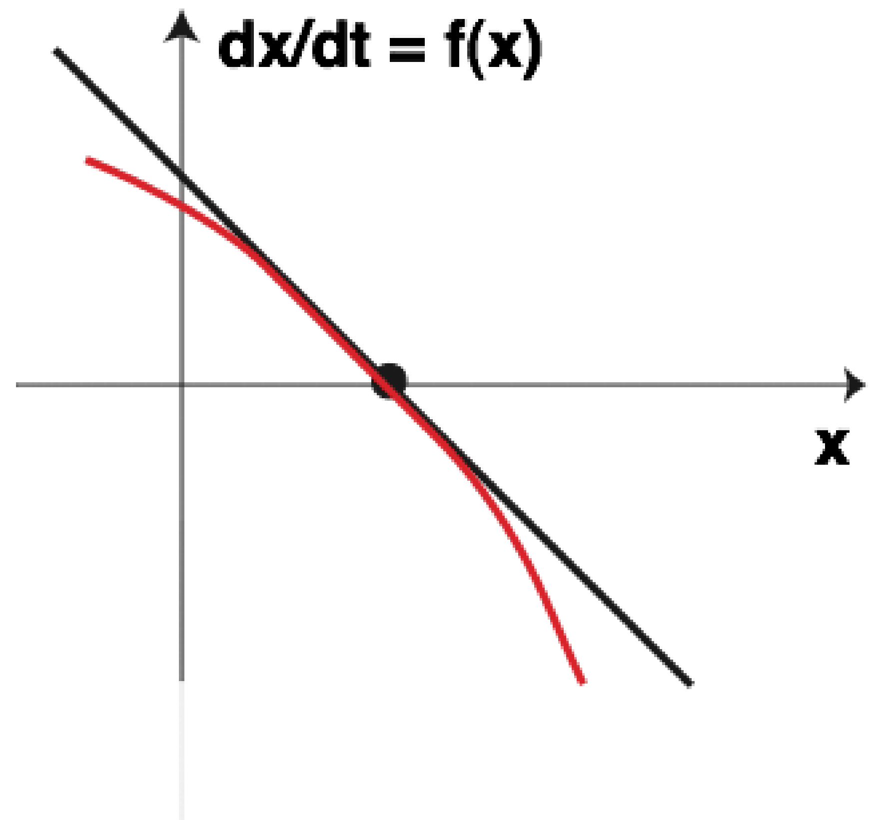
attractor

- fixed point, to which neighboring initial conditions converge = attractor



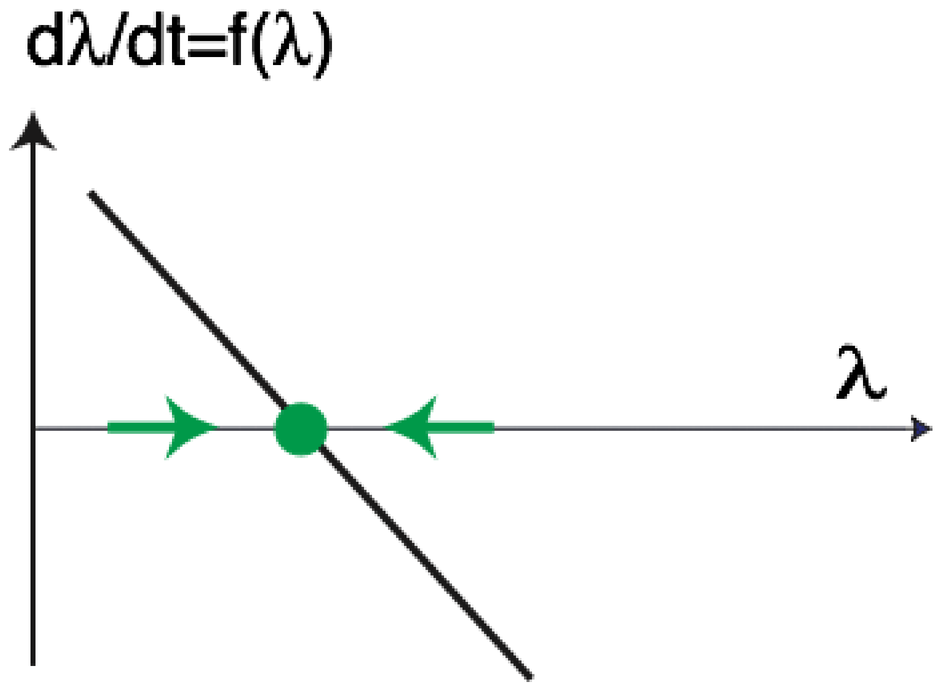
linear approximation near attractor

- non-linearity as a small perturbation/deformation of linear system
- => non-essential non-linearity



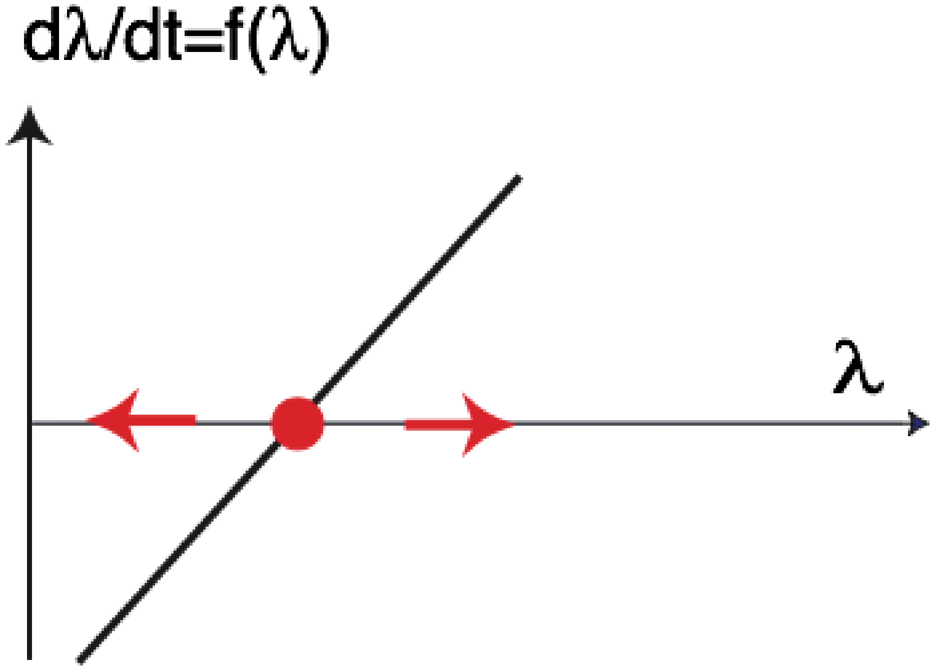
stability in a linear system

- if the slope of the linear system is negative, the fixed point is (asymptotically stable)
- => attractor



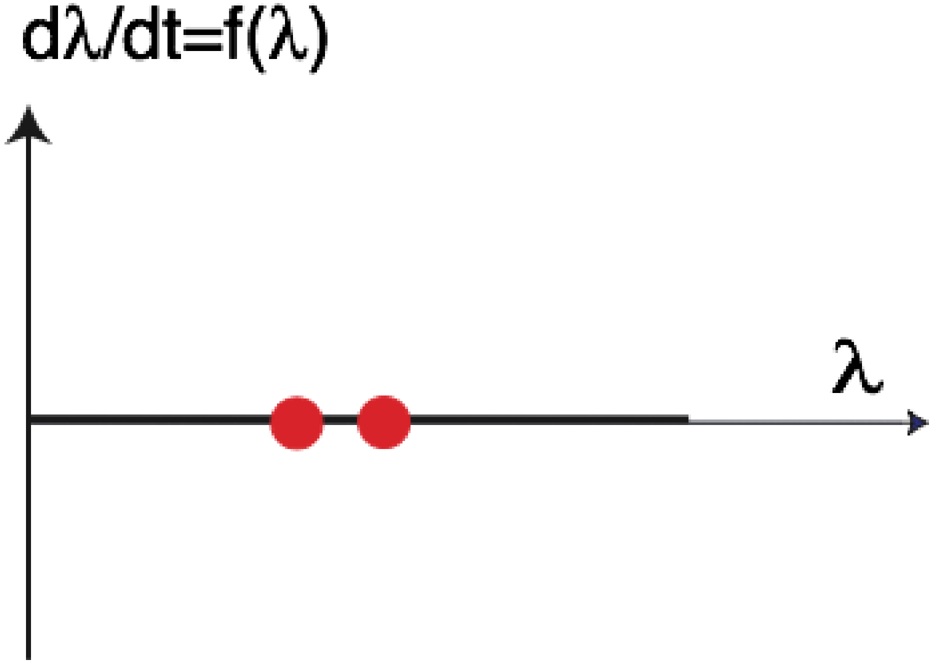
stability in a linear system

- if the slope of the linear system is positive, then the fixed point is unstable
- => repellor

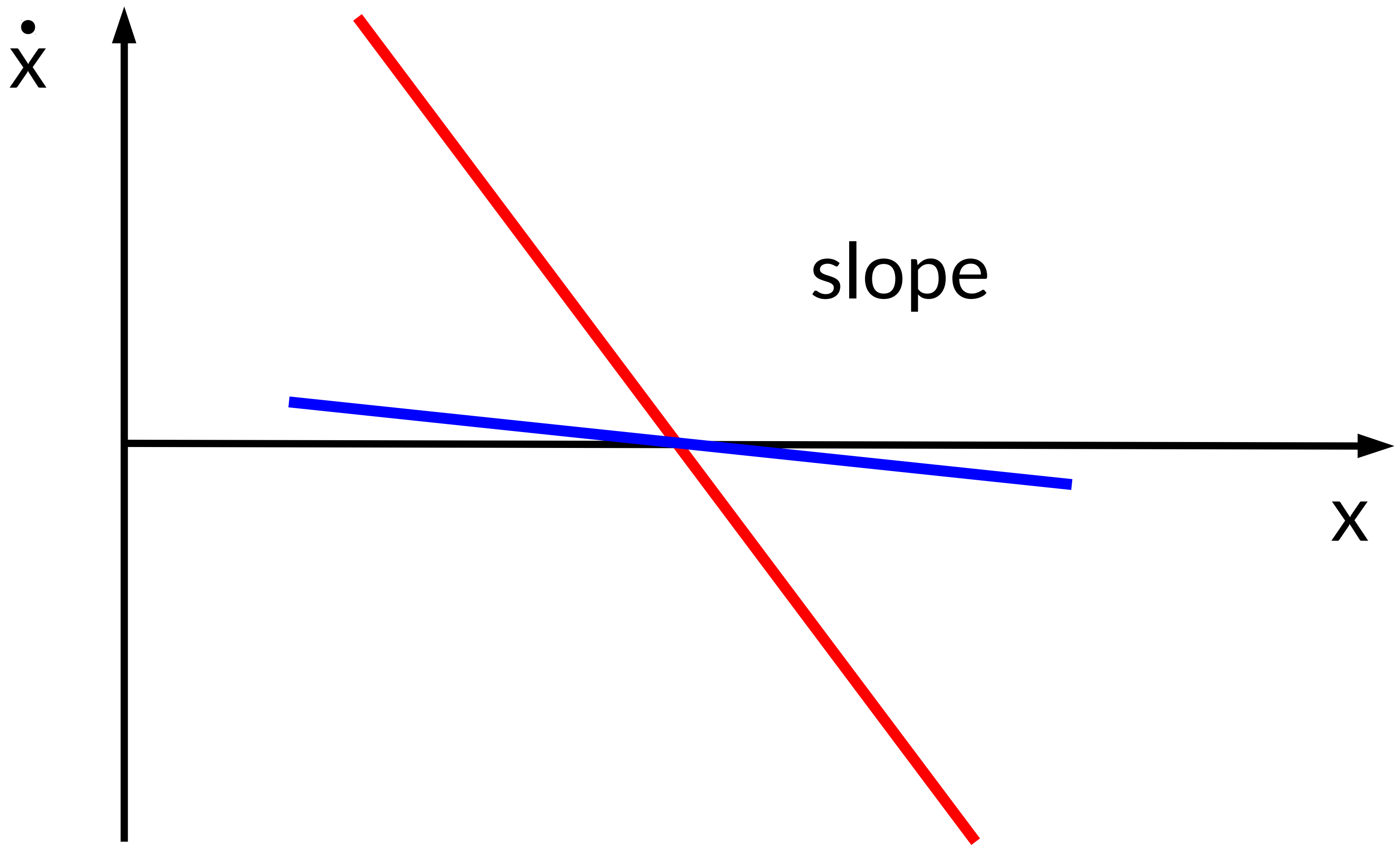


stability in a linear system

- if the slope of the linear system is zero, then the system is indifferent (marginally stable: stable but not asymptotically stable)



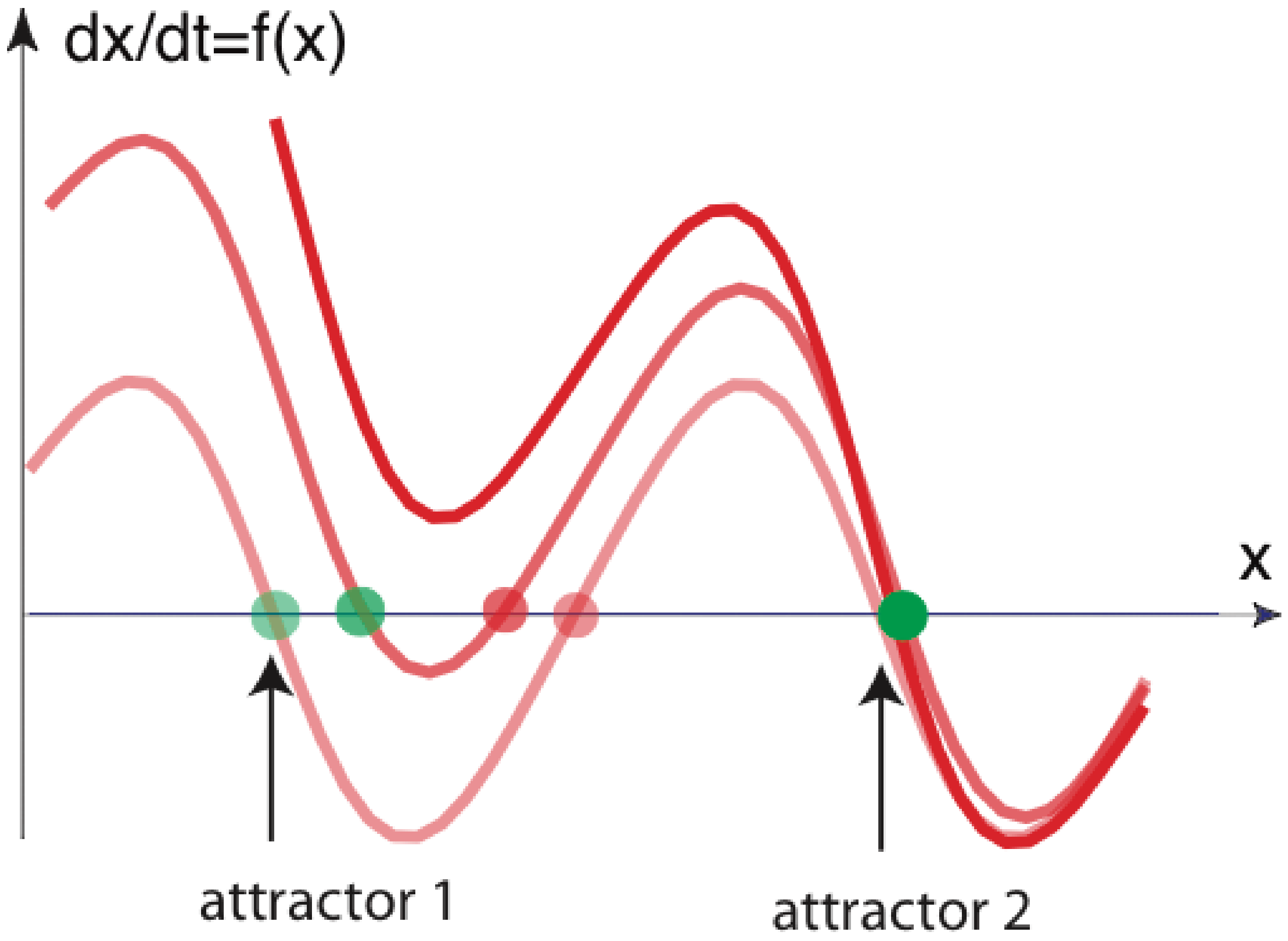
degree of stability



bifurcations

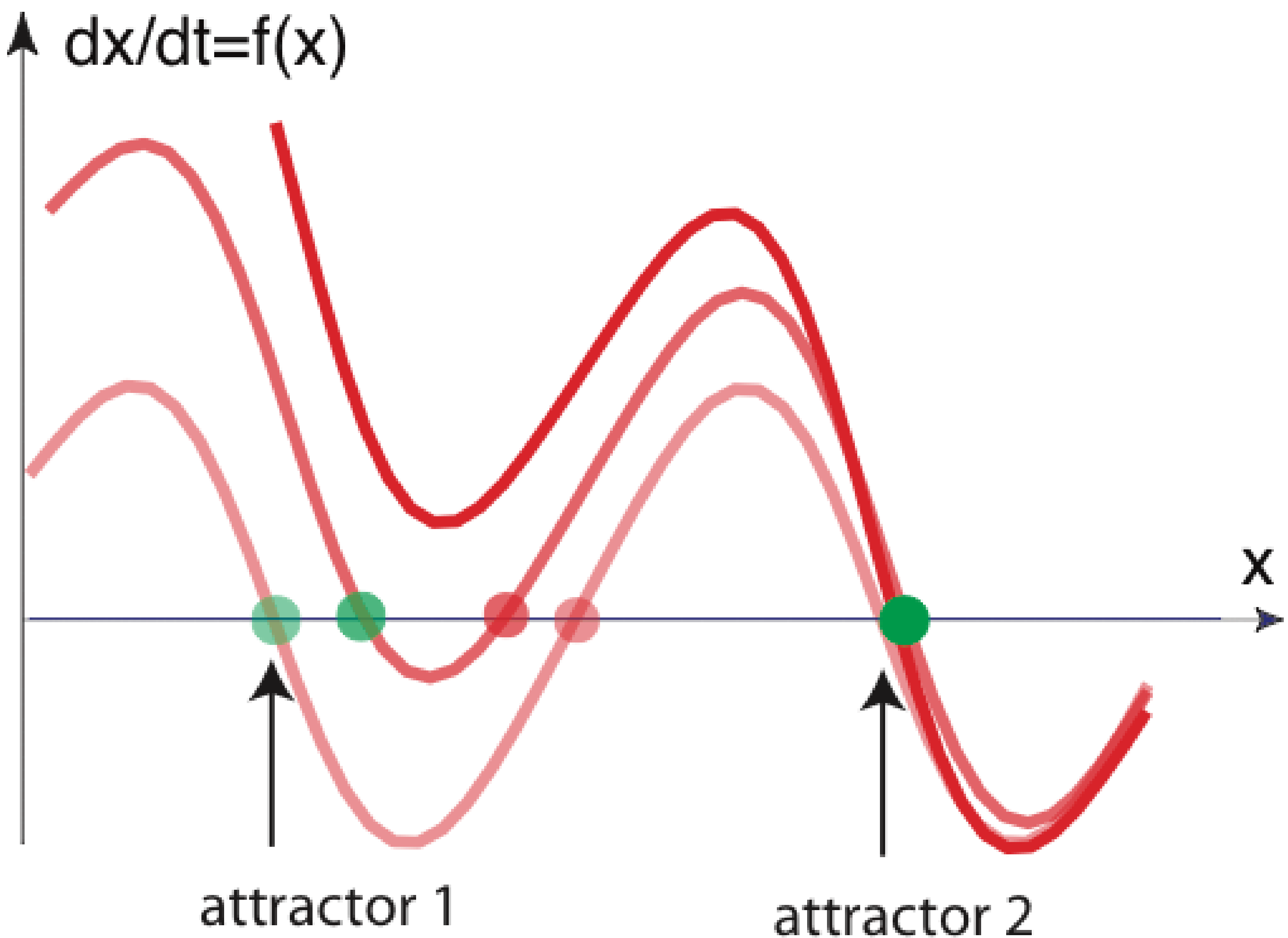
- look now at families of dynamical systems, which depend (smoothly) on parameters
- ask: as the parameters change (smoothly), how do the solutions change (smoothly?)
  - smoothly: topological equivalence of the dynamical systems at neighboring parameter values
  - bifurcation: dynamical systems NOT topological equivalent as parameter changes infinitesimally

bifurcation



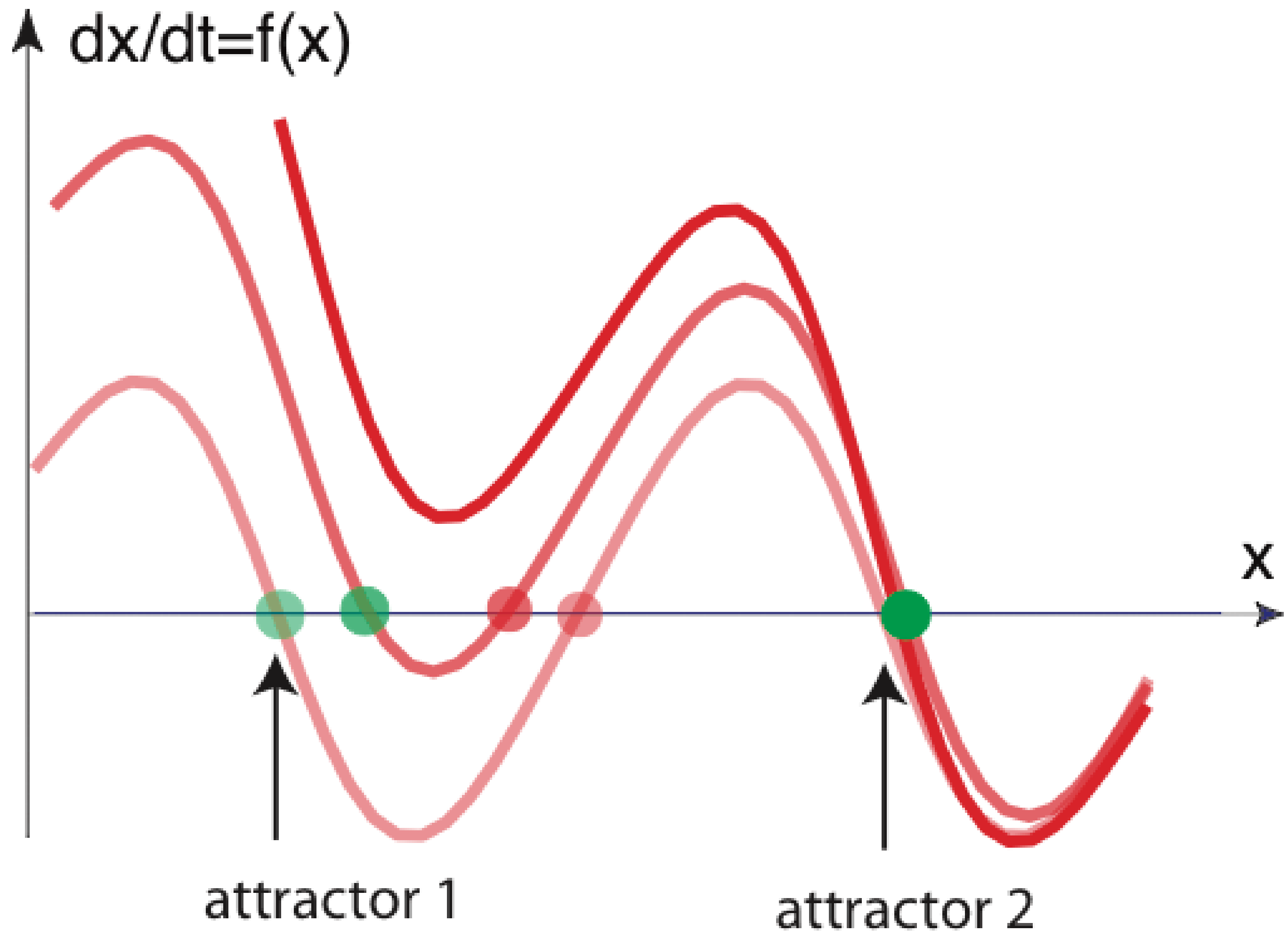
bifurcation

- bifurcation=qualitative change of dynamics (change in number, nature, or stability of fixed points) as the dynamics changes smoothly



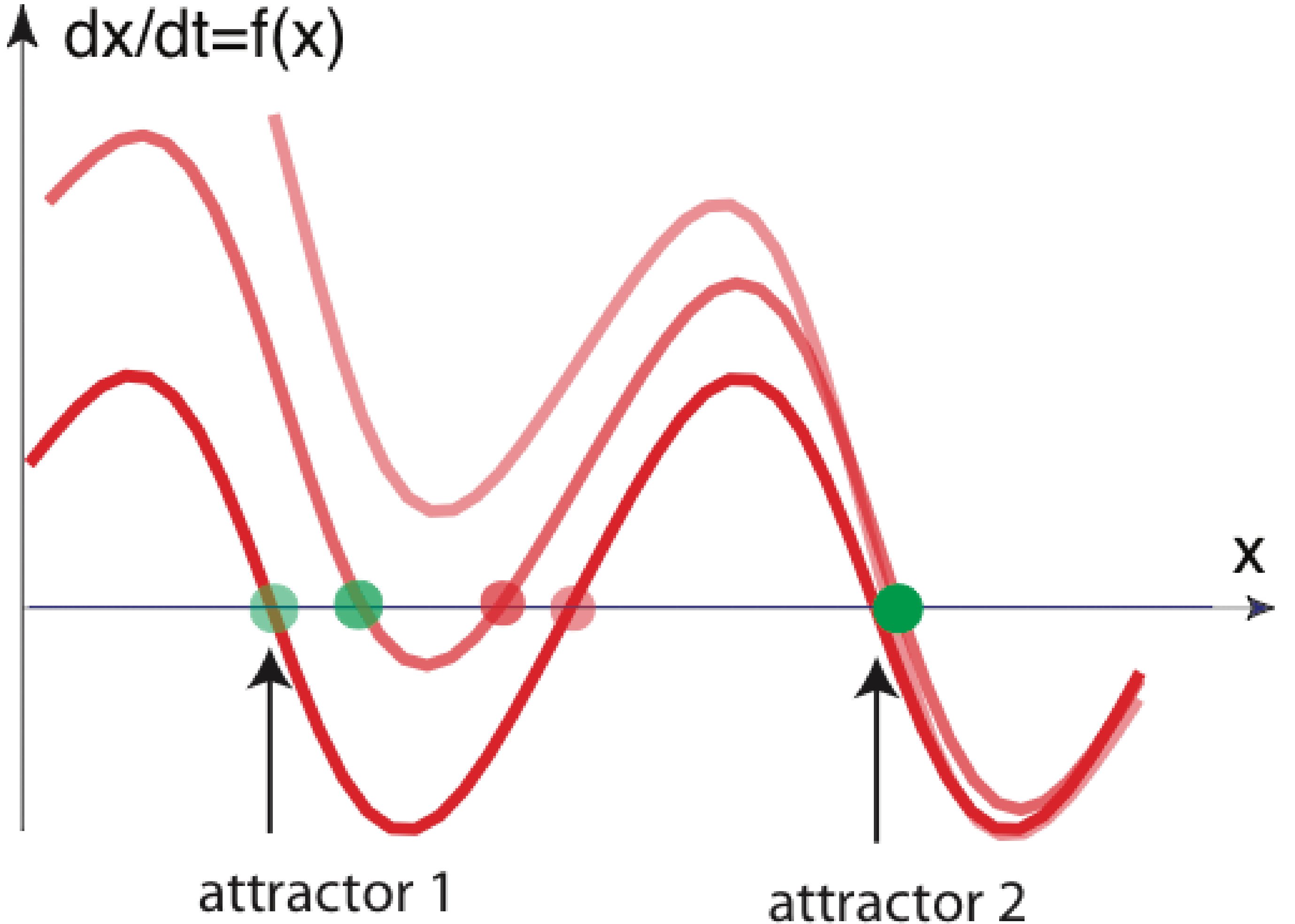
tangent bifurcation

- the simplest bifurcation: an attractor collides with a repellor and the two annihilate



reverse bifurcation

- changing the dynamics in the opposite direction



bifurcations are instabilities

- that is, an attractor becomes unstable before disappearing
- (or the attractor appears with reduced stability)
- formally: a zero-real part is a necessary condition for a bifurcation to occur

tangent bifurcation

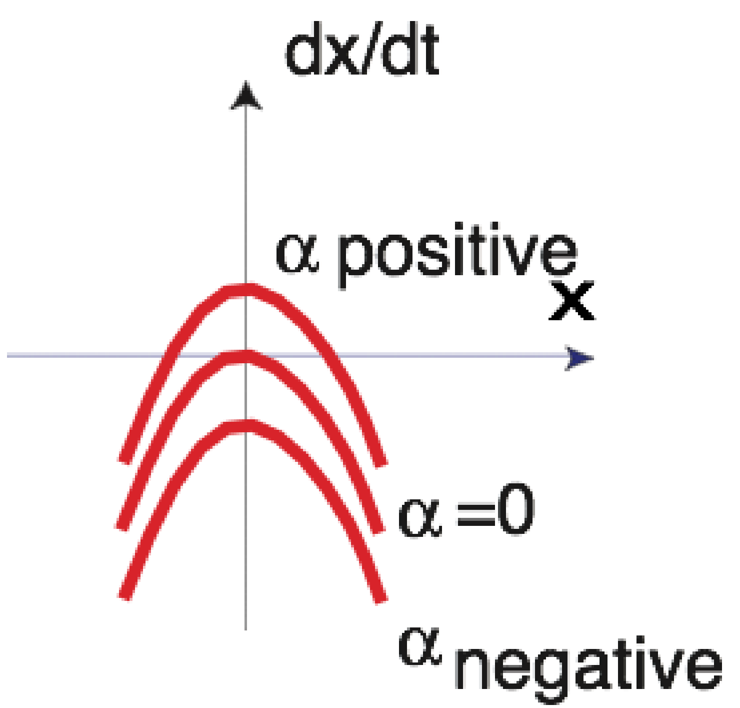
- => simulation

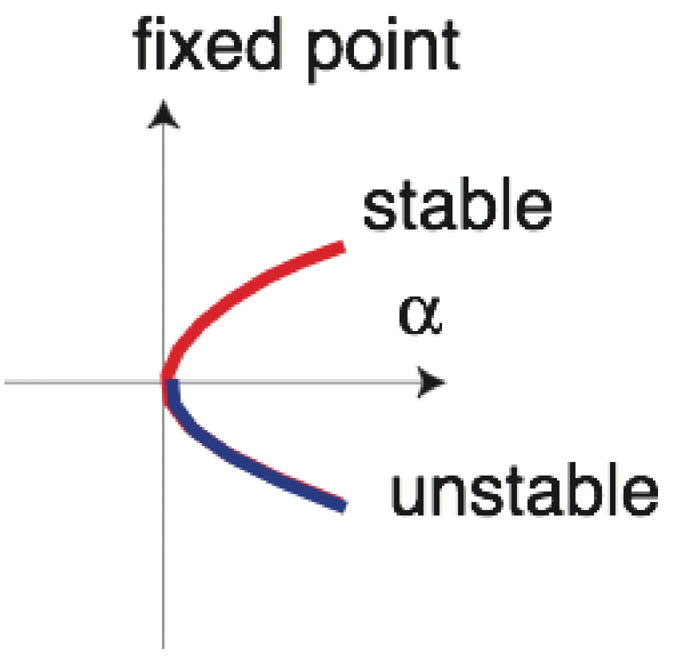
tangent bifurcation

- normal form of tangent bifurcation

$$\dot{x} = \alpha - x^2$$

- (=simplest polynomial equation whose flow is topologically equivalent to the bifurcation)



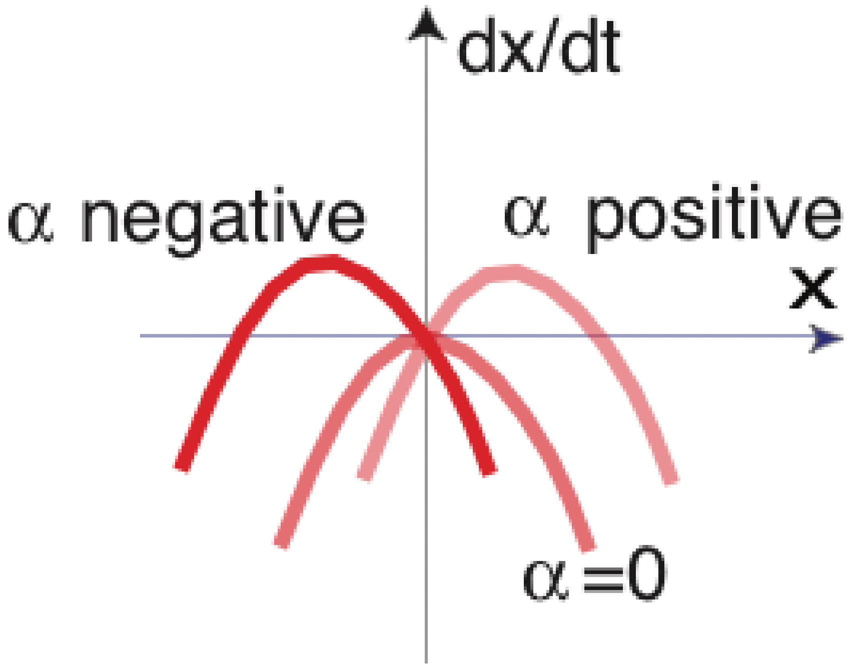


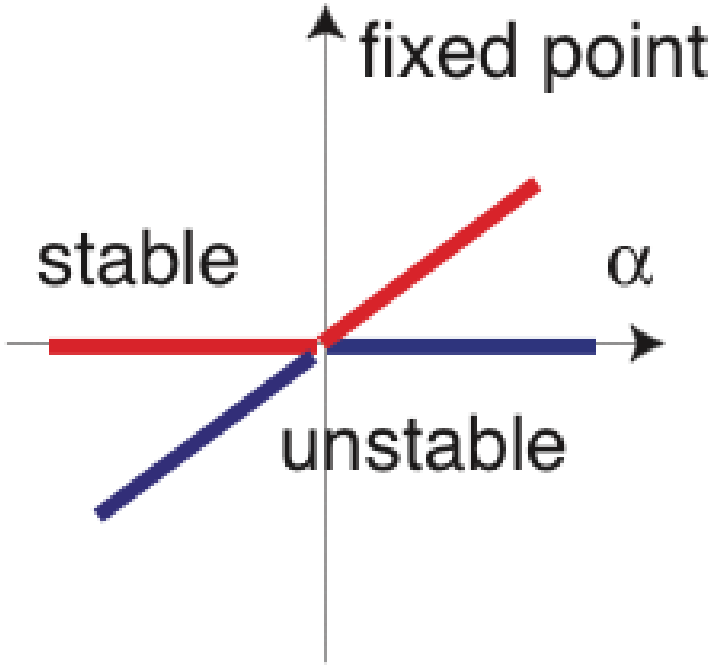
Hopf theorem

- when a single (or pair of complex conjugate) eigenvalue crosses the imaginary axis, one of four bifurcations occur
  - tangent bifurcation
  - transcritical bifurcation
  - pitchfork bifurcation
  - Hopf bifurcation

transcritical bifurcation

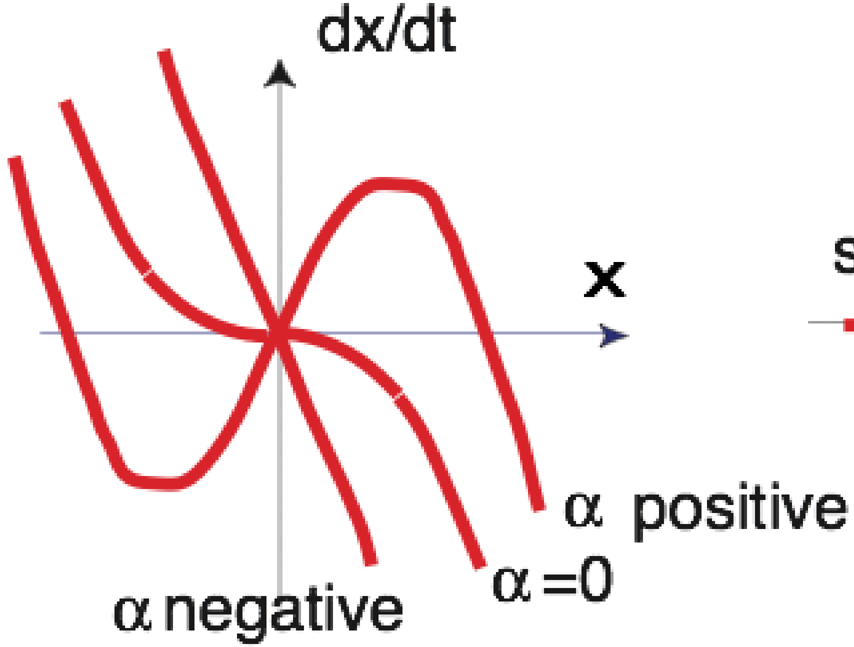
- normal form $\dot{x} = \alpha x - x^2$

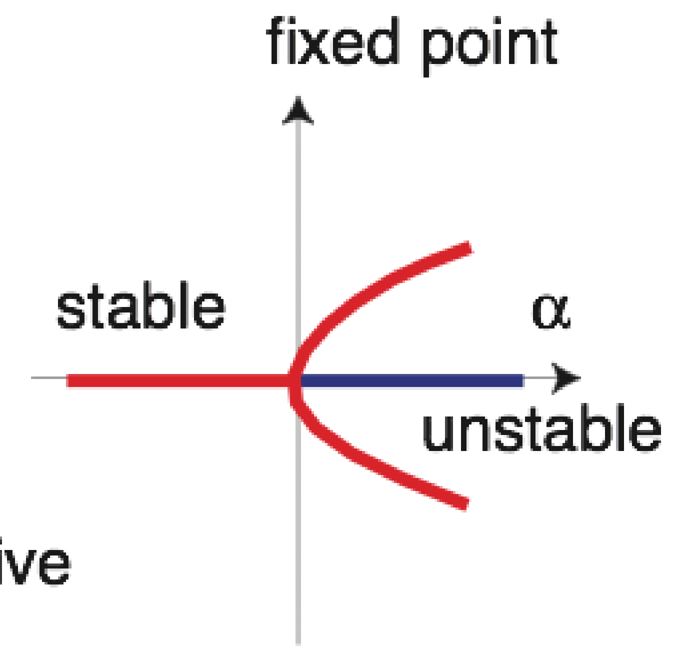




pitchfork bifurcation

- normal form $\dot{x} = \alpha x - x^3$





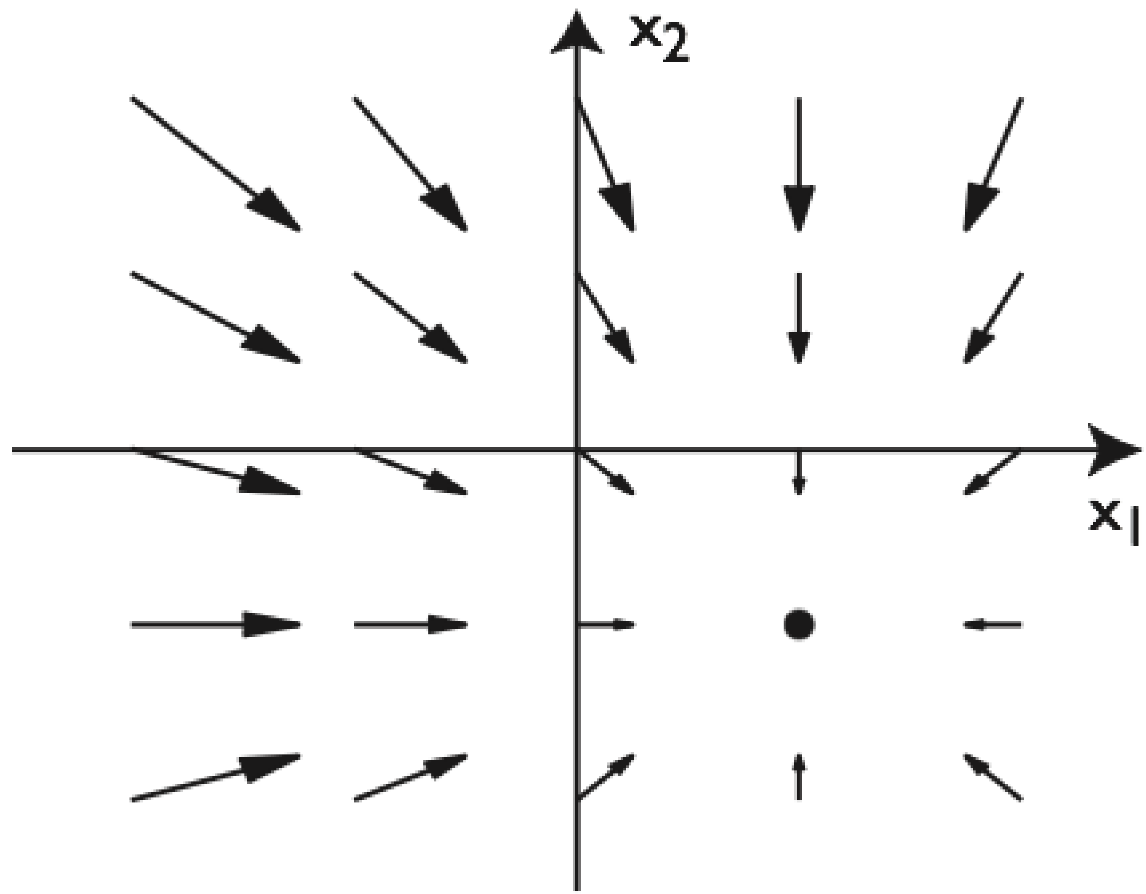
pitchfork bifurcation

=> simulation

Hopf: need higher dimensions

2D dynamical system: vector-field

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$



vector-field

$$\dot{x}_1 = f_1(x_1, x_2)$$
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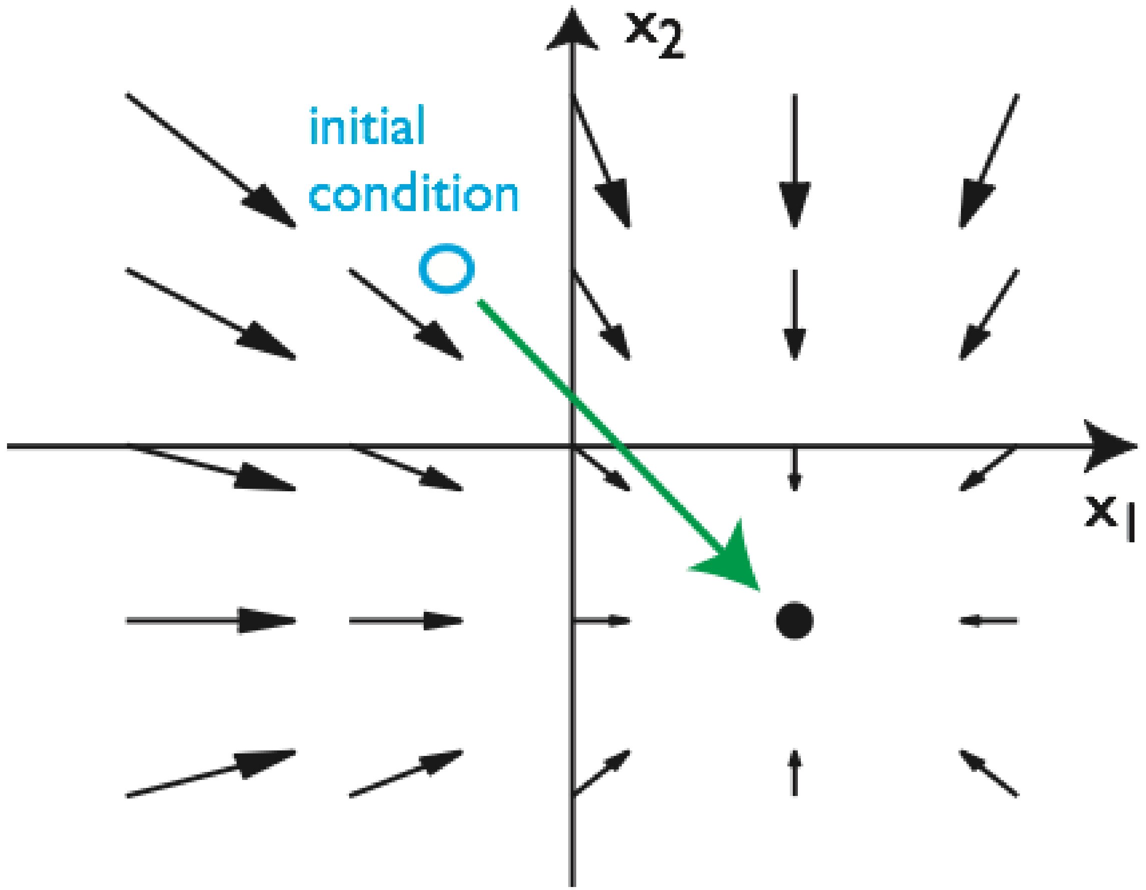
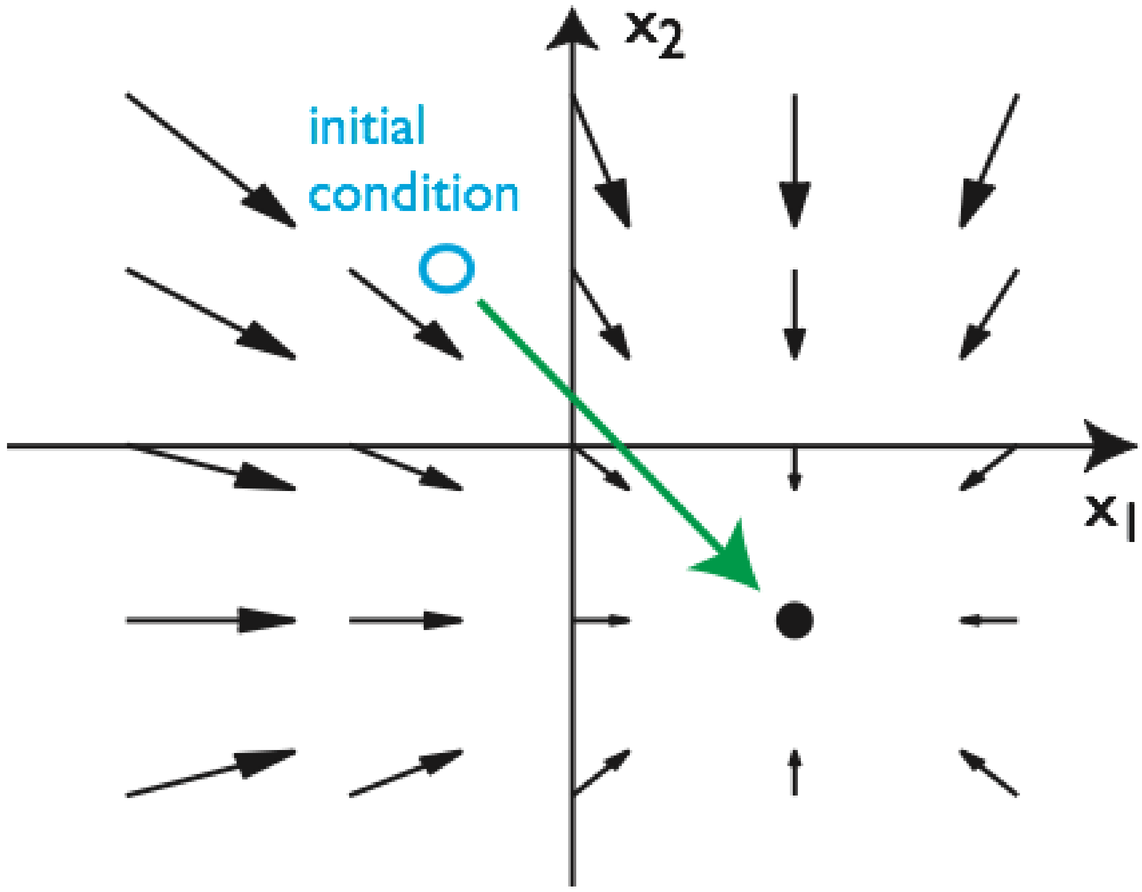


fixed point, stability, attractor

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

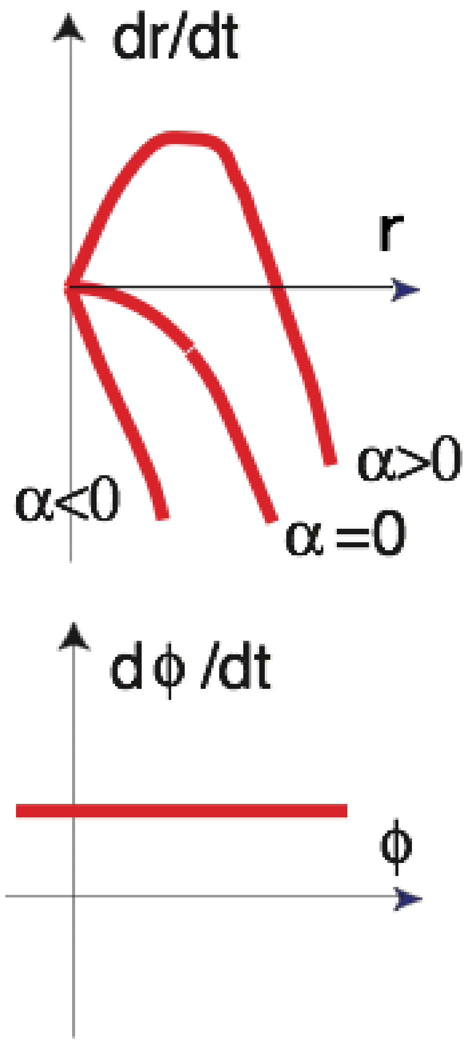


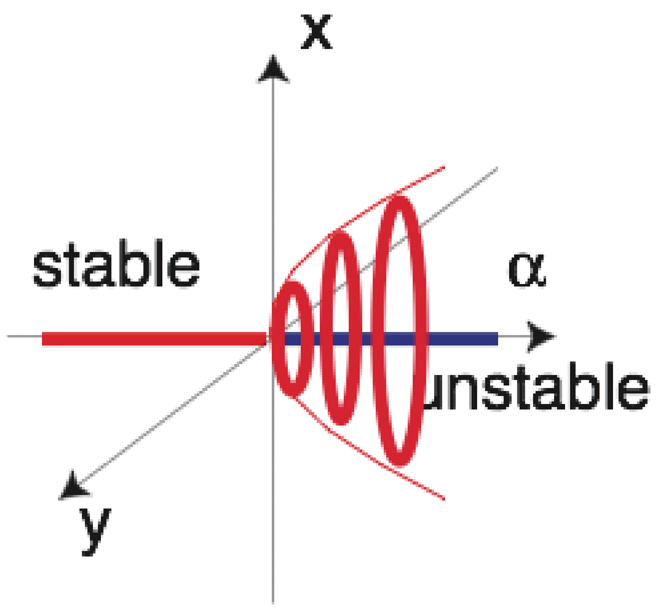
Hopf bifurcation

- normal form

$$\dot{r} = \alpha r - r^3$$

$$\dot{\phi} = \omega$$





forward dynamics

- given known equation, determined fixed points /limit cycles and their stability
- more generally: determine invariant solutions (stable, unstable and center manifolds)

inverse dynamics

- given solution, find the equation…
- this is the problem faced in design of behavioral dynamics…

inverse dynamics: design

- in the design of behavioral dynamics… you may be given:
- attractor solutions/stable states
- and how they change as a function of parameters/conditions
- => identify the class of dynamical systems using the 4 elementary bifurcations
- and use normal form to provide an exemplary representative of the equivalence class of dynamics

important concepts

time-variation

rate of change

dynamical system

phase plot vs. time course plot

present determines the future

numerical solutions

fixed points (attractors, repellors)

stability

bifurcations & instabilities