Lecture 5
Integration

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Computer Science and Mathematics
Preparatory Course

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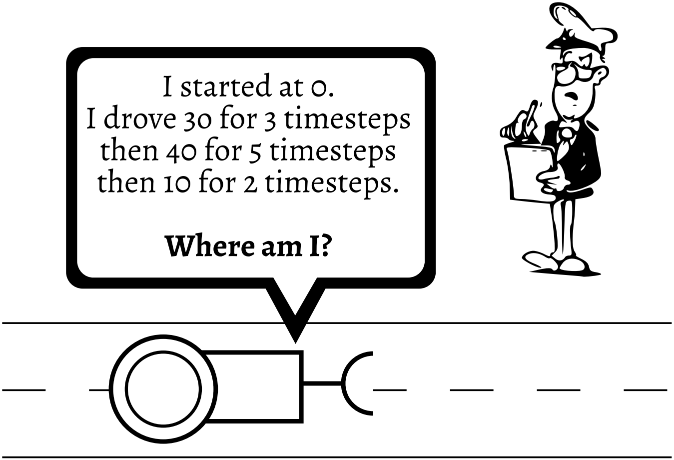
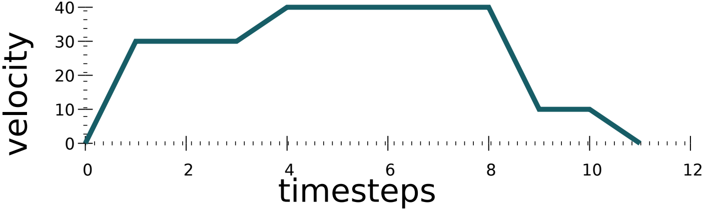
Reverting Differentiation



From Velocity to position



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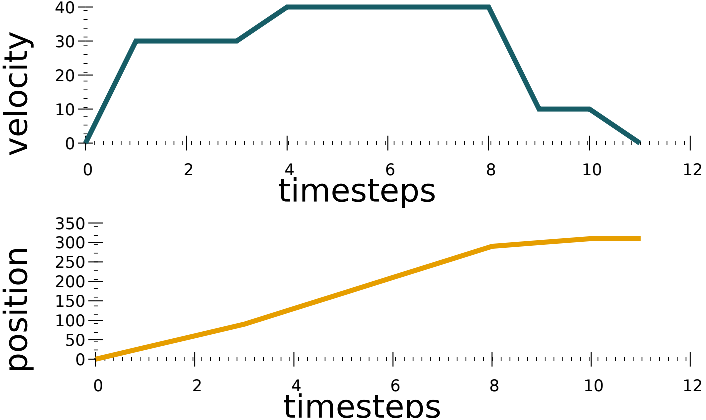
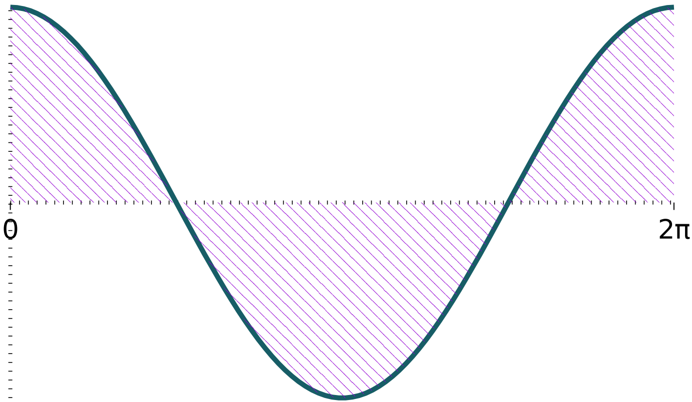


Overview

Integral as Area

$f(x) = cos(x)$ $\int_0^{2\pi} cos(x)$



Integral as Function

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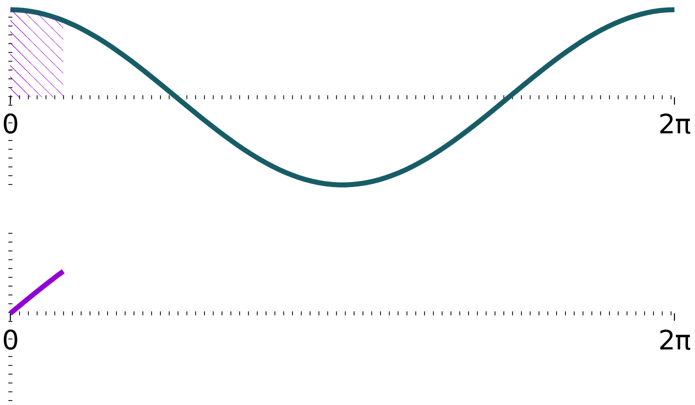

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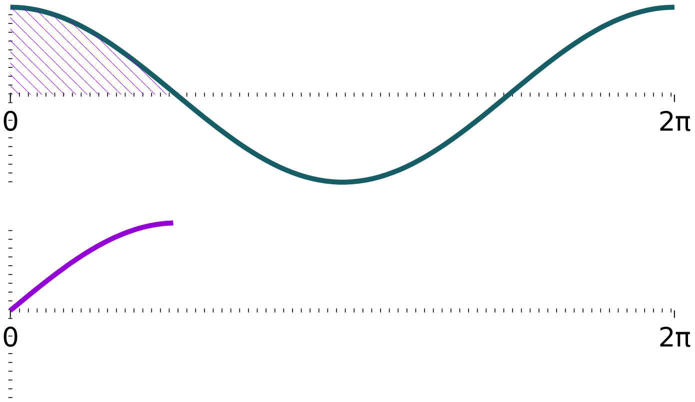

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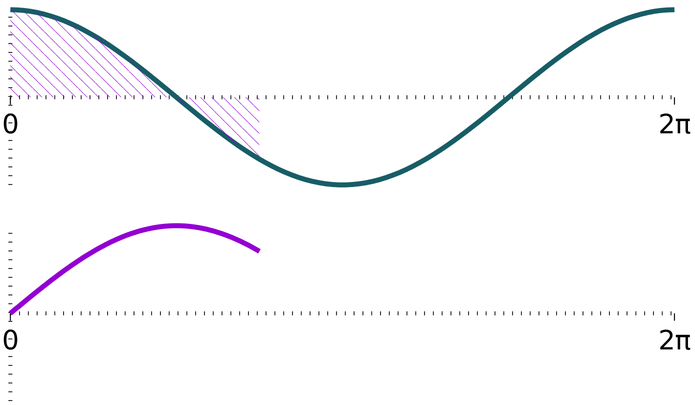

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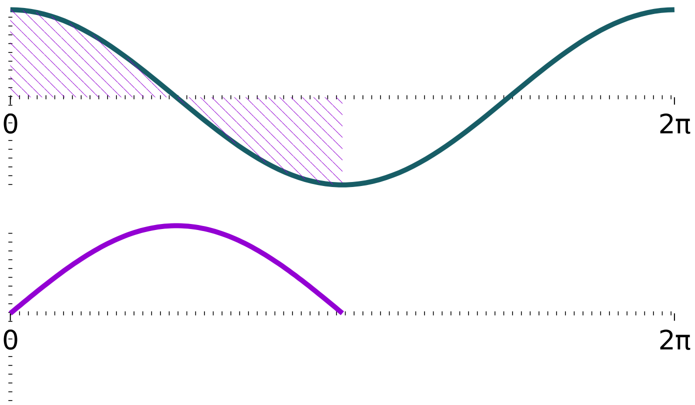

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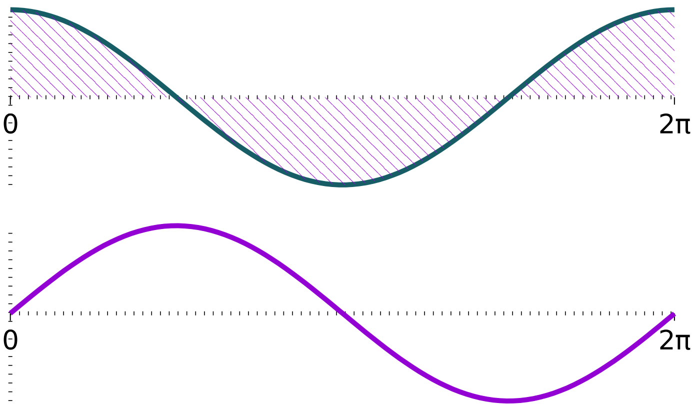
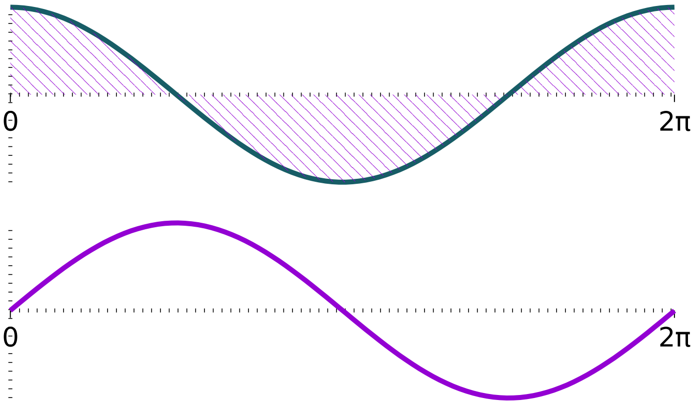


Integral as Function

$f(x) = cos(x)$ $\int_0^{2\pi} cos(x) = sin(2\pi)$



Geometric Definition

Definite Integral

The **definite integral** of a function $f(x)$ between the **lower boundary** a and the **upper boundary** b

$$\int_a^b f(x)$$

is defined as the size of the area between f and the x-axis inside the boundaries. Areas above the x-Axis are considered positive and areas below negative.

The Antiderivative

Definition

If f is a function with domain $[a, b] \to \mathbb{R}$ and there is a function F, which is differentiable in the interval $[a, b]$ with the property that

$$F'(x) = f(x),$$

then F is considered the **antiderivative** of f

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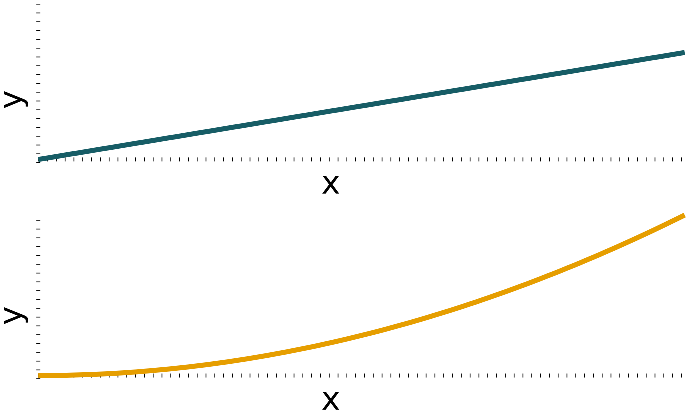
Properties of the antiderivative

- Differentiation removes constants, because of that an antiderivative is described by a family of functions $F(x) + c$
- Unlike with differentiation there are no fixed rules to compute an antiderivative from a given f

A function and its antiderivative

$f(x) = x$ $\quad$ $F(x) = \frac{1}{2}x^2$



Calculating the Integral

Calculating the area in an interval

If f is integrable and continuous in $[a, b]$. Then the following holds for each antiderivative F of f

$$\int_a^b f(x)dx = \int_a^b F'(x)dx = [F(x)]_a^b = F(b) - F(a)$$

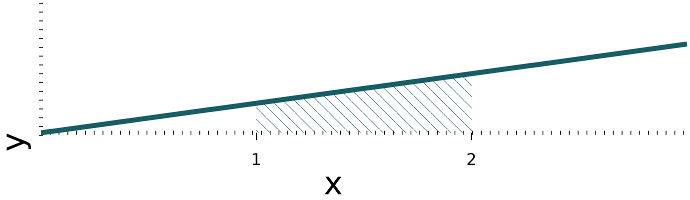
Example:

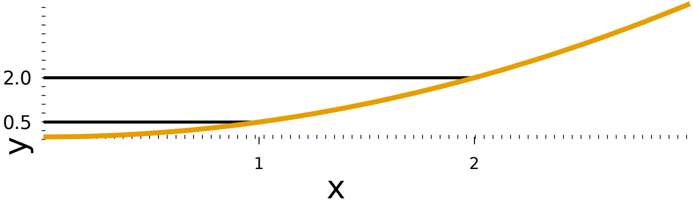
- Area under $f(x)$ between values 1 and 2

$$\int_1^2 xdx = \left[\frac{1}{2}x^2\right]_1^2 = \frac{1}{2}2^2 - \frac{1}{2}1^2 = 1.5$$

Integral as area underneath a function

$f(x) = x$ $\quad F(x) = \frac{1}{2}x^2$ $\quad \int_1^2 f(x)dx = F(2) - F(1)$





The Integral as Linear Operator

Integration Rules

- **Summation**

$$\int_a^b f(x) + g(x) = \int_a^b f(x) + \int_a^b g(x)$$

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$$\int_a^b cf(x) = c\int_a^b f(x)$$

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- **Boundary Transformations**

$$\int_a^b f(x) + \int_b^c f(x) = \int_a^c f(x) \quad \wedge \quad \int_a^b f(x) = -\int_b^a f(x)$$

Improper Integrals

Infinite Intervals

It is possible to calculate the area in infinitely large intervals. Intervals with an infinite boundary are called **Improper Integrals**

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

Example:

- Convergent improper integral

$$\int_1^\infty x^{-2}dx = \lim_{b\to\infty} \int_1^b x^{-2}dx = \lim_{b\to\infty} \left[-x^{-1}\right]_1^b = \lim_{b\to\infty} \left(-b^{-1} + 1\right) = 1$$

Numerical Approximation

- It is not trivial to find the antiderivative to a given function or a given dataset

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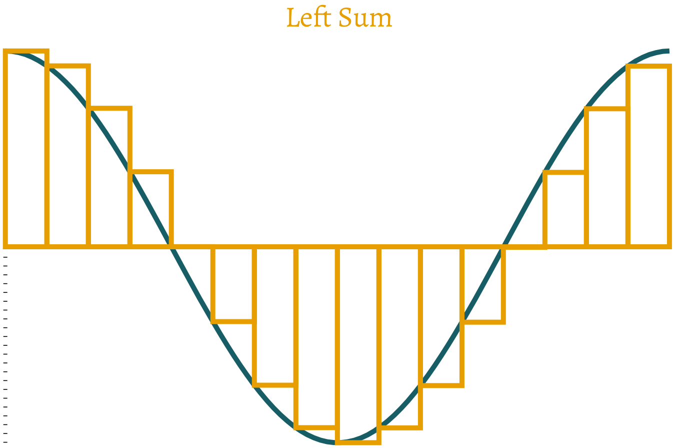
Partioning an Interval

Let $(x_i)_{i \in [a,b]}$ be a sequence of n increasing numbers in $[a, b]$ with fixed distance h between x_i and $x_i + 1$ for all x_i.

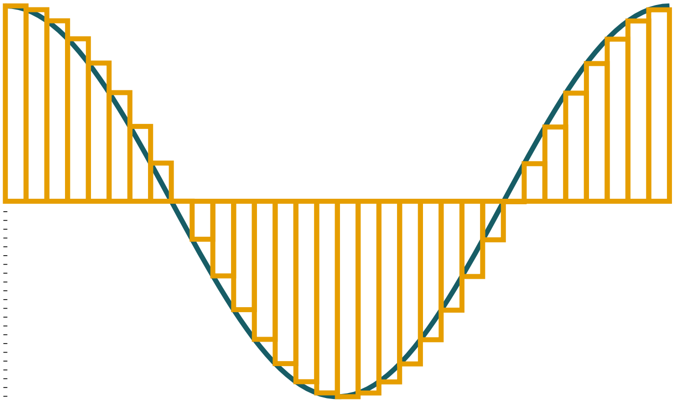
$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

Riemann Sums



Riemann Sums

Left Sum



Riemann Sums

Left and Right Sum

- For an interval $[x_i, x_{i+1}]$ and a function f the functions

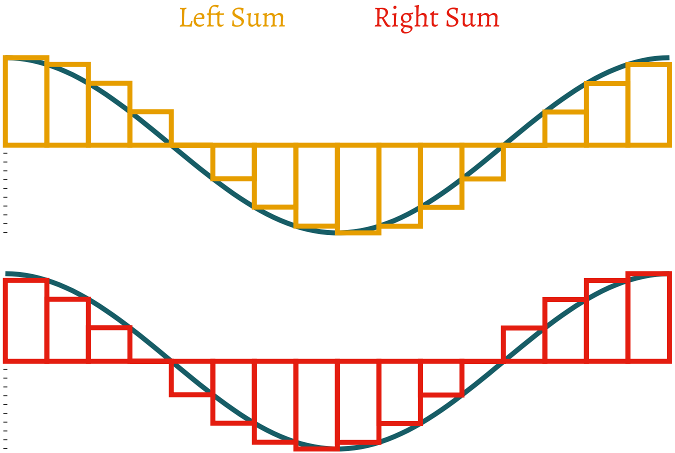
$$\text{Left}(f, [x_i, x_{i+1}[) = f(x_i) \text{ and } \text{Right}(f,]x_i, x_{i+1}]) = f(x_{i+1})$$

  are defined to return the leftmost or rightmost value of the function in the interval.

- **Left and Right Sum** are defined as the Sums of Left and Right across whole partitioned interval $(x_i)_{i \in [a,b]}$

$$I_L = \sum_i^n \text{Left}(f, x_i, x_{i+1}) \text{ and } I_R = \sum_i^n \text{Right}(f, x_i, x_{i+1})$$

Left and Right Sum



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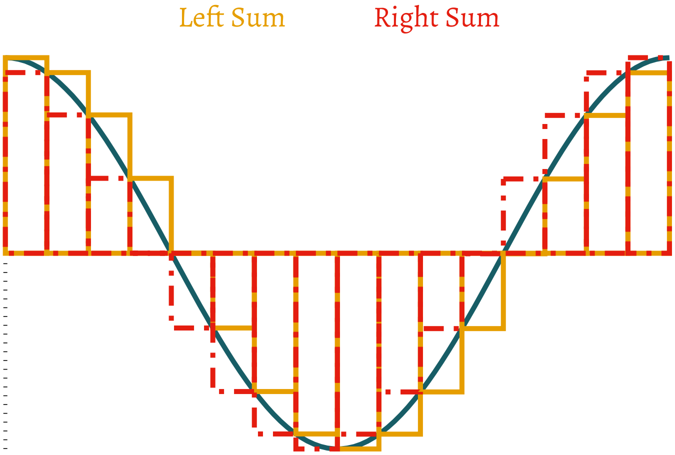

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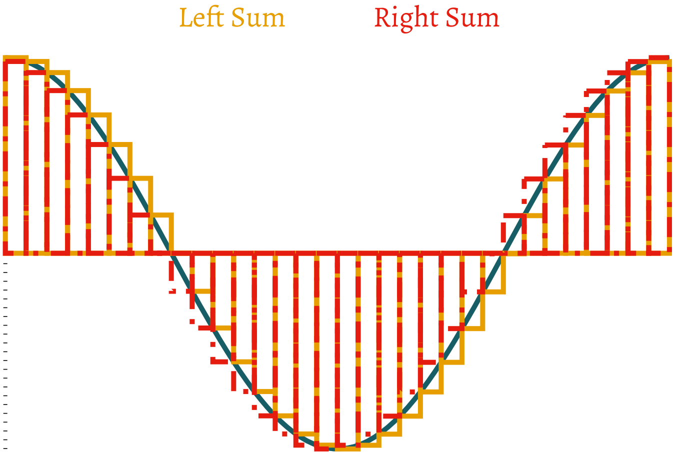


Estimation of the True Integral

- Left and Right Sums for a partition $(x_i)_{i\in[a,b]}$ give us an estimate of the integral
$$I_L \leq \int_a^b f(x)dx \leq I_R,$$
if the function in the interval is increasing and
$$I_R \leq \int_a^b f(x)dx \leq I_L,$$
if the function in the interval is decreasing.

Midpoint Method

Calculating Midpoints

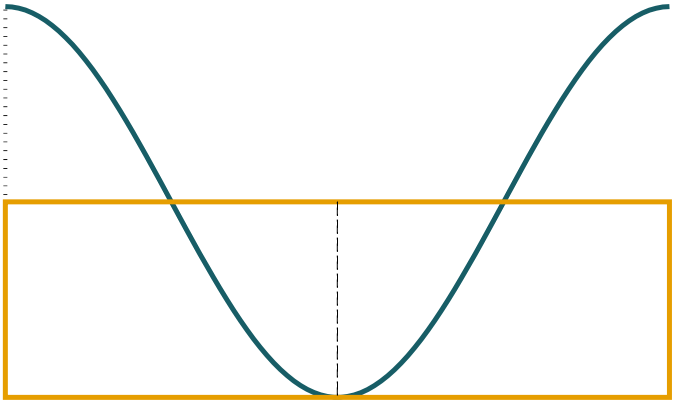
Another way of approximating an integral with finite sums is the **Midpoint Method**, which uses the function value in the middle of a given interval $[x_i, x_{i+1}]$

$$\text{Mid}(f, [x_i, x_{i+1}]) = f(\frac{x_i + x_{i+1}}{2})$$

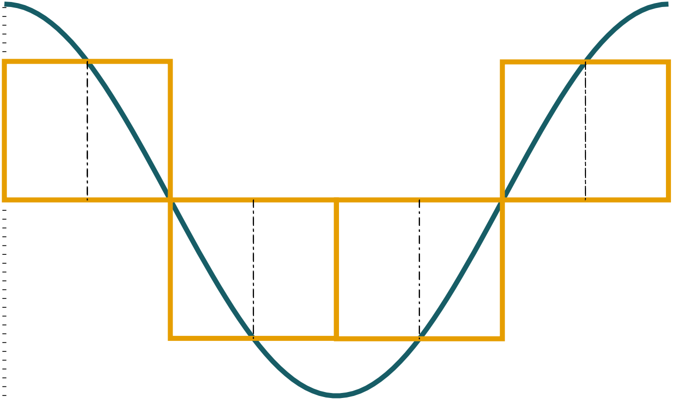
The sum of Midpoints also yields an estimation of the area under the curve

$$I_M = \int_i^n Mid(f, [x_i, x_{i+1}])$$

Midpoint Sums



Midpoint Sums



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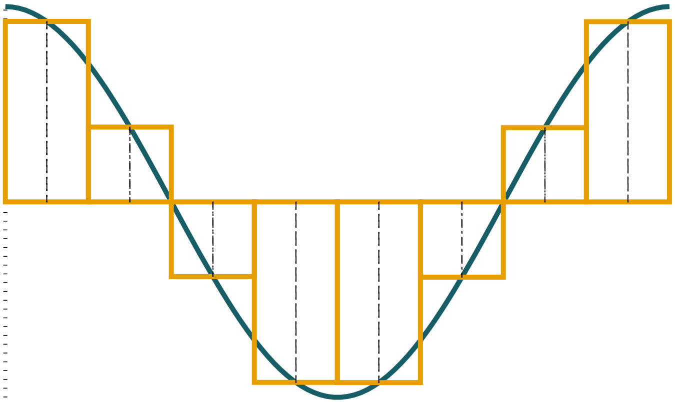
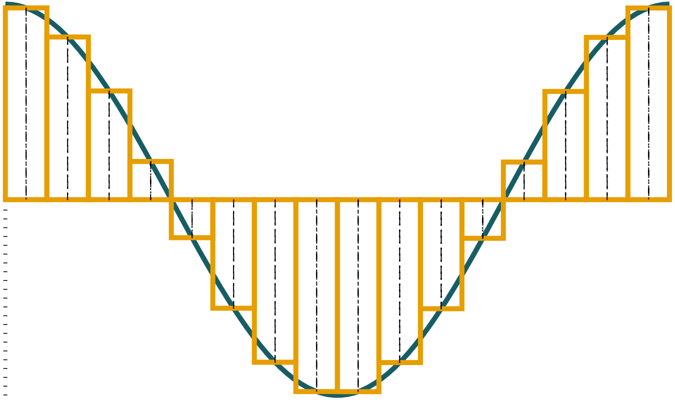
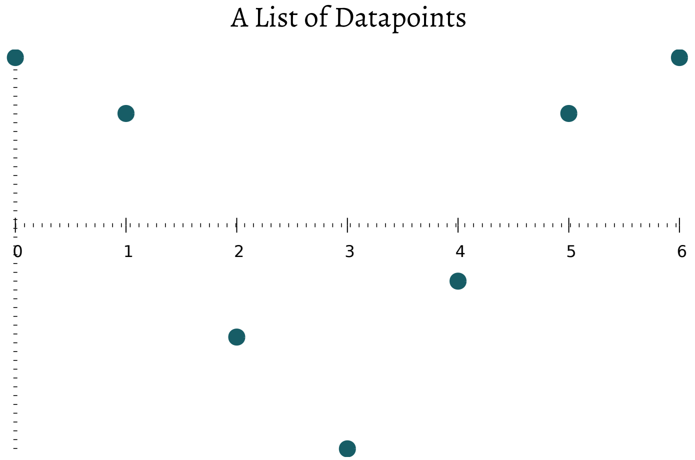


Midpoint Sums



(Simple) Numerical Integration



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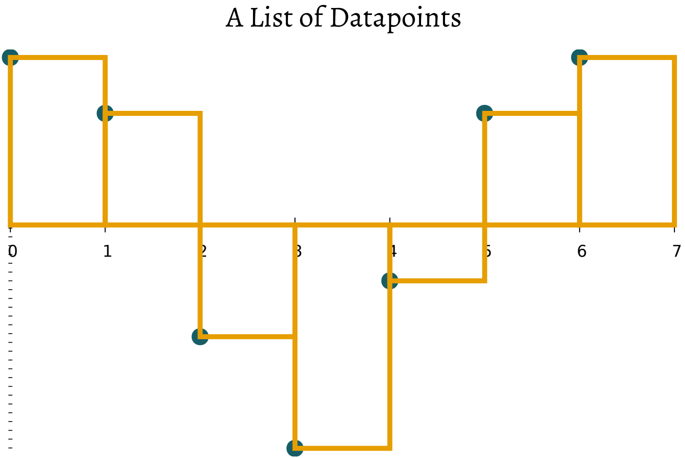

Integrating a List of Datapoints

- From a sensor we receive the following velocity values $v(x_i)$:

x_i	0	1	2	3	4	5	6
$v(x_i)$	3	2	-2	-4	-1	2	3

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- The integrated position for x_3 and startpoint $s = 2$ equals:

$$V(x_3) = s + x_0 + x_1 + x_2 + x_3 = 2 + 3 + 2 + (-2) + (-4) = 1$$

- The position at time-step x_3 is 1

Reading Files

- Opening a file

```
fileObject = open('file.txt', 'r')
#The option r stands for read
```

- Reading the file contents

```
#readlines creates a list containing each line
lines = fileObject.readlines()
for line in lines:
    print(line)
```

- Close the file after usage:

```
fileObj.close()#This can be done right after readlines()
```

Details on Strings

- Useful string operations

```
#Strip removes the new-line character '\n'
line = line.strip()
#Split tokenizes the string at the given character
line = line.split(' ')# 'Hello you' to ['Hello','you']
line = line.split('o')# 'Hello you' to ['Hell',' y','u']
line = line.replace('l','b')# 'Hello you' to 'Hebbo you'
```

Tasks

1. Download the file *velocity_series.txt* from the course page and write a script that reads its contents and stores them as a list of floating values. Plot the list with *Pyplot*.
   - Use *file.readlines*() to receive a list of strings containing each line
   - Extract the velocity in each line by applying the *split*() method in a for-loop
   - In the loop typecast the velocity into a float and append it to a second list
2. Write a script that takes a list of velocities and uses simple numerical integration to calculate a list of positions. Assume a starting position of your choice.
   - Initialize a position variable with your starting position and create an empty position list.
   - Loop through your velocity list. In each loop add the current velocity to your position variable and append the result to your position list.
3. *(Optional)* Compute the numerical integral of $cos(x)$ in the interval $[0, 2\pi]$ using the midpoint method. Vary the number of subintervals. Plot your results together with $sin(x)$ to verify your integrated data.