Dynamical systems tutorial

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Dynamical systems: Tutorial

- the word “dynamics”
  - time-varying measures
  - range of a quantity
  - forces causing/accounting for movement => dynamical systems

- dynamical systems are the universal language of science
  - physics, engineering, chemistry, theoretical biology, economics, quantitative sociology, ...
time-variation and rate of change

- variable \( x(t) \);
- rate of change \( \frac{dx}{dt} \)
dynamical system

\[
dx/dt = f(x)\]
dynamical system

\[ \frac{dx}{dt} = f(x) \]
dynamical system

- present determines the future
- given initial condition
- predict evolution (or predict the past)

\[ \frac{dx}{dt} = f(x) \]
dynamical systems

- $x$: spans the state space (or phase space)
- $f(x)$: is the “dynamics” of $x$ (or vector-field)
- $x(t)$ is a solution of the dynamical systems to the initial condition $x_0$
  - if its rate of change = $f(x)$
  - and $x(0)=x_0$
Dynamical systems

as differential equations: initial state determines the future

\[ \dot{x} = f(x) \]
Dynamical systems

A vector of initial states determines the future: systems of differential equations:

\[ \dot{x} = f(x) \quad \text{where} \quad x = (x_1, x_2, \ldots, x_n) \]
Dynamical systems

- Continuously many variables $x(y)$ determine the future =
- An initial function $x(y)$ determines the future

- Partial differential equations
  $$\dot{x}(y, t) = f \left( x(y, y), \frac{\partial x(y, t)}{\partial y}, \ldots \right)$$

- Functional differential equations
  $$\dot{x}(y, t) = \int dy' g \left( x(y, t), x(y', t) \right)$$
a piece of past trajectory determines the future

Dynamical systems

delay differential equations

functional differential equations

\[ \dot{x}(t) = f(x(t - \tau)) \]

\[ \dot{x}(t) = \int_0^t dt' f(x(t')) \]
sample time discretely

compute solution by iterating through time

\[ \dot{x} = f(x) \]

\[ t_i = i \times \Delta t; \quad x_i = x(t_i) \]

\[ \dot{x} \approx \frac{\Delta x}{\Delta t} = \frac{x_{i+1} - x_i}{\Delta t} \]

\[ x_{i+1} = x_i + \Delta t \times f(x_i) \]

[forward Euler]
linear dynamics

=> simulation
fixed point, to which neighboring initial conditions converge = attractor
A fixed point is a constant solution of the dynamical system

$$\dot{x} = f(x)$$

$$\dot{x} = 0 \Rightarrow f(x_0) = 0$$
stability

- mathematically really: asymptotic stability
- defined: a fixed point is asymptotically stable, when solutions of the dynamical system that start nearby converge in time to the fixed point
stability

- the mathematical concept of stability (which we do not use) requires only that nearby solutions stay nearby

- Definition: a fixed point is **unstable** if it is not stable in that more general sense,
  
  - that is: if nearby solutions do not necessarily stay nearby (may diverge)
linear approximation near attractor

- non-linearity as a small perturbation/deformation of linear system
- => non-essential non-linearity

\[
\frac{dx}{dt} = f(x)
\]
if the slope of the linear system is negative, the fixed point is (asymptotically stable)
if the slope of the linear system is positive, then the fixed point is unstable
stability in a linear system

If the slope of the linear system is zero, then the system is indifferent (marginally stable: stable but not asymptotically stable)
stability in linear systems

generalization to multiple dimensions

- if the real-parts of all Eigenvalues are negative: stable
- if the real-part of any Eigenvalue is positive: unstable
- if the real-part of any Eigenvalue is zero: marginally stable in that direction (stability depends on other eigenvalues)
stability in nonlinear systems

- Stability is a local property of the fixed point.

- \[ \Rightarrow \text{linear stability theory} \]

  - The eigenvalues of the linearization around the fixed point determine stability.

  - All real-parts negative: stable.

  - Any real-part positive: unstable.

  - Any real-part zero: undecided: now nonlinearity decides (non-hyperpolic fixed point).
stability in nonlinear systems

- all real-parts negative: stable
- any real-part positive: unstable
stability in nonlinear systems

any real-part zero: undecided: now nonlinearity decides (non-hyperpolic fixed point)
bifurcations

Look now at families of dynamical systems, which depend (smoothly) on parameters.

Ask: as the parameters change (smoothly), how do the solutions change (smoothly?)

Smoothly: topological equivalence of the dynamical systems at neighboring parameter values.

Bifurcation: dynamical systems NOT topological equivalent as parameter changes infinitesimally.
bifurcation

\[ \frac{dx}{dt} = f(x) \]

Attractor

Attractor

Attractor 1

Attractor 2
bifurcation

bifurcation = qualitative change of dynamics (change in number, nature, or stability of fixed points) as the dynamics changes smoothly.
the simplest bifurcation (co-dimension 0): an attractor collides with a repellor and the two annihilate

dx/dt=f(x)
local bifurcation

dx/dt = f(x)

attractor 1

attractor 2
reverse bifurcation

changing the dynamics in the opposite direction

dx/dt = f(x)

attractor 1

attractor 2
bifurcations are instabilities

that is, an attractor becomes unstable before disappearing

(or the attractor appears with reduced stability)

formally: a zero-real part is a necessary condition for a bifurcation to occur
tangent bifurcation

- normal form of tangent bifurcation
  \[ \dot{x} = \alpha - x^2 \]

- (=simplest polynomial equation whose flow is topologically equivalent to the bifurcation)

\[ x_0 = \sqrt{\alpha} \]
Hopf theorem

When a single (or pair of complex conjugate) eigenvalue crosses the imaginary axis, one of four bifurcations occur:

- tangent bifurcation
- transcritical bifurcation
- pitchfork bifurcation
- Hopf bifurcation
transcritical bifurcation

\[ \dot{x} = \alpha x - x^2 \]

Normal form

\( \alpha \) negative
\( \alpha \) positive
\( \alpha = 0 \)

Fixed point

Stable
Unstable
pitchfork bifurcation

normal form

\[ \dot{x} = \alpha x - x^3 \]

fixed point

\[ \dot{x} = -2x_0 x = -2\sqrt{\alpha}x \]

\( \alpha \) positive

\( \alpha \) negative

\( \alpha = 0 \)

stable

unstable
Hopf: need higher dimensions
2D dynamical system: vector-field

\[ \dot{x}_1 = f_1(x_1, x_2) \]
\[ \dot{x}_2 = f_2(x_1, x_2) \]
\[
\dot{x}_1 = f_1(x_1, x_2) \\
\dot{x}_2 = f_2(x_1, x_2)
\]
fixed point, stability, attractor

\[ \dot{x}_1 = f_1(x_1, x_2) \]
\[ \dot{x}_2 = f_2(x_1, x_2) \]
Hopf bifurcation

\[ \dot{r} = \alpha r - r^3 \]
\[ \dot{\phi} = \omega \]

Normal form

\( \alpha < 0 \)
\( \alpha = 0 \)
\( \alpha > 0 \)
forward dynamics

- given known equation, determined fixed points / limit cycles and their stability
- more generally: determine invariant solutions (stable, unstable and center manifolds)
inverse dynamics

given solution, find the equation…

this is the problem faced in design of behavioral dynamics…
inverse dynamics: design

- in the design of behavioral dynamics... you may be given:
  - attractor solutions/stable states
  - and how they change as a function of parameters/conditions
  
  => identify the class of dynamical systems using the 4 elementary bifurcations
  and use normal form to provide an exemplary representative of the equivalence class of dynamics