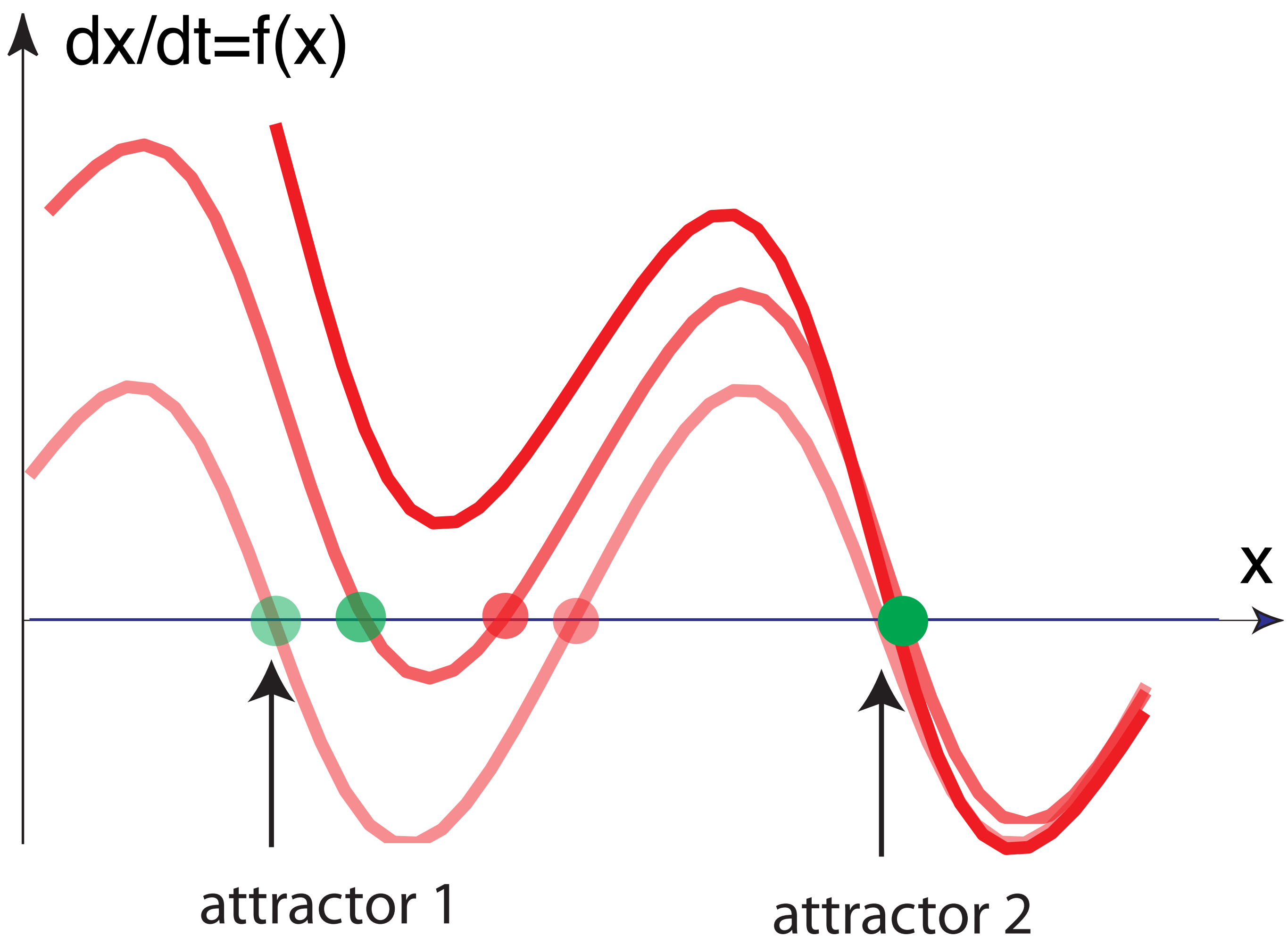
Dynamical systems tutorial

Gregor Schöner, INI, RUB

bifurcations

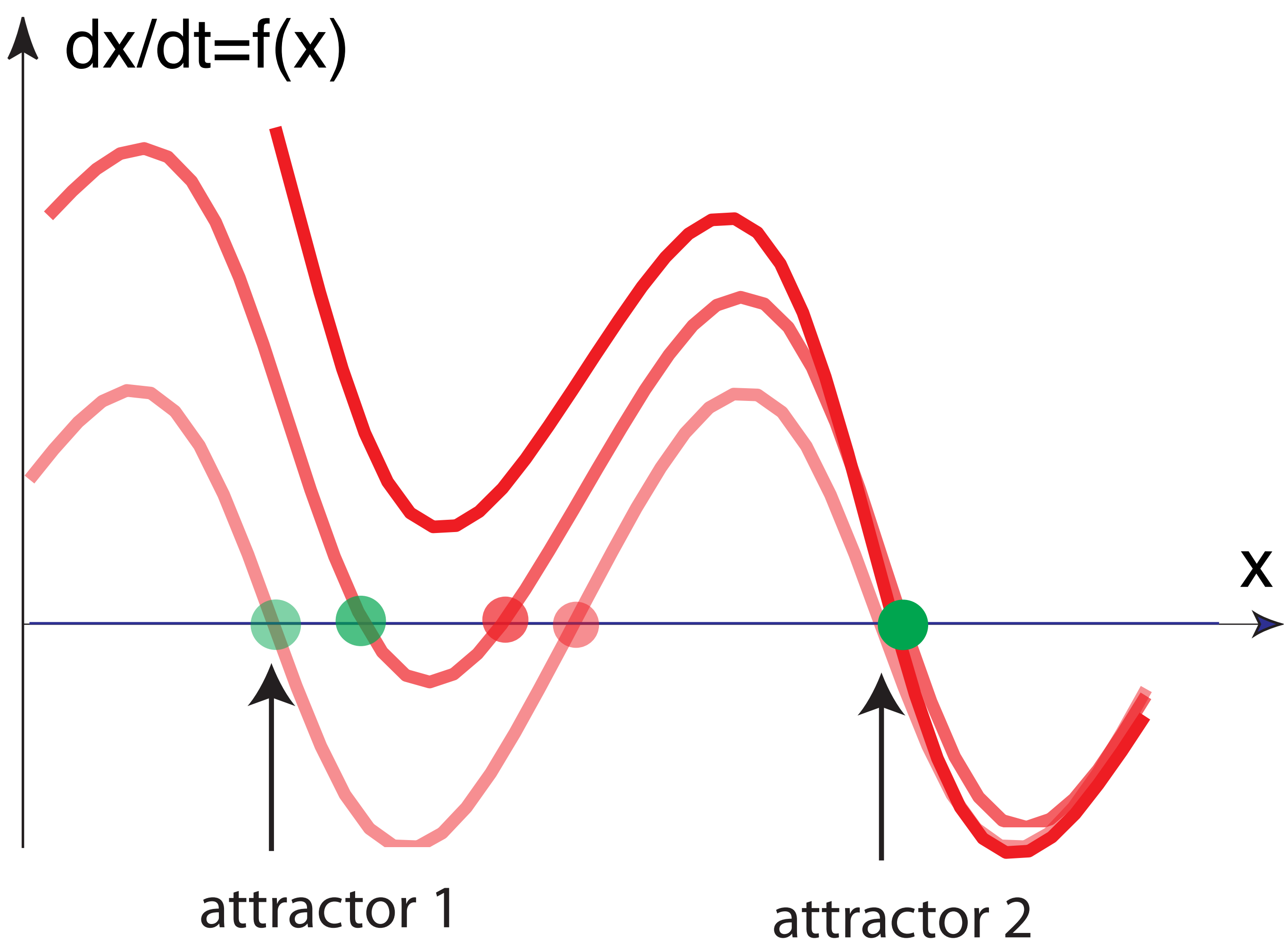
- look now at families of dynamical systems, which depend (smoothly) on parameters
- ask: as the parameters change (smoothly), how do the solutions change (smoothly?)
  - smoothly: topological equivalence of the dynamical systems at neighboring parameter values
  - bifurcation: dynamical systems NOT topological equivalent as parameter changes infinitesimally

bifurcation



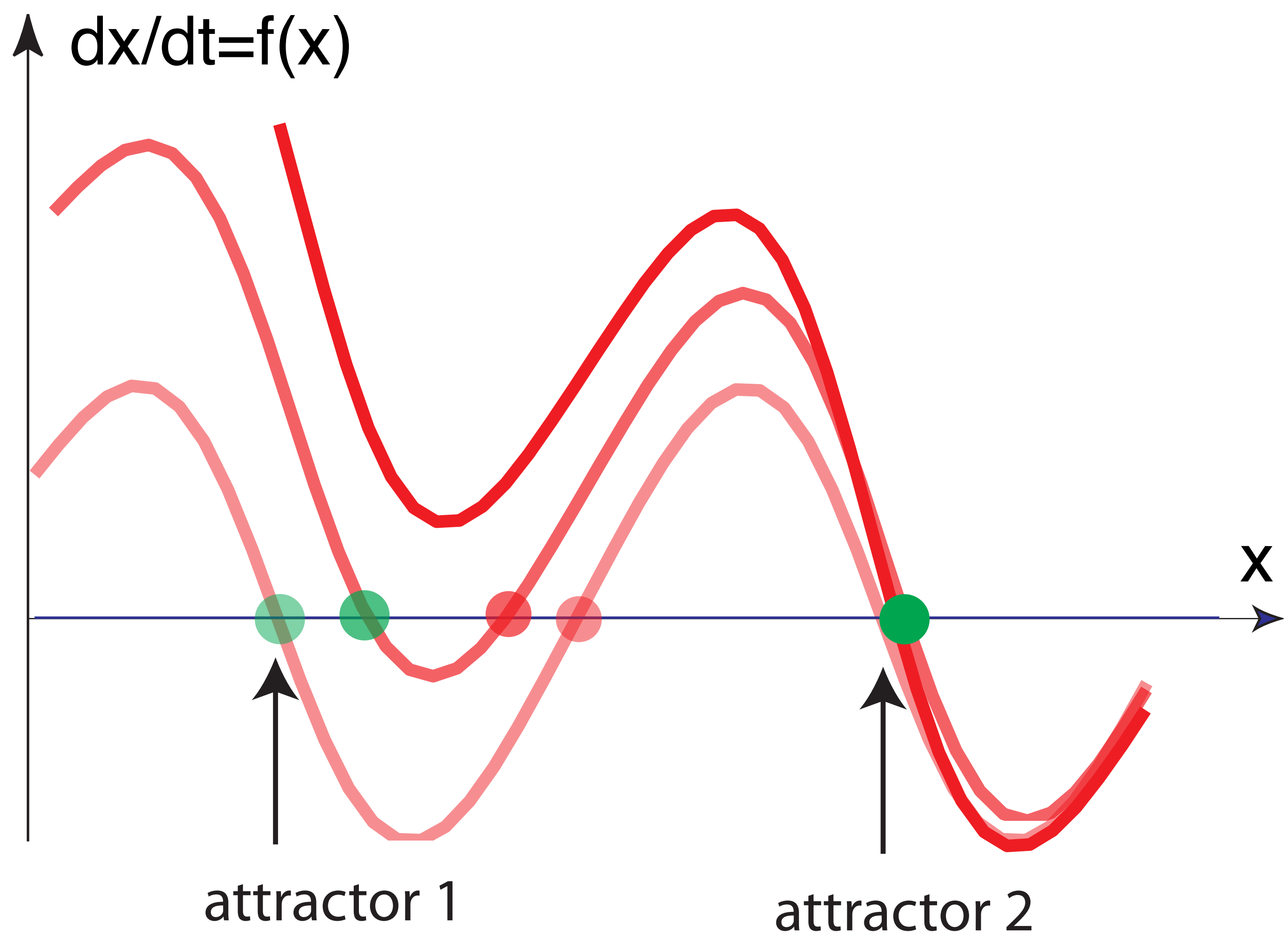
bifurcation

- bifurcation=qualitative change of dynamics (change in number, nature, or stability of fixed points) as the dynamics changes smoothly

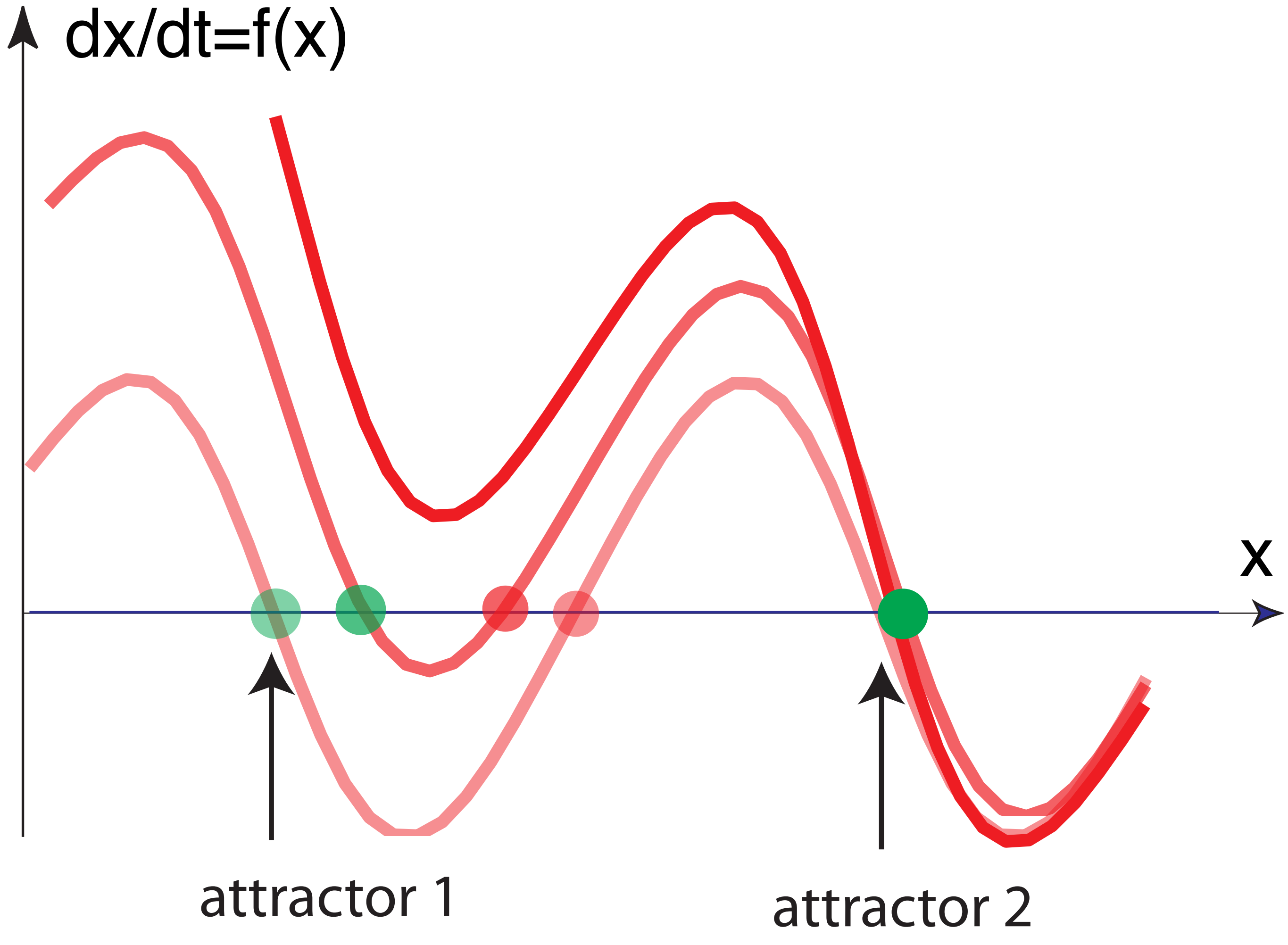


tangent bifurcation

- the simplest bifurcation (co-dimension 0): an attractor collides with a repellor and the two annihilate

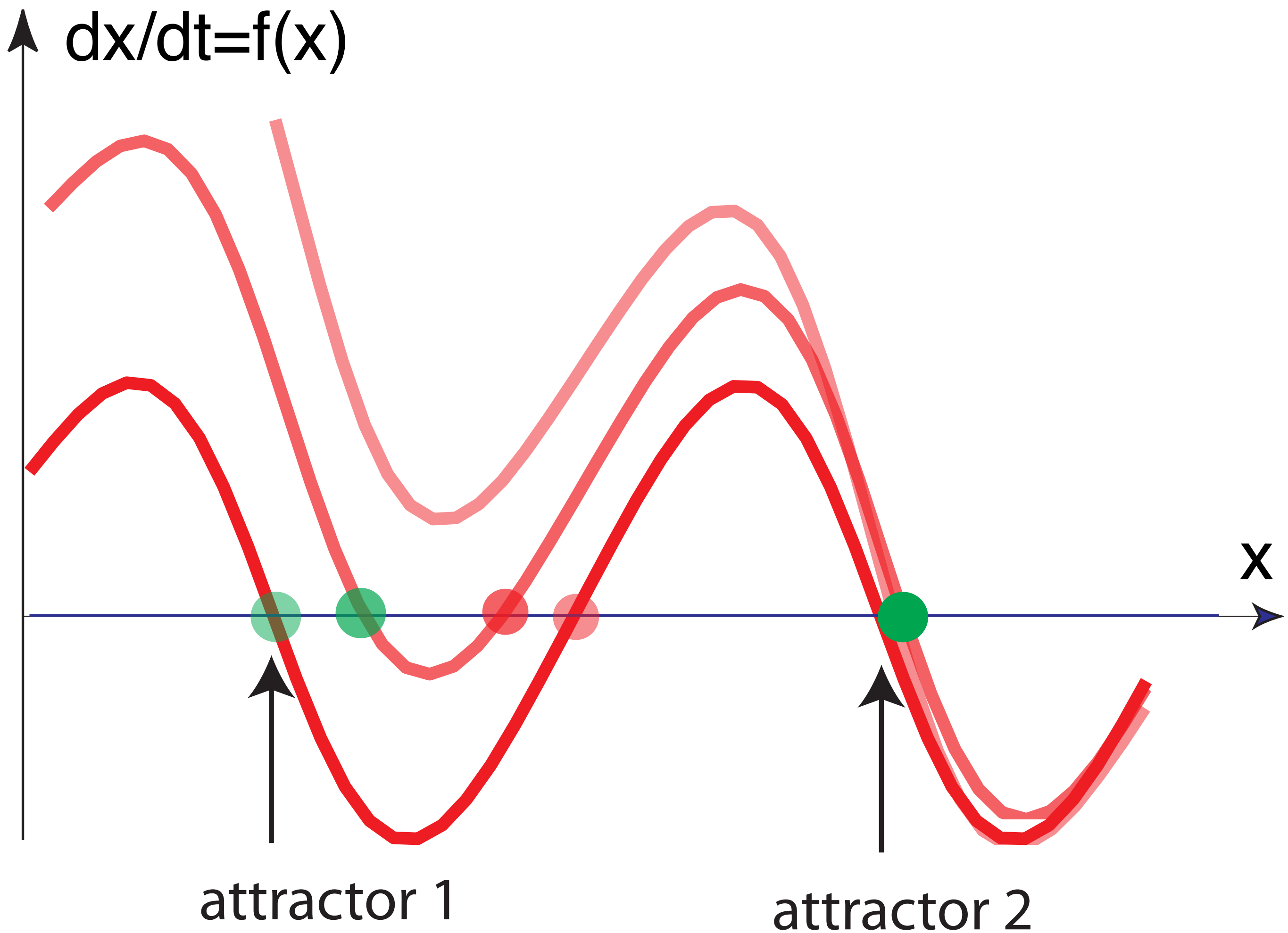


local bifurcation



reverse bifurcation

- changing the dynamics in the opposite direction



bifurcations are instabilities

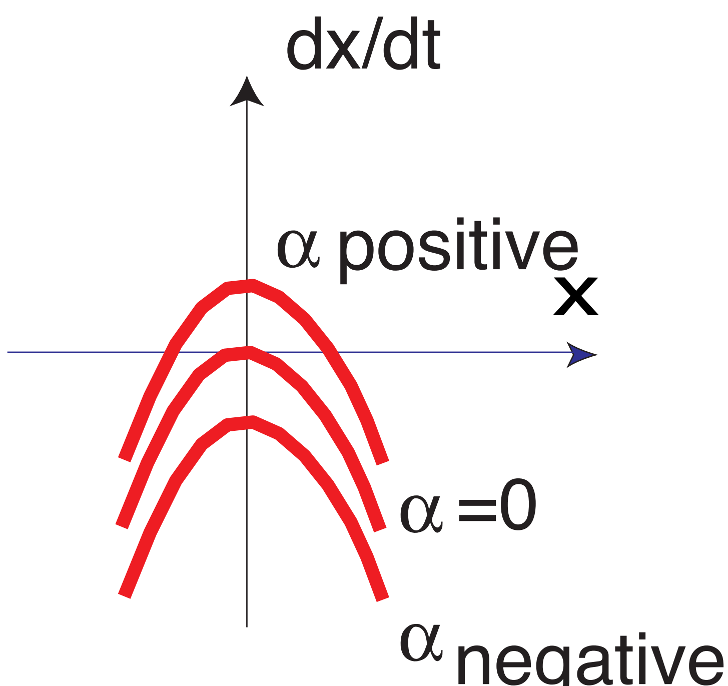
- that is, an attractor becomes unstable before disappearing
- (or the attractor appears with reduced stability)
- formally: a zero-real part is a necessary condition for a bifurcation to occur

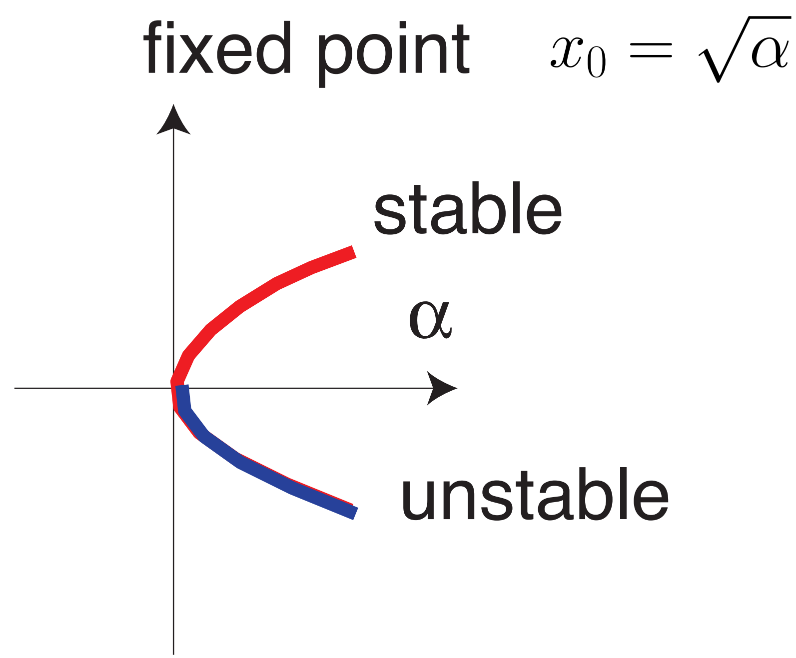
tangent bifurcation

- normal form of tangent bifurcation

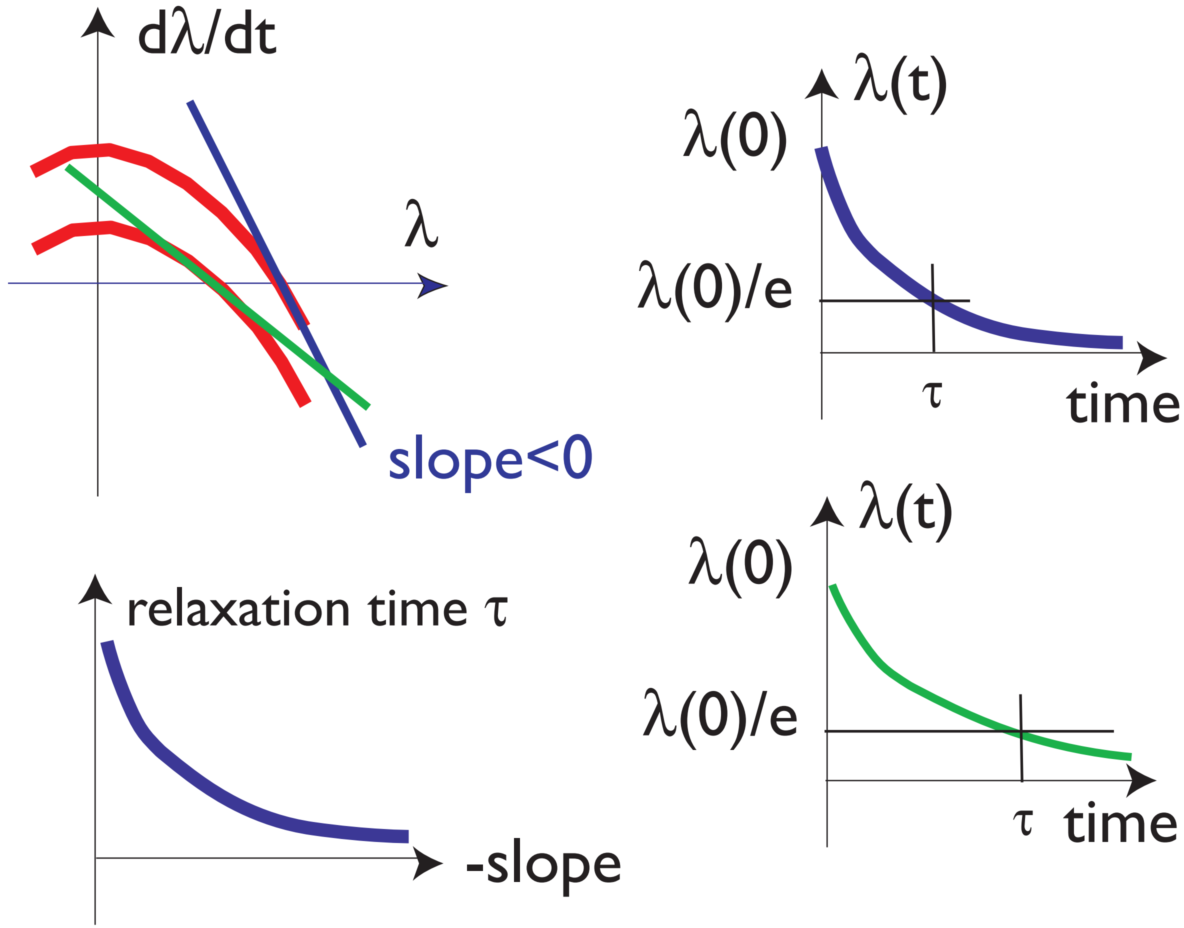
  $$\dot{x} = \alpha - x^2$$

- (=simplest polynomial equation whose flow is topologically equivalent to the bifurcation)





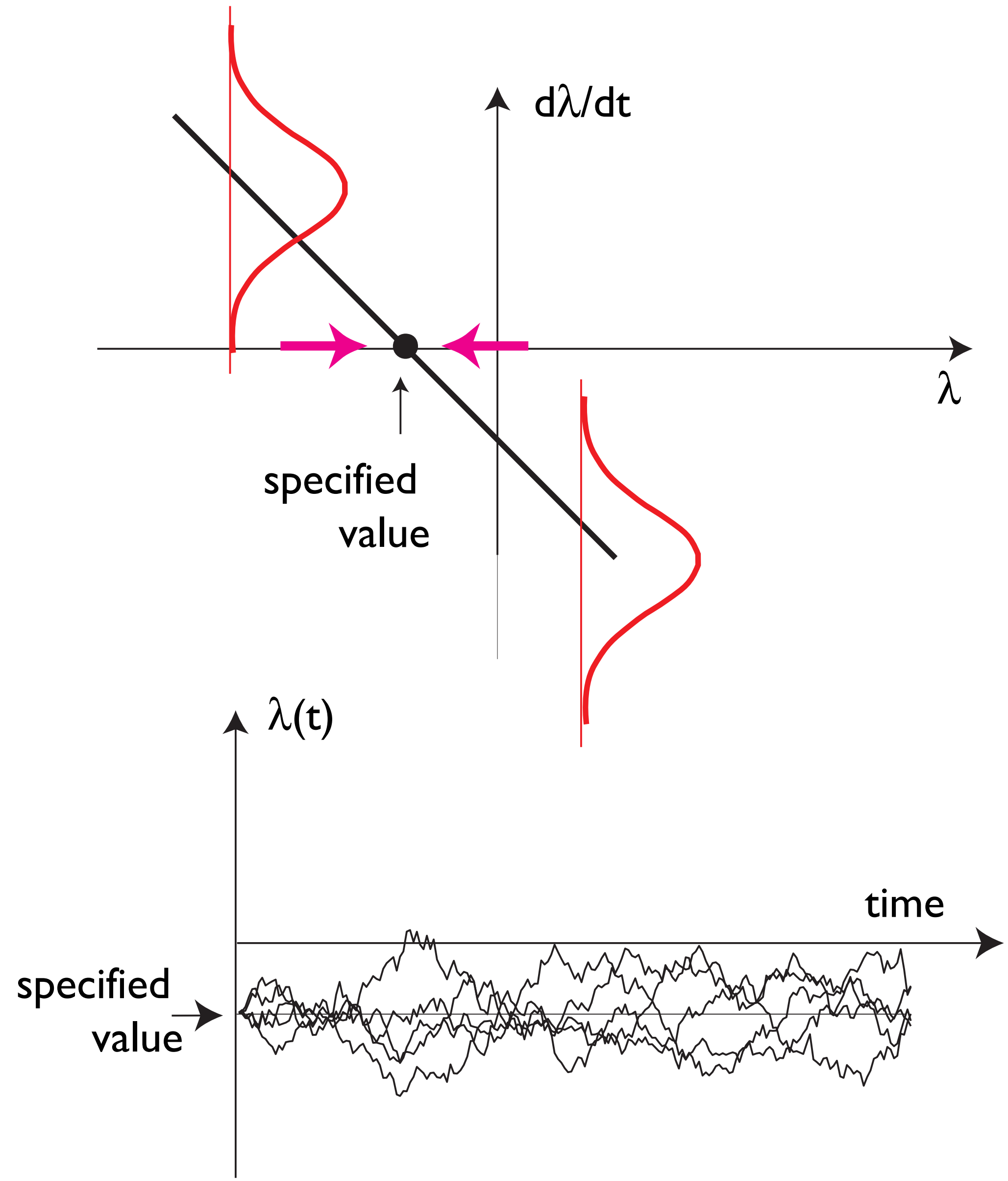
bifurcations are instabilities



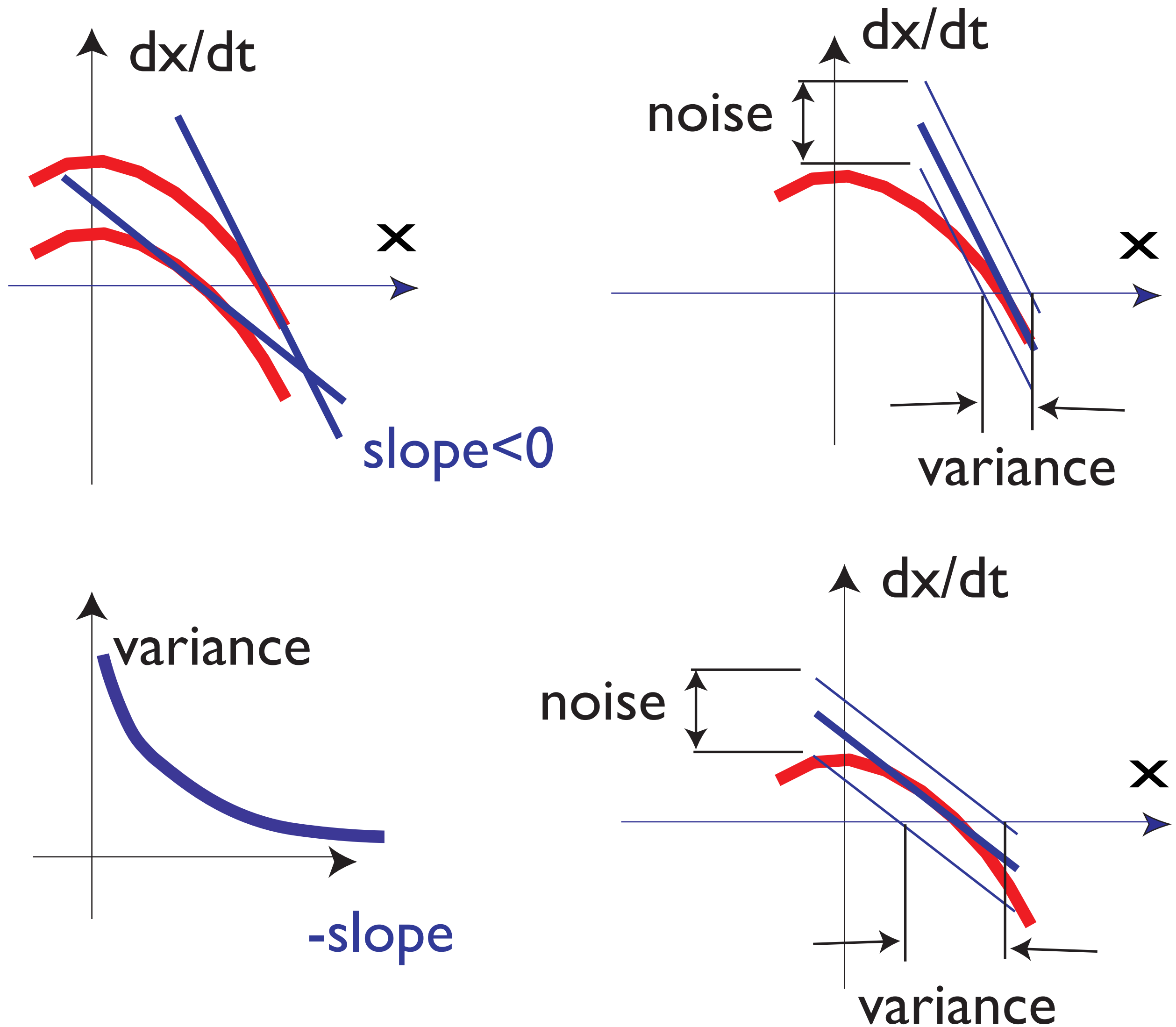
measures of stability: fluctuations

$$\dot{\lambda} = -\alpha(\lambda - \lambda_0) + \text{noise}$$

- with stable fixed-point dynamics: fluctuations are constrained

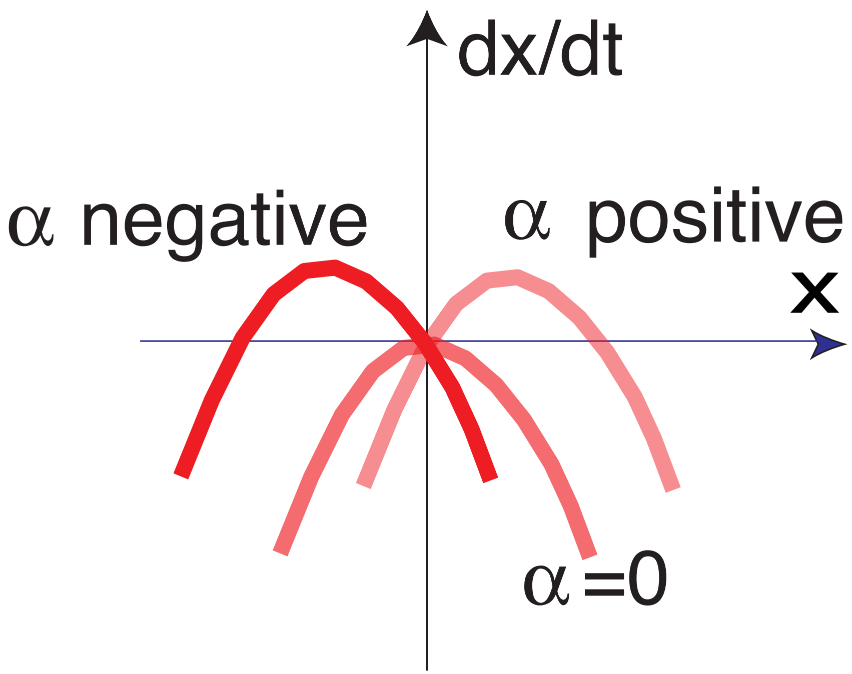


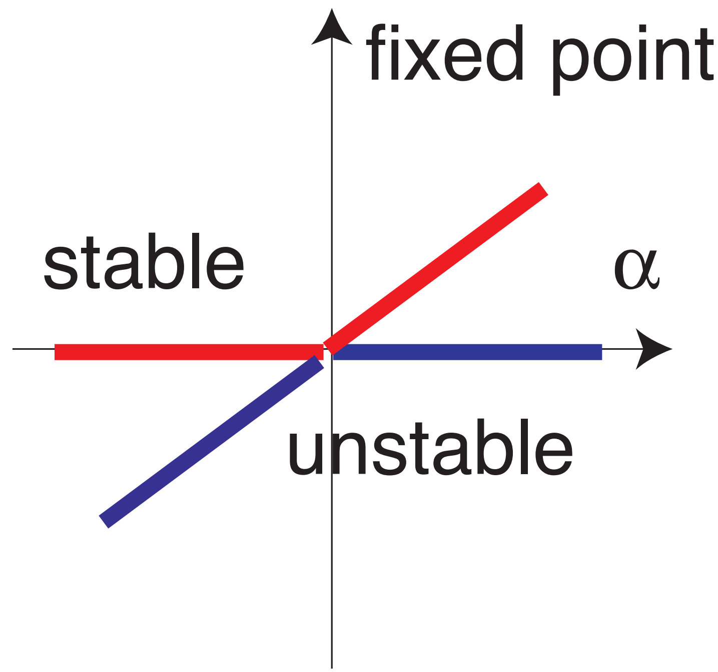
bifurcations are instabilities



transcritical bifurcation

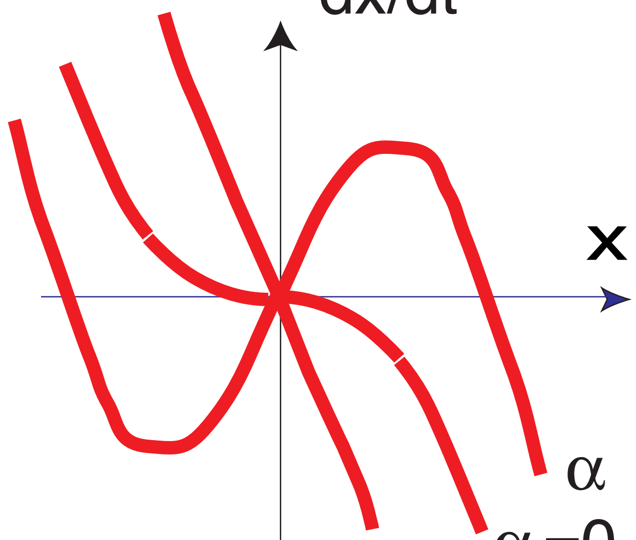
- normal form $\dot{x} = \alpha x - x^2$

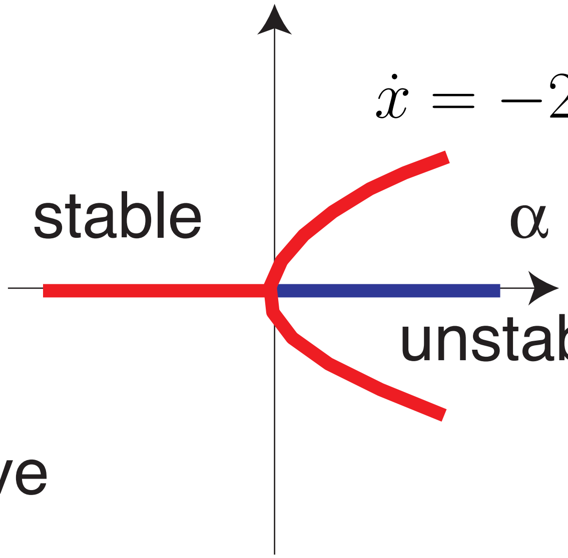




pitchfork bifurcation

- normal form $\dot{x} = \alpha x - x^3$





Hopf theorem

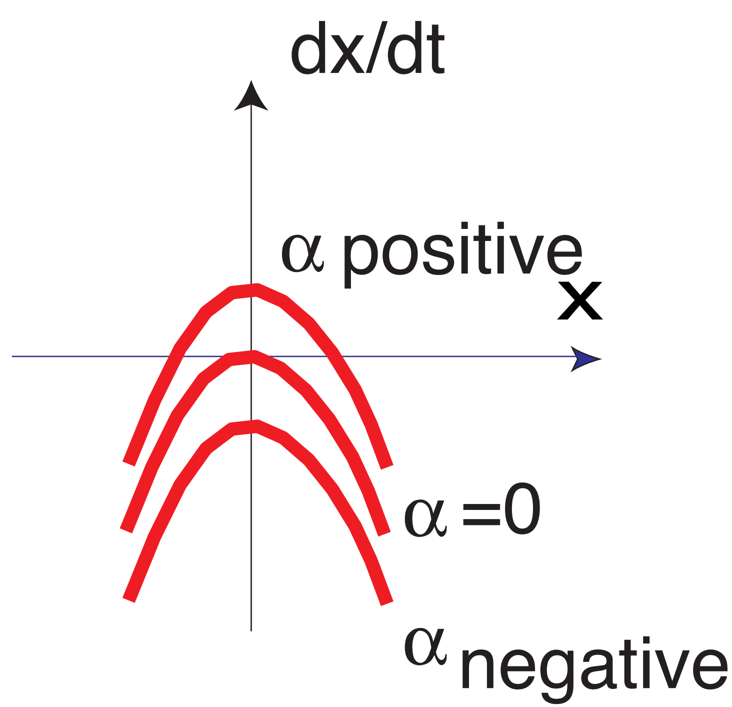
- when a single (or pair of complex conjugate) eigenvalue crosses the imaginary axis, one of four bifurcations occur
  - tangent bifurcation
  - transcritical bifurcation
  - pitchfork bifurcation
  - Hopf bifurcation

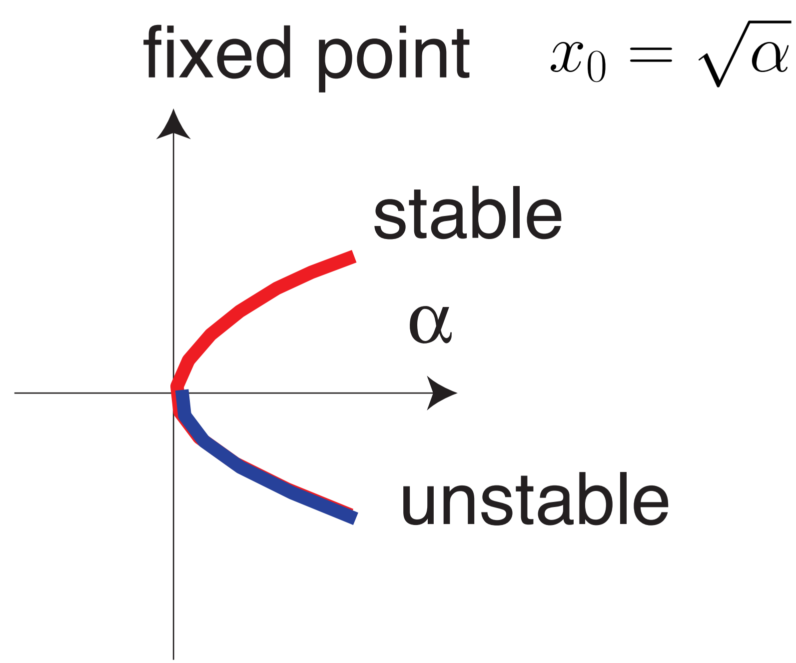
tangent bifurcation

- normal form of tangent bifurcation

$$\dot{x} = \alpha - x^2$$

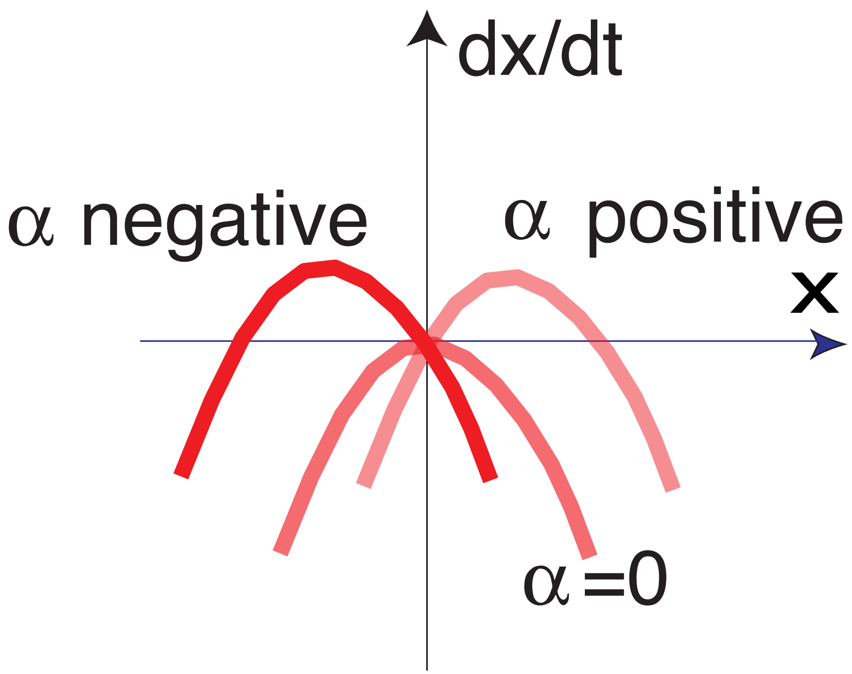
- (=simplest polynomial equation whose flow is topologically equivalent to the bifurcation)

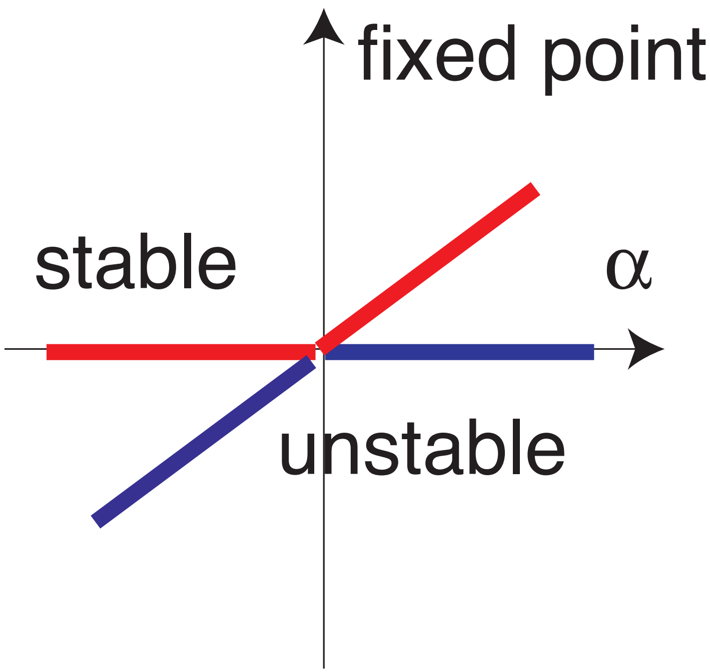




transcritical bifurcation

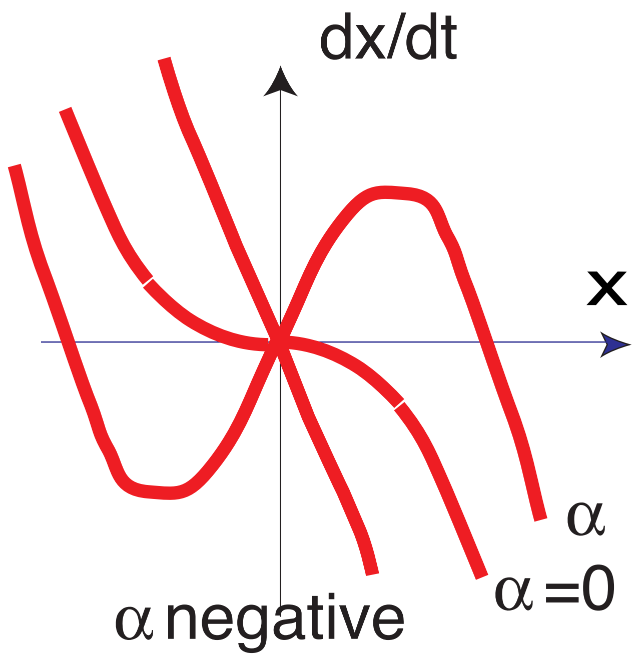
- normal form $\dot{x} = \alpha x - x^2$

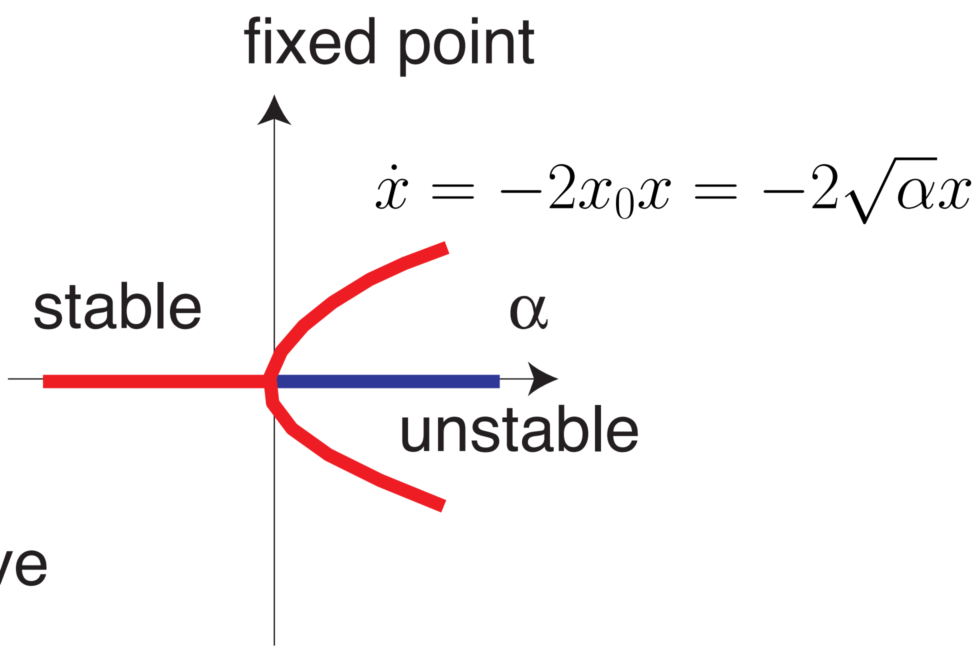




pitchfork bifurcation

- normal form $\dot{x} = \alpha x - x^3$

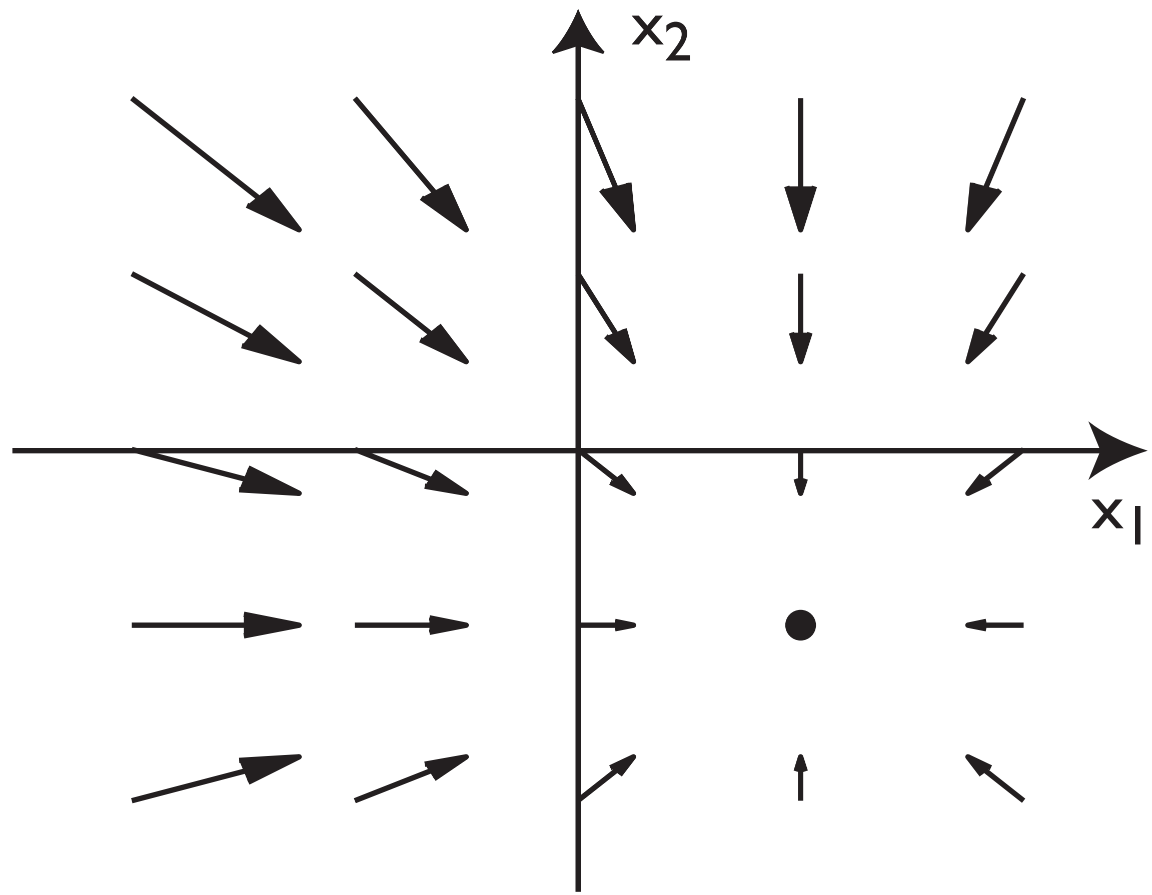




Hopf: need higher dimensions

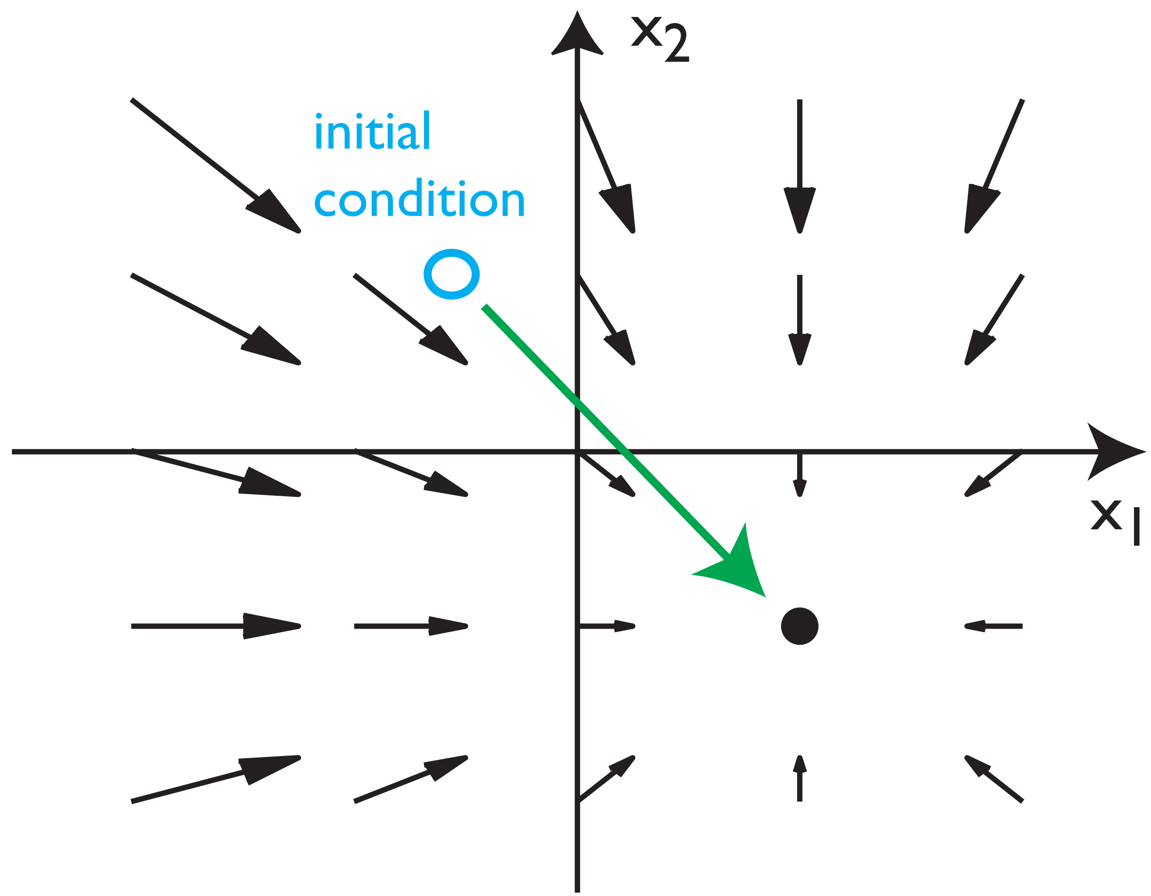
2D dynamical system: vector-field

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$



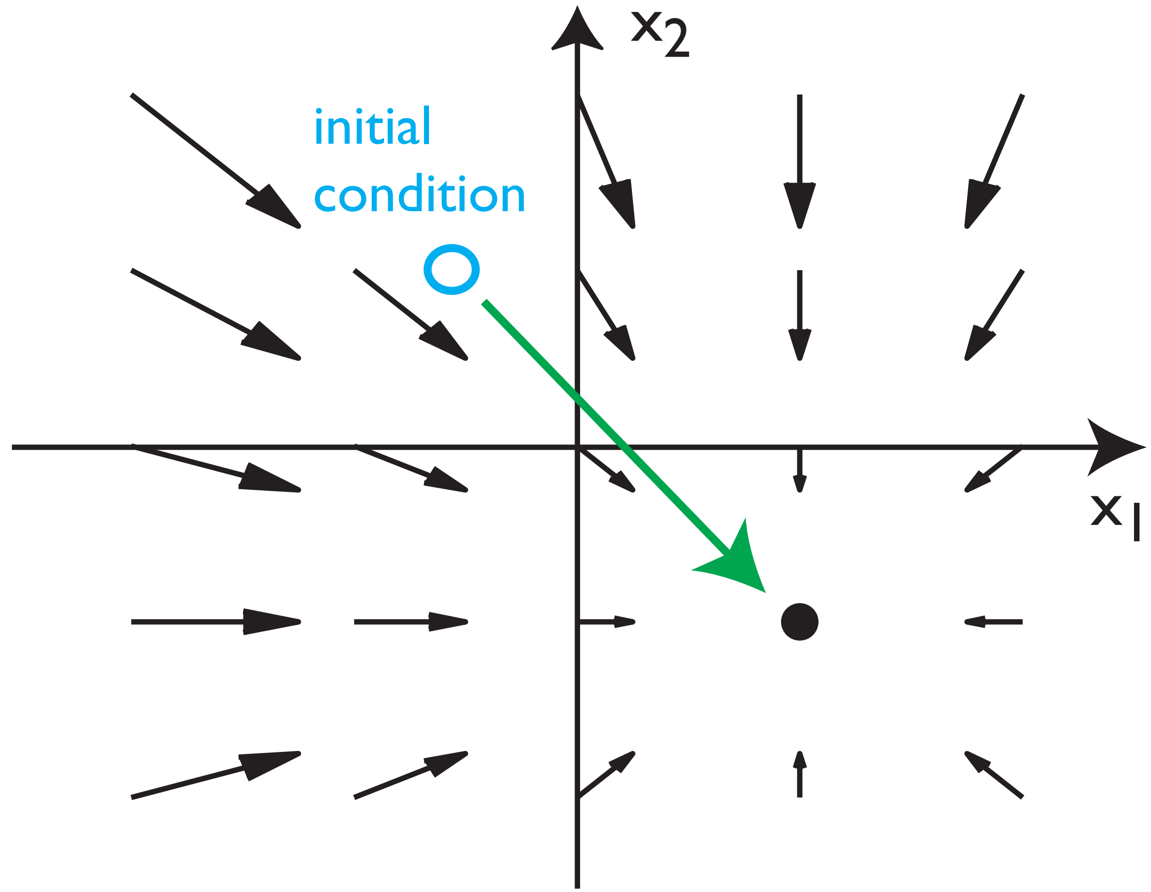
vector-field

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$



fixed point, stability, attractor

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

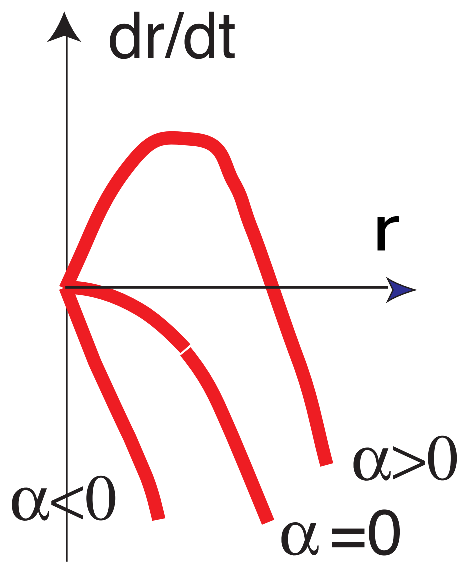


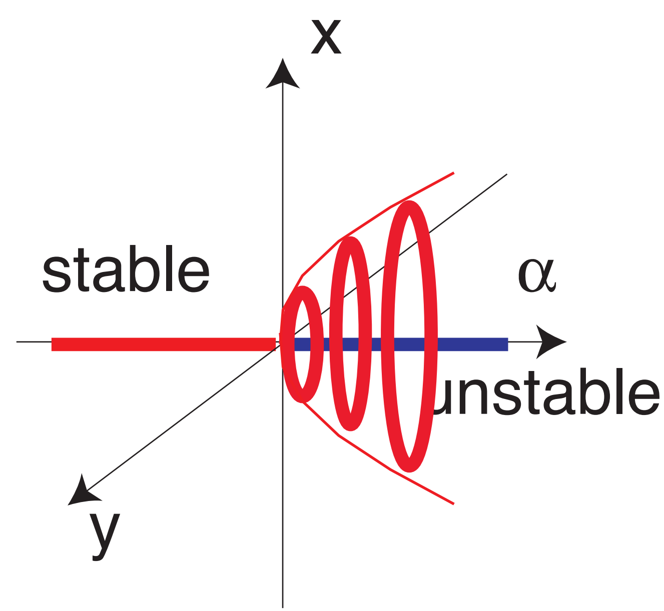
Hopf bifurcation

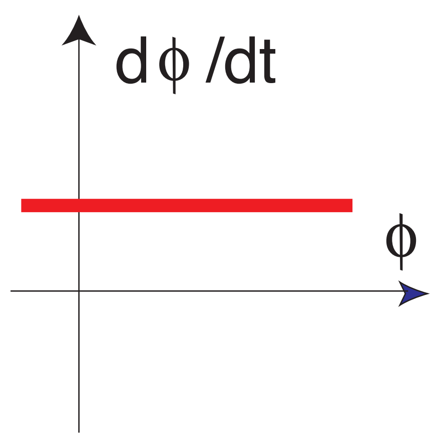
- normal form

$$\dot{r} = \alpha r - r^3$$

$$\dot{\phi} = \omega$$





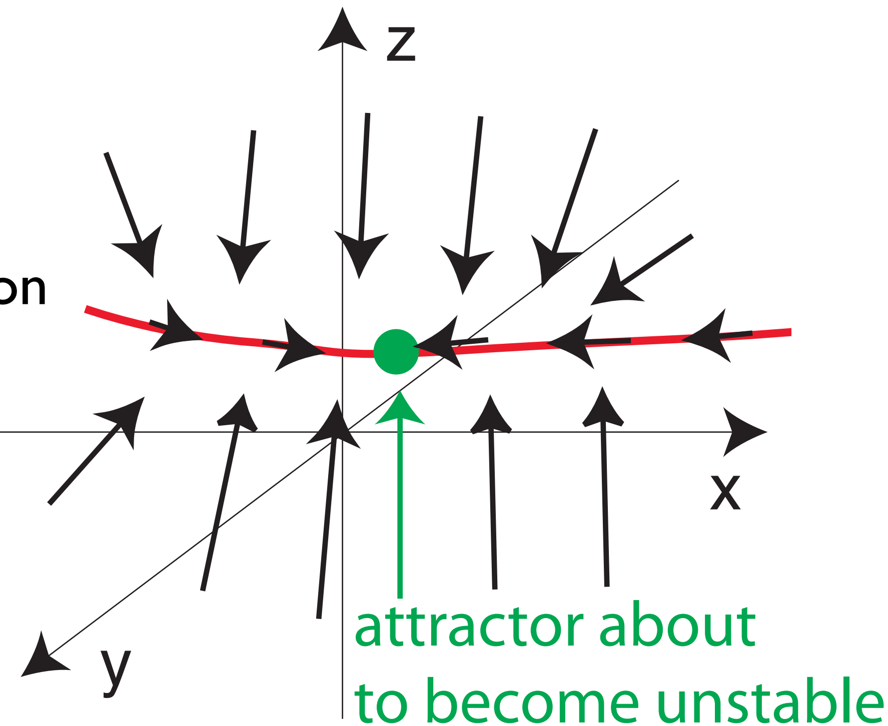


higher bifurcations

- e.g., degenerate (non-linear terms simultaneously zero with real-part of an eigenvalue)
- e.g. higher co-dimension ...

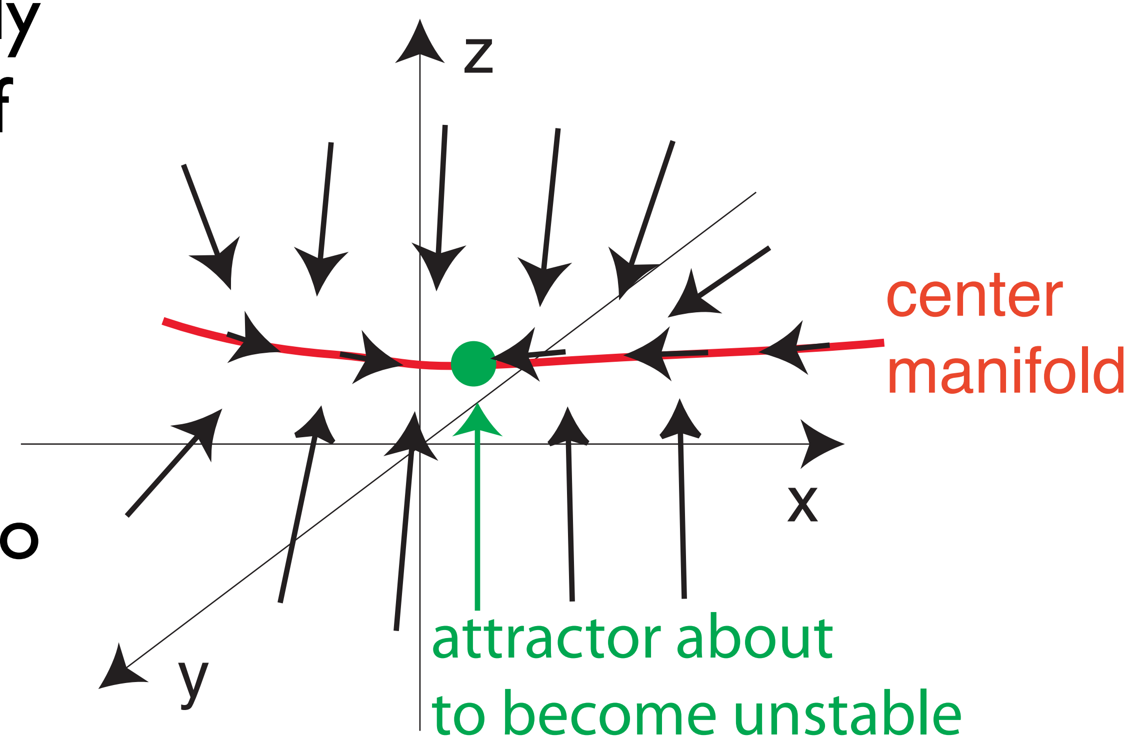
the center manifold

- at bifurcation: real part of one eigenvalue=0
  - corresponding eigenvector: critical direction
- near bifurcation: real part of one
  - eigenvalue close to zero
- => Center manifold theorem



center manifold

- = manifold which is locally tangent to eigenvector of zero real part eigenvalue
- CM theorem: dynamics within the CM is topologically equivalent to original dynamics
- => enormous dimensional reduction



center manifold

- => the 4 elementary bifurcations are “fair” representatives of any dynamical system with the same qualitative flow

forward dynamics

- given known equation, determined fixed points / limit cycles and their stability
  - more generally: determine invariant solutions (stable, unstable and center manifolds)
  - use CMF to simplify dynamics

inverse dynamics

- given classification of solutions (stable states) and their dependence on parameters, determine the dynamical system! That's the modeler's job
- practical approach
  - identify the class of dynamical systems using the 4 elementary bifurcations
  - and use normal form to provide an exemplary representative of the equivalence class of dynamics