Lecture 6
Differential Equations

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Computer Science and Mathematics
Preparatory Course

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Differential Equation as Rule System

- A differential equation describes how the rate of change of a system depends on its current state. For example:

$$f'(x) = d(f(x)) \quad \text{or} \quad \frac{dy}{dx} = d(y)$$

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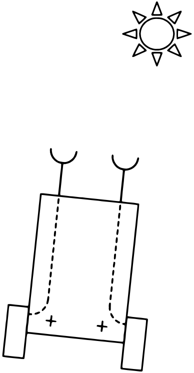
- A differential equation describes how a system should change in a given state.
- Brief oversimplification:

A differential equation describes rules for the future

The Vehicle's Behavior as Function of Angle Change



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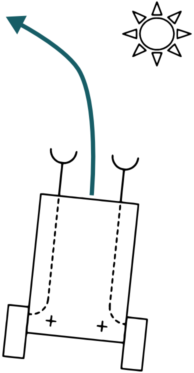

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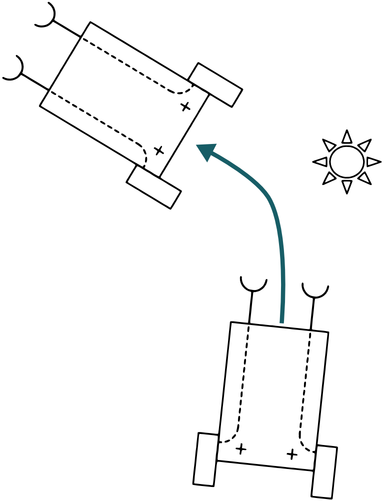

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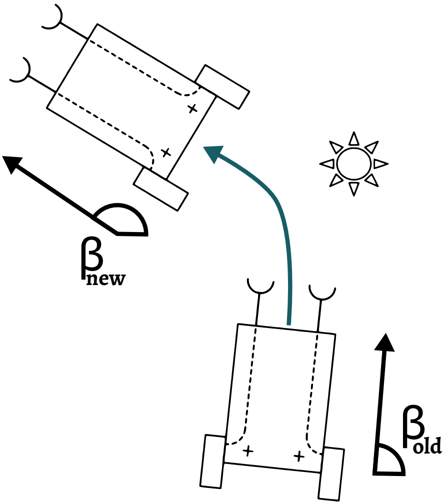

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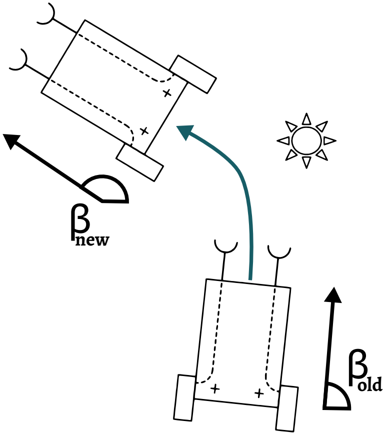

- The vehicle's change in angle depends on its current sensor input

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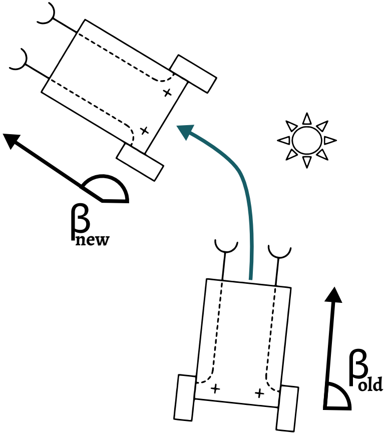

- The vehicle's change in angle depends on its current sensor input
- The following equation may describe its behavior

$$\frac{d\beta}{dt} = -S_L + S_R,$$

where t describes time and S_L, S_R left and right sensor values.

Overview

1. **Motivation**

2. **Mathematics**
   - Solving Differential Equations
   - Qualitative Analysis
   - Numerical Approximation

3. **Tasks**

Solving Differential Equations

- Given a differential equation of the form $f'(x) = d(f(x)) \ldots$ the original function $f(x)$ is usually not known.
- Solving a differential equation describes the process of finding an $f(x)$ that follows the above rule for all x
- Differential equations entail two equations
  1. The function $d(f(x))$ governing the rate of change
  2. The function $f(x)$ describing the overall behavior

Derivative vs. Differential equation

- $f'(x) = cx$

- $f'(x) = cf(x)$

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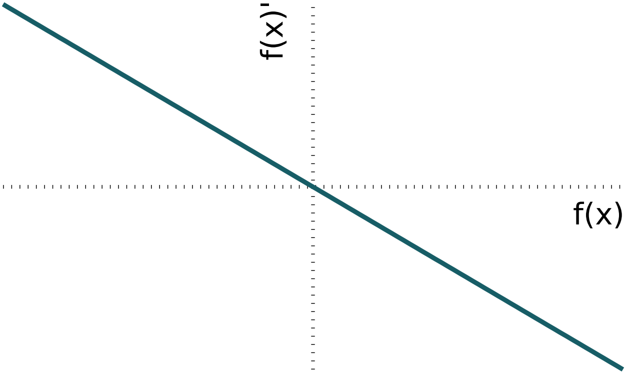
  - Considering the chain rule the derivative of e^{cx} is exactly ce^{cx} therefore $f(x) = ce^{cx}$
  - Usually a differential equation is not that easily solvable

Dynamical Systems Theory

- Mathematicians want to find solutions to particular differential equations
- **Dynamical Systems Theory** is concerned with analyzing the qualitative behavior of the system

Qualitative Behavior of Differential Equations

$$f'(x) = \frac{dy}{dx} = -f(x)$$



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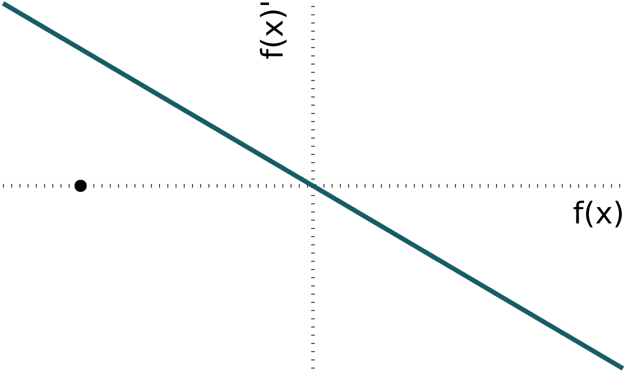

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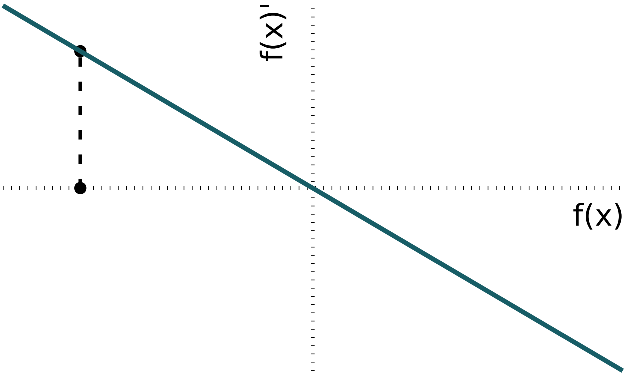

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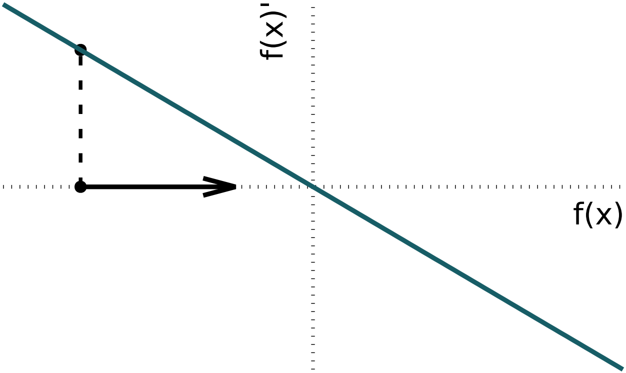

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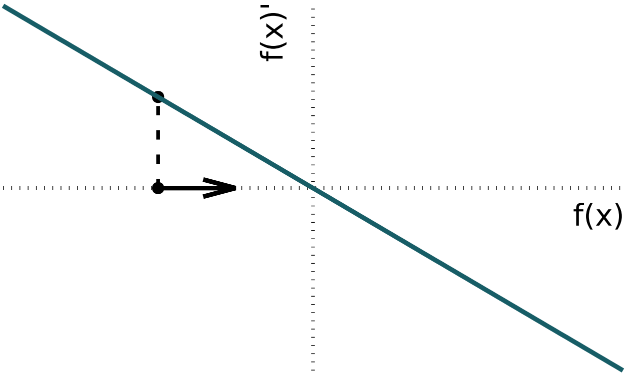

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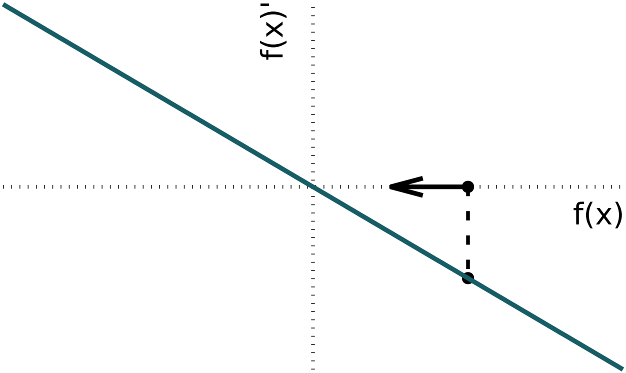

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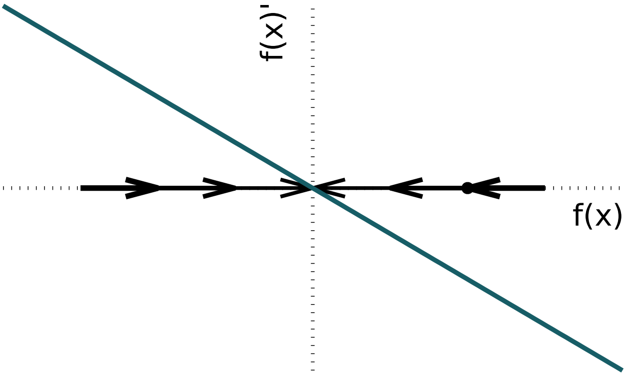
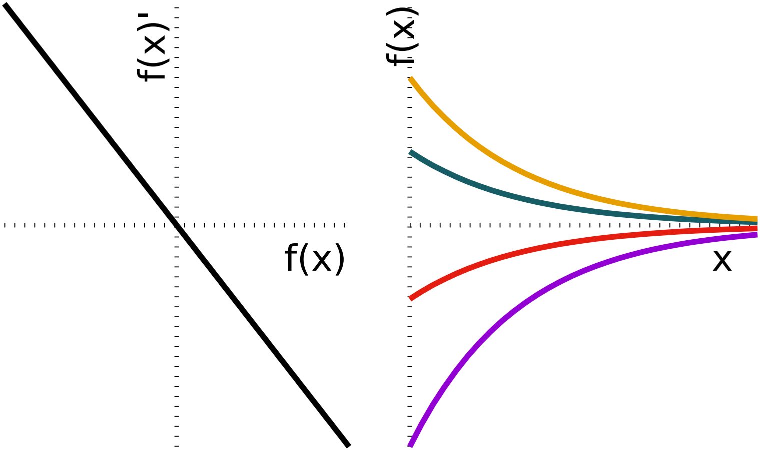


Attractors



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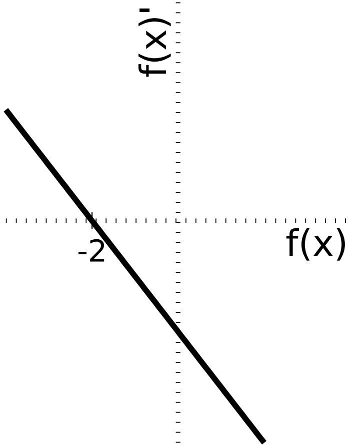

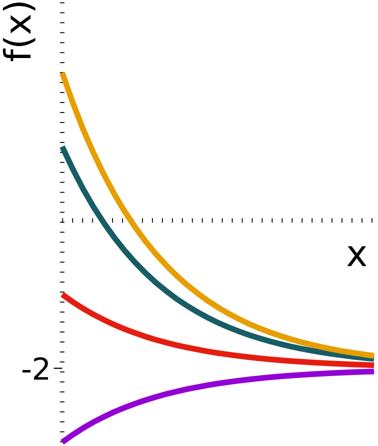

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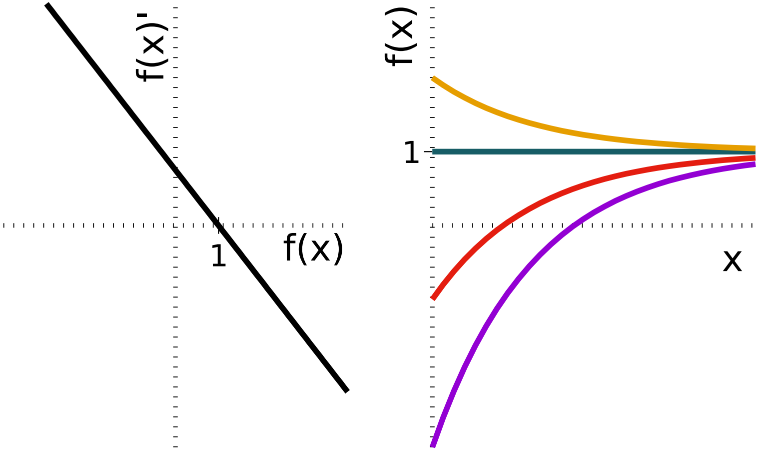
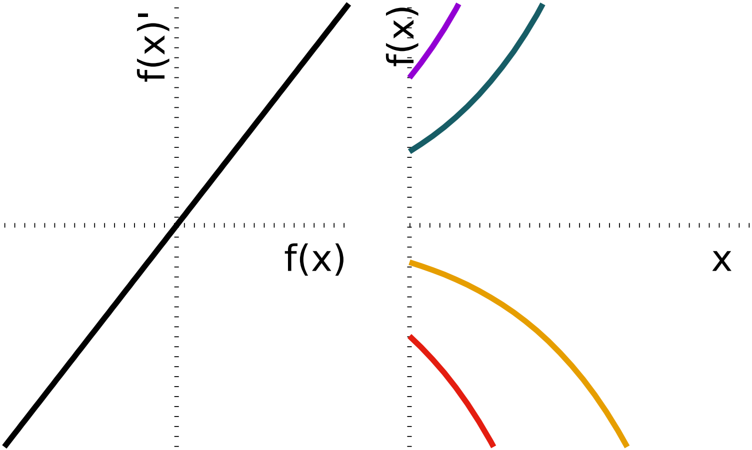
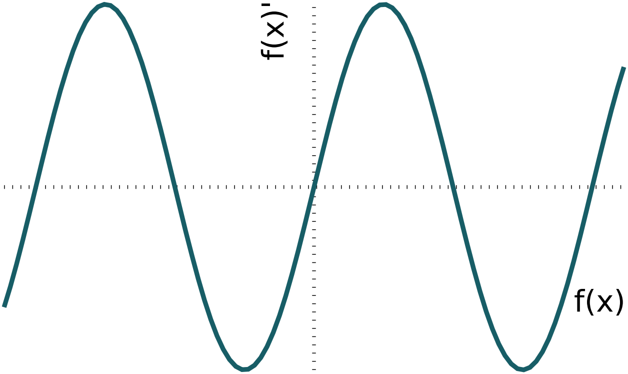


Repellors



Initial Condition Matters

$$f'(x) = \frac{dy}{dx} = sin(x)$$



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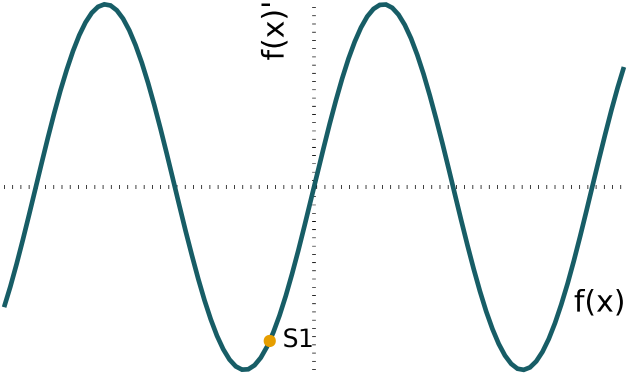

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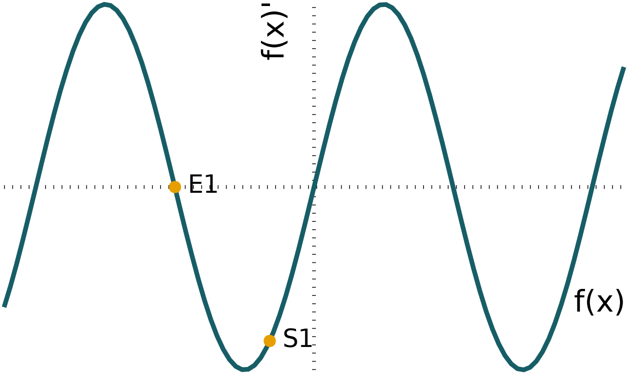

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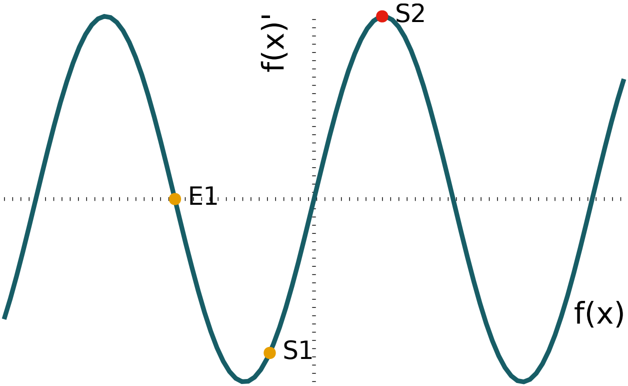

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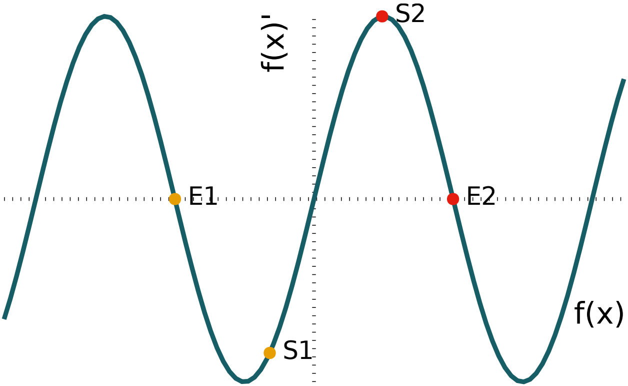

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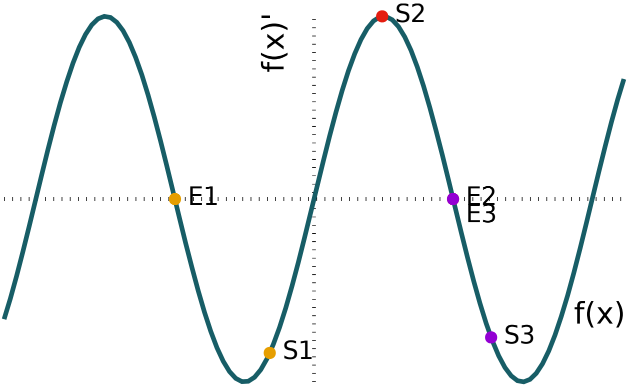
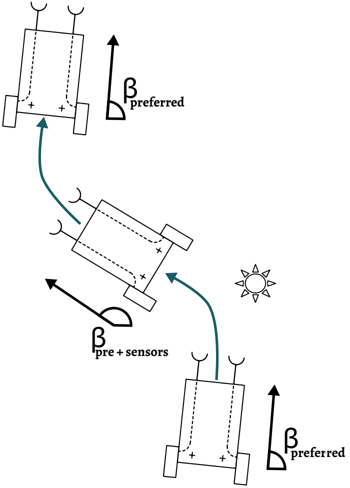


Back to the Braitenberg Vehicle



- We govern the vehicles behavior with a differential equation

$$\frac{d\beta}{dt} = -\beta - S_L + S_R,$$

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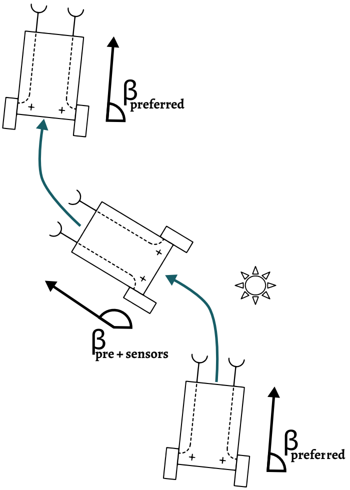


- We govern the vehicles behavior with a differential equation

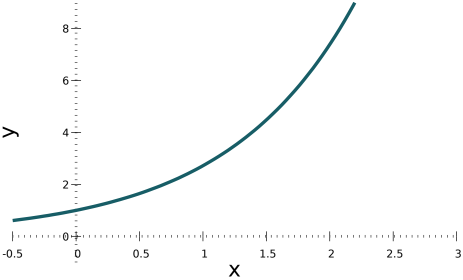
$$\frac{d\beta}{dt} = -\beta - S_L + S_R,$$

- Adding an attractor gives the vehicle a preferred orientation

Euler Approximation

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$y(x) = e^x$



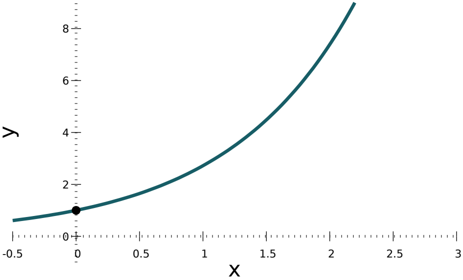
Euler Approximation

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$\Delta x = 1$$

x	y	$\frac{dy}{dx}$
0	1	1

$y(x) = e^x$

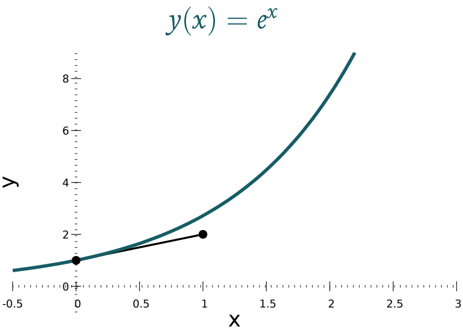


Euler Approximation

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$\Delta x = 1$$

x	y	$\frac{dy}{dx}$
0	1	1
1	2	2



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$$\Delta x = 1$$

x	y	$\frac{dy}{dx}$
0	1	1
1	2	2
2	4	4

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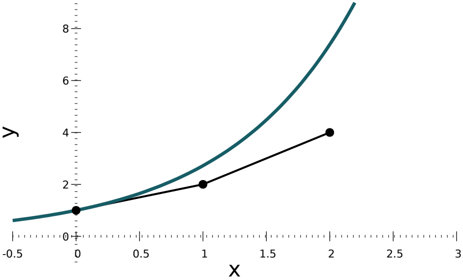

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$$\Delta x = 1$$

x	y	$\frac{dy}{dx}$
0	1	1
1	2	2
2	4	4
3	8	8

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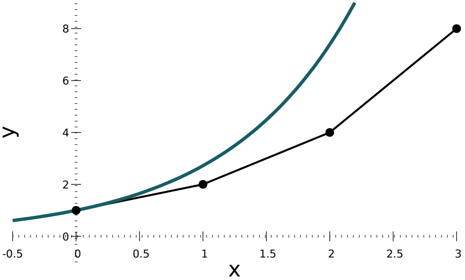
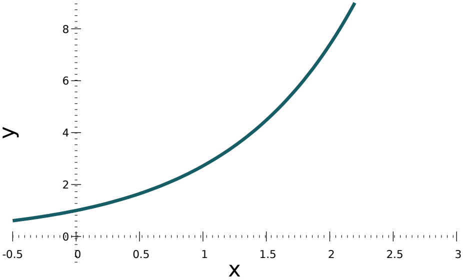


Varying the stepsize

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$y(x) = e^x$$



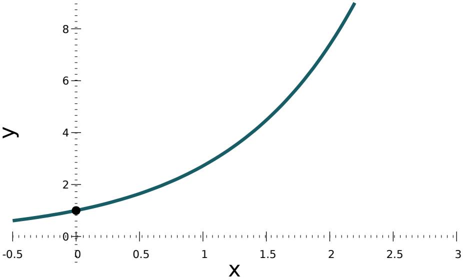
Varying the stepsize

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$\mathbf{\Delta x = 0.5}$$

x	y	$\frac{dy}{dx}$
0	1	1

$y(x) = e^x$



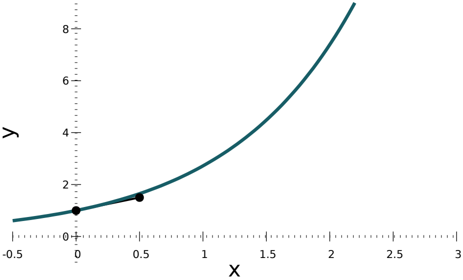
Varying the stepsize

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$\boldsymbol{\Delta x = 0.5}$$

x	y	$\frac{dy}{dx}$
0	1	1
0.5	1.5	1.5

$y(x) = e^x$



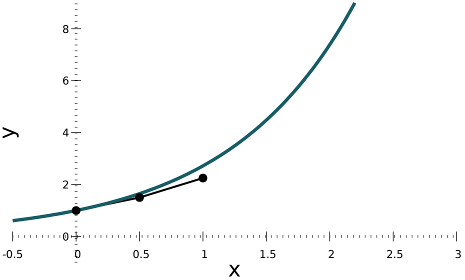
Varying the stepsize

$$\frac{dy}{dx} = y \quad y(0) = 1$$

$$\boldsymbol{\Delta x = 0.5}$$

x	y	$\frac{dy}{dx}$
0	1	1
0.5	1.5	1.5
1	2.25	2.25

$y(x) = e^x$



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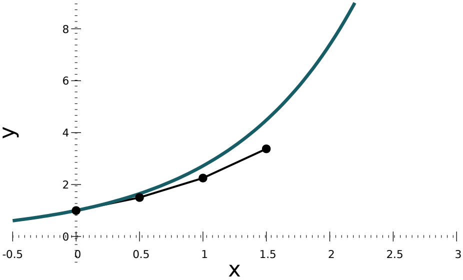
$$\Delta x = 0.5$$

x	y	$\frac{dy}{dx}$
0	1	1
0.5	1.5	1.5
1	2.25	2.25
1.5	3.375	3.375

$$y(x) = e^x$$



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1.5	3.375	3.375
2	5.0625	5.0625

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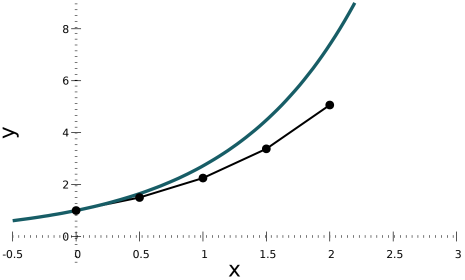

Euler Approximation in Words

1. Start with a certain value for x and y and the differential equation $\frac{dy}{dx} = \ldots$ you want to approximate
2. Decide for a step size that determines the accuracy of your approximation
3. Repeat as long as you like:
   - 3.1 Use the current y-value to calculate the current rate of change $\frac{dy}{dx}$
   - 3.2 Calculate the next y-value by taking the current y-value and adding to it the rate of change times the step size
   - 3.3 Increase x by the step size

Tasks

1. Download the *task_7_1_template.py* from the course page and implement a rule for the angle change depending on the sensor values.
   - Start by implementing the simple rule from the motivation slides
   - Try your rule by running the template. You can add obstacles by clicking on the screen.
   - Let your change in the angle depend on the current heading itself. How can you set any preferred driving angle?
   - What do you need to change to make the vehicle go towards obstacles?
2. Write a script that uses the euler approximation method used above to approximate the differential equation $f'(x) = f(x)$
   - Start with f(0) = 1 and choose your own step size and number of steps
   - Look at the previous slide to implement the algorithm
   - (Optional) How can you change your script to easily switch to the equation $f'(x) = f(x)^2$?