Dynamical systems tutorial

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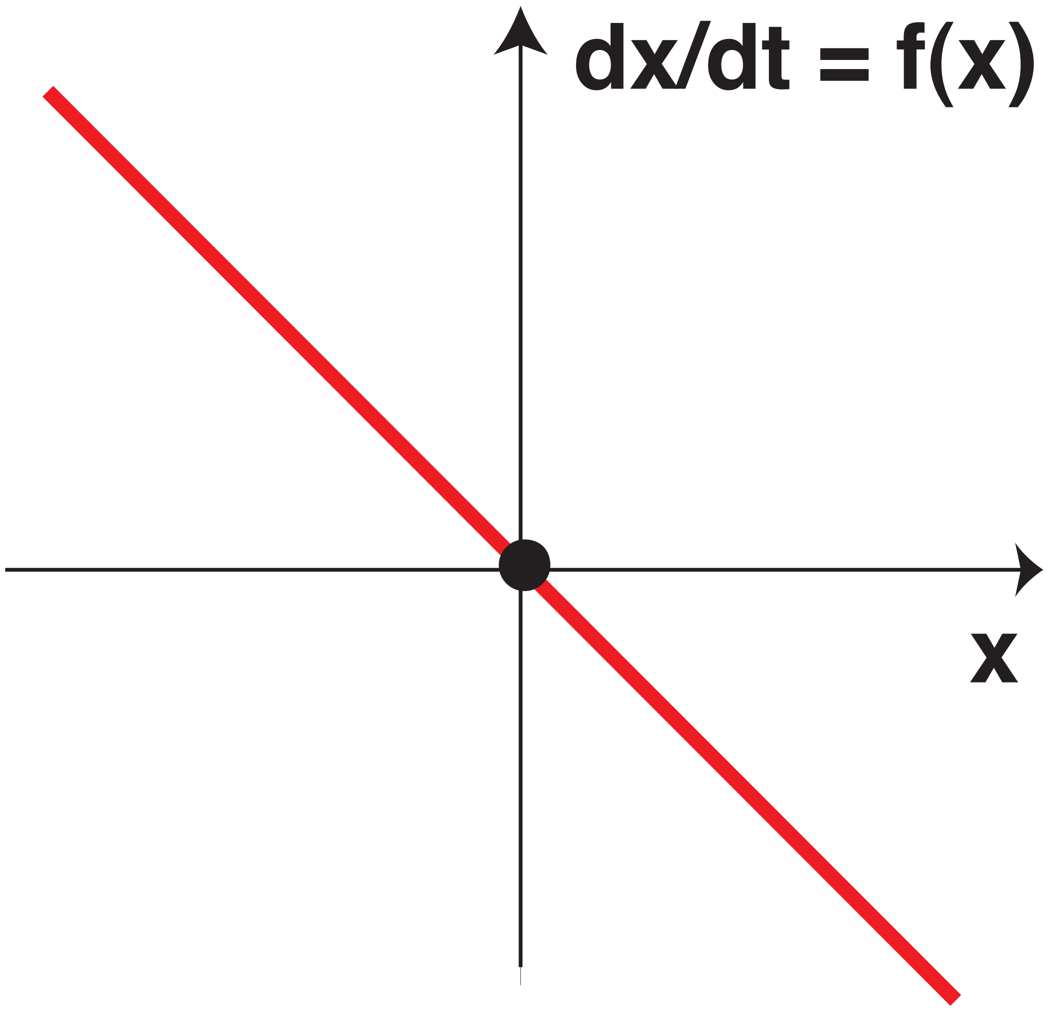
Dynamical systems: Tutorial

- the word “dynamics”
  - time-varying measures
  - range of a quantity
  - forces causing/accounting for movement => dynamical systems
- dynamical systems are the universal language of science
  - physics, engineering, chemistry, theoretical biology, economics, quantitative sociology, ...

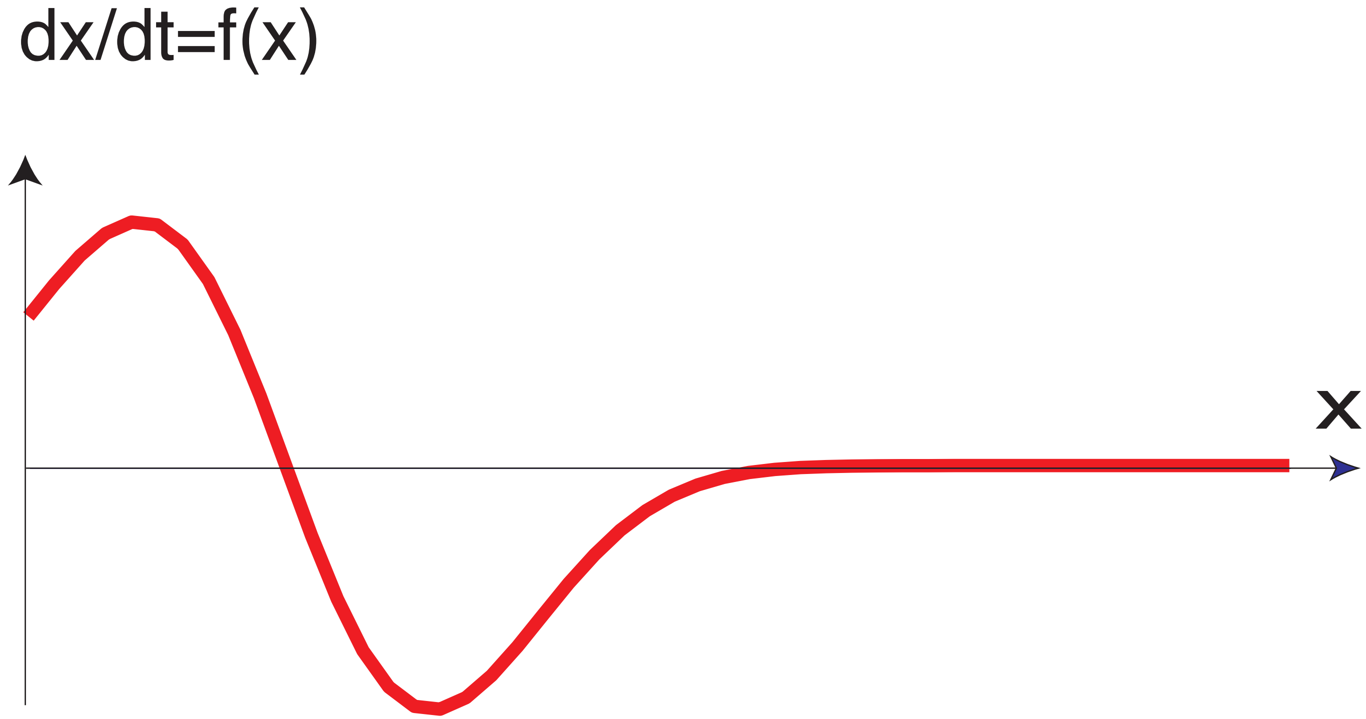
time-variation and rate of change

- variable $x(t)$;
- rate of change dx/dt

dynamical system



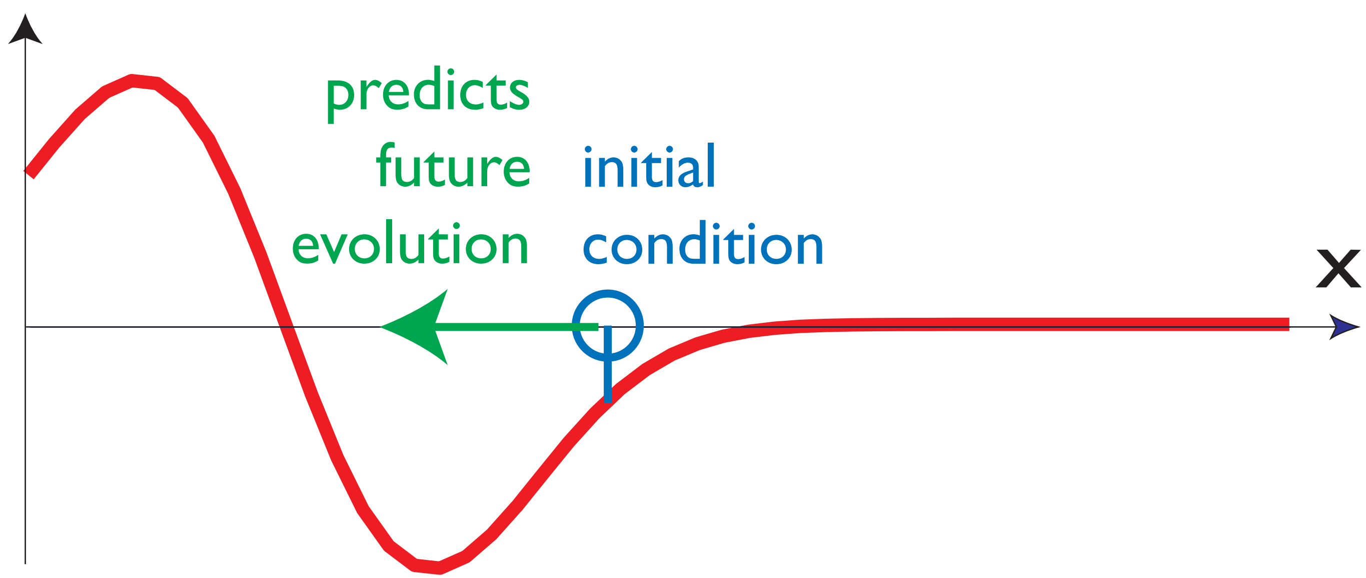
dynamical system



dynamical system

- present determines the future
  - given initial condition
  - predict evolution (or predict the past)

$dx/dt=f(x)$



dynamical systems

- x: spans the state space (or phase space)
- f(x): is the “dynamics” of x (or vector-field)
- x(t) is a solution of the dynamical systems to the initial condition x_0
    - if its rate of change = f(x)
    - and x(0)=x_0

numerics

- sample time discretely
- compute solution by iterating through time

$$\dot{x} = f(x)$$

$$t_i = i * \Delta t; \qquad x_i = x(t_i)$$

$$\dot{x} = \frac{dx}{dt} \approx \frac{\Delta x}{\Delta t} = \frac{x_{i+1} - x_i}{\Delta t}$$

$$x_{i+1} = x_i + \Delta t * f(x_i)$$

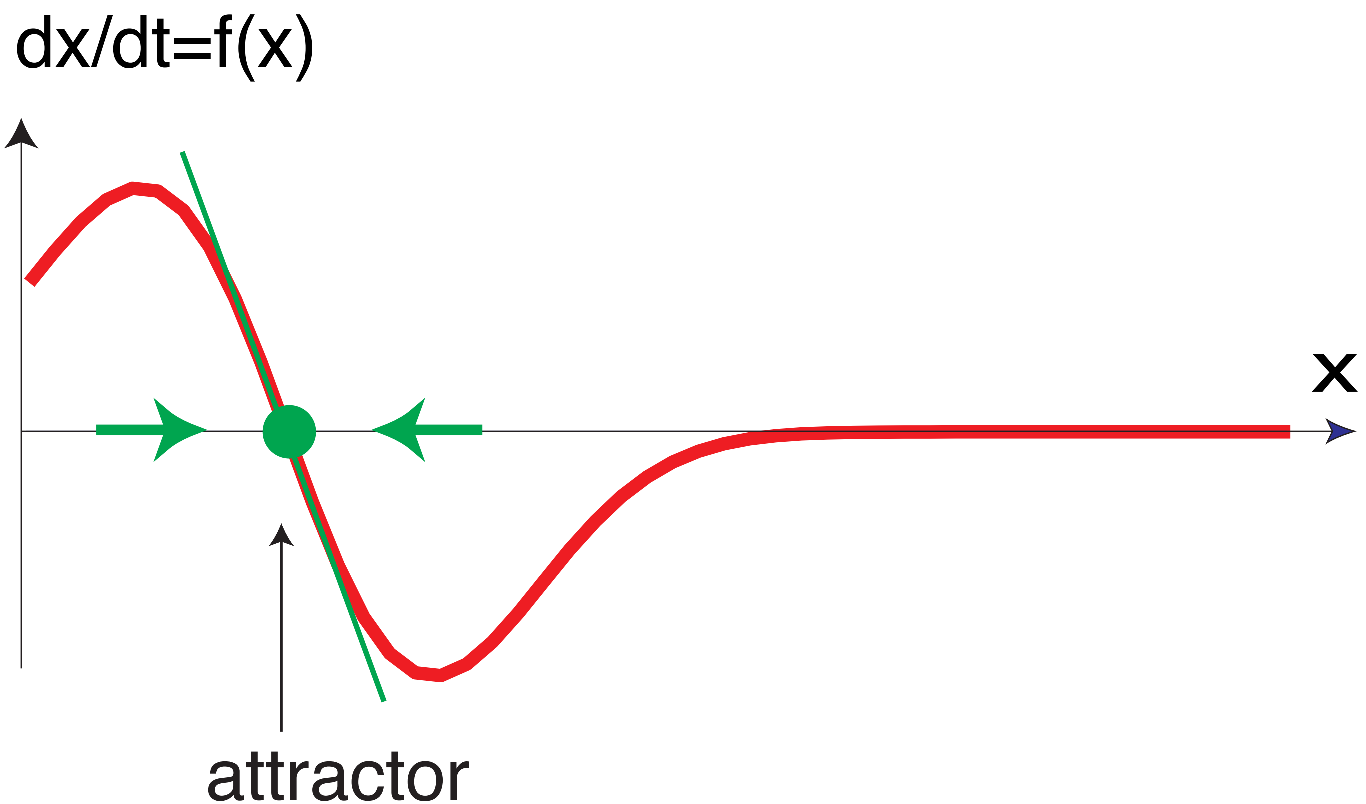
[forward Euler]

linear dynamics

- => simulation

attractor

- fixed point, to which neighboring initial conditions converge = attractor



fixed point

- is a constant solution of the dynamical system

$$\dot{x} = f(x)$$

$$\dot{x} = 0 \Rightarrow f(x_0) = 0$$

stability

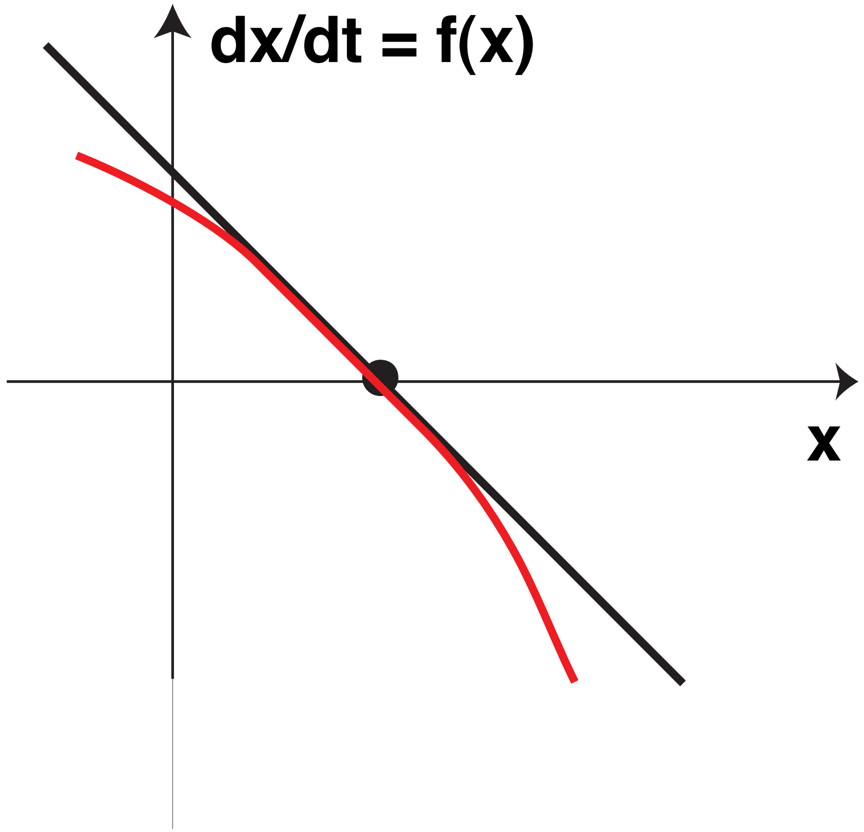
- mathematically really: asymptotic stability
- defined: a fixed point is asymptotically stable, when solutions of the dynamical system that start nearby converge in time to the fixed point

stability

- the mathematical concept of stability (which we do not use) requires only that nearby solutions stay nearby
- Definition: a fixed point is unstable if it is not stable in that more general sense,
  - that is: if nearby solutions do not necessarily stay nearby (may diverge)

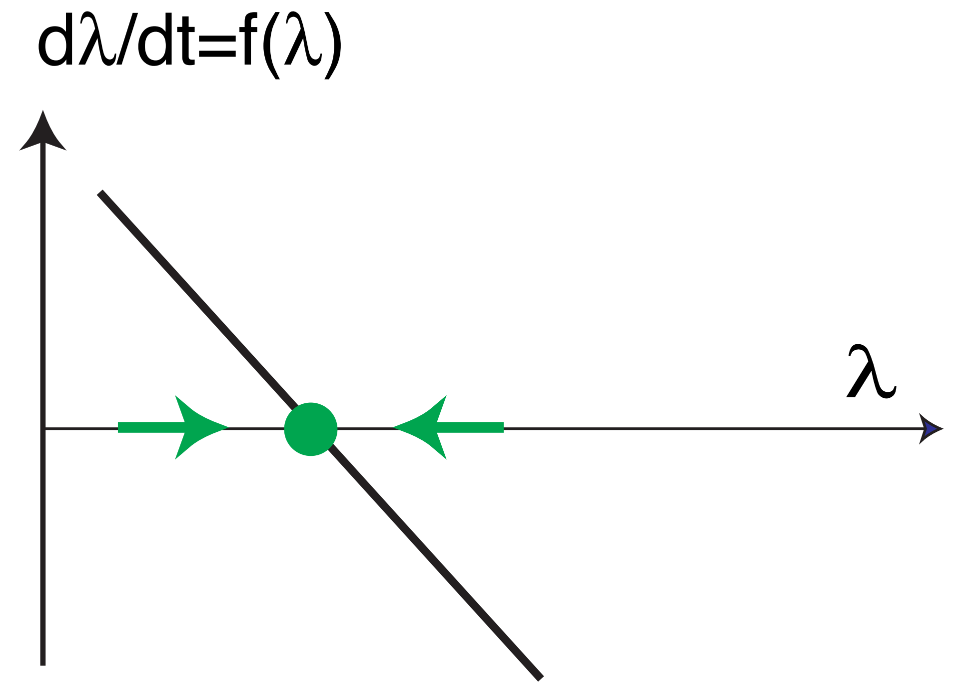
linear approximation near attractor

- non-linearity as a small perturbation/ deformation of linear system
- => non-essential non-linearity



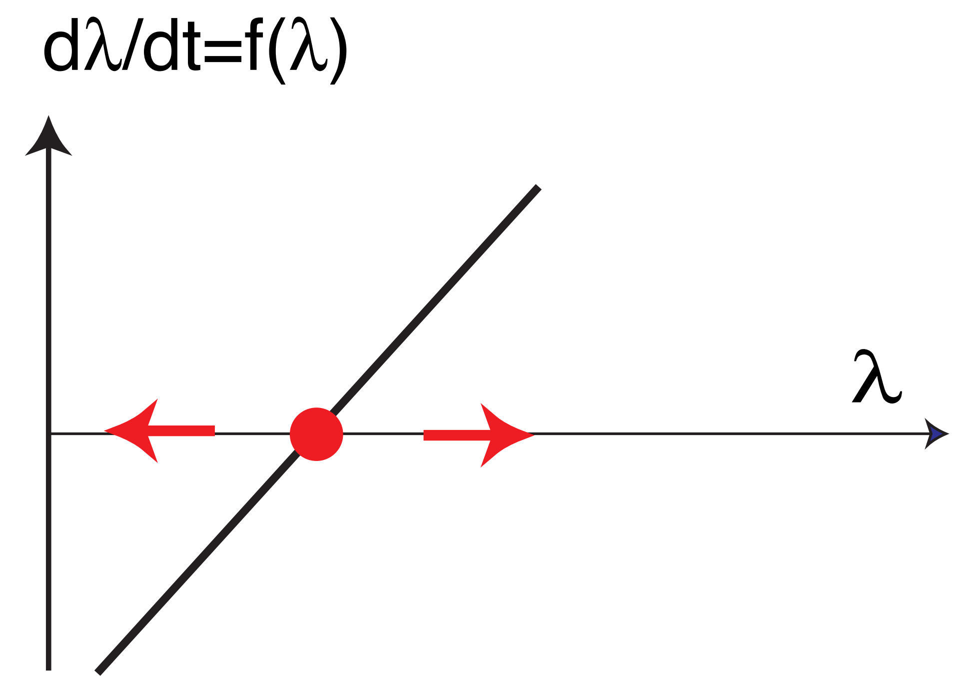
stability in a linear system

- if the slope of the linear system is negative, the fixed point is (asymptotically stable)



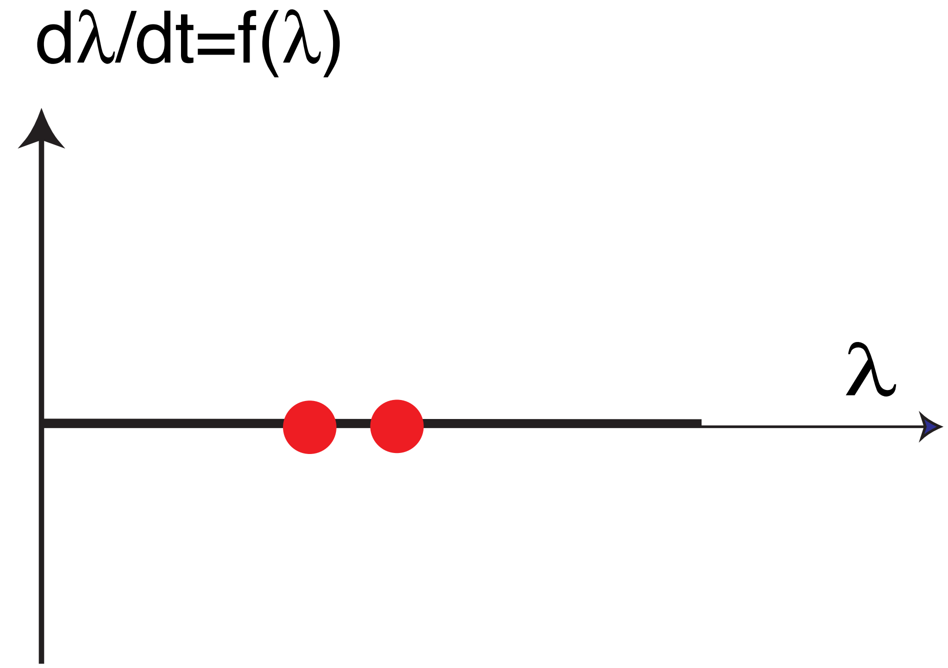
stability in a linear system

- if the slope of the linear system is positive, then the fixed point is unstable



stability in a linear system

- if the slope of the linear system is zero, then the system is indifferent (marginally stable: stable but not asymptotically stable)



stability in linear systems

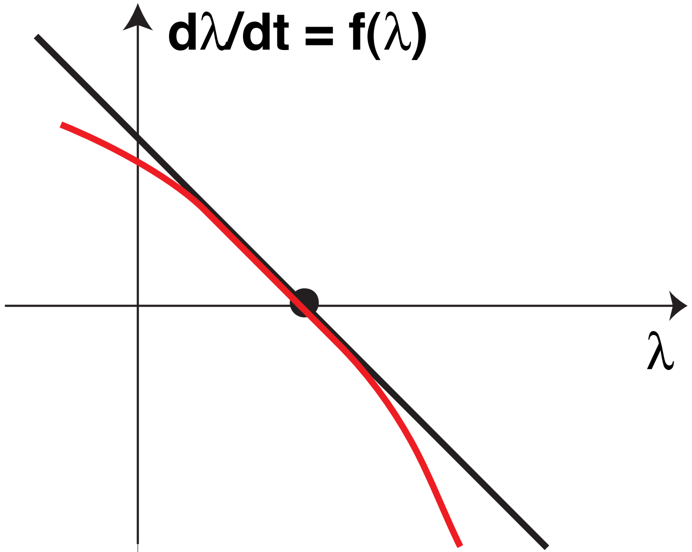
- generalization to multiple dimensions
  - if the real-parts of all Eigenvalues are negative: stable
  - if the real-part of any Eigenvalue is positive: unstable
  - if the real-part of any Eigenvalue is zero: marginally stable in that direction (stability depends on other eigenvalues)

stability in nonlinear systems

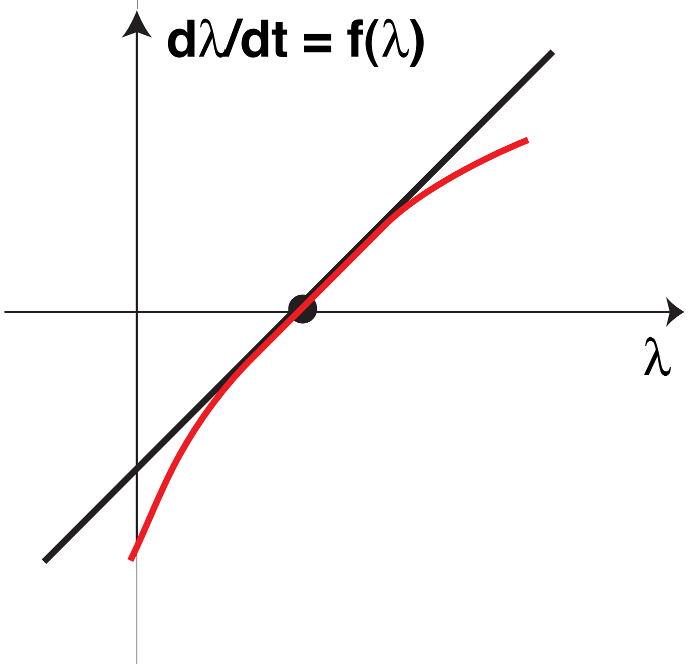
- stability is a local property of the fixed point
- => linear stability theory
  - the eigenvalues of the linearization around the fixed point determine stability
  - all real-parts negative: stable
  - any real-part positive: unstable
  - any real-part zero: undecided: now nonlinearity decides (non-hyberpolic fixed point)

stability in nonlinear systems

- all real-parts negative: stable

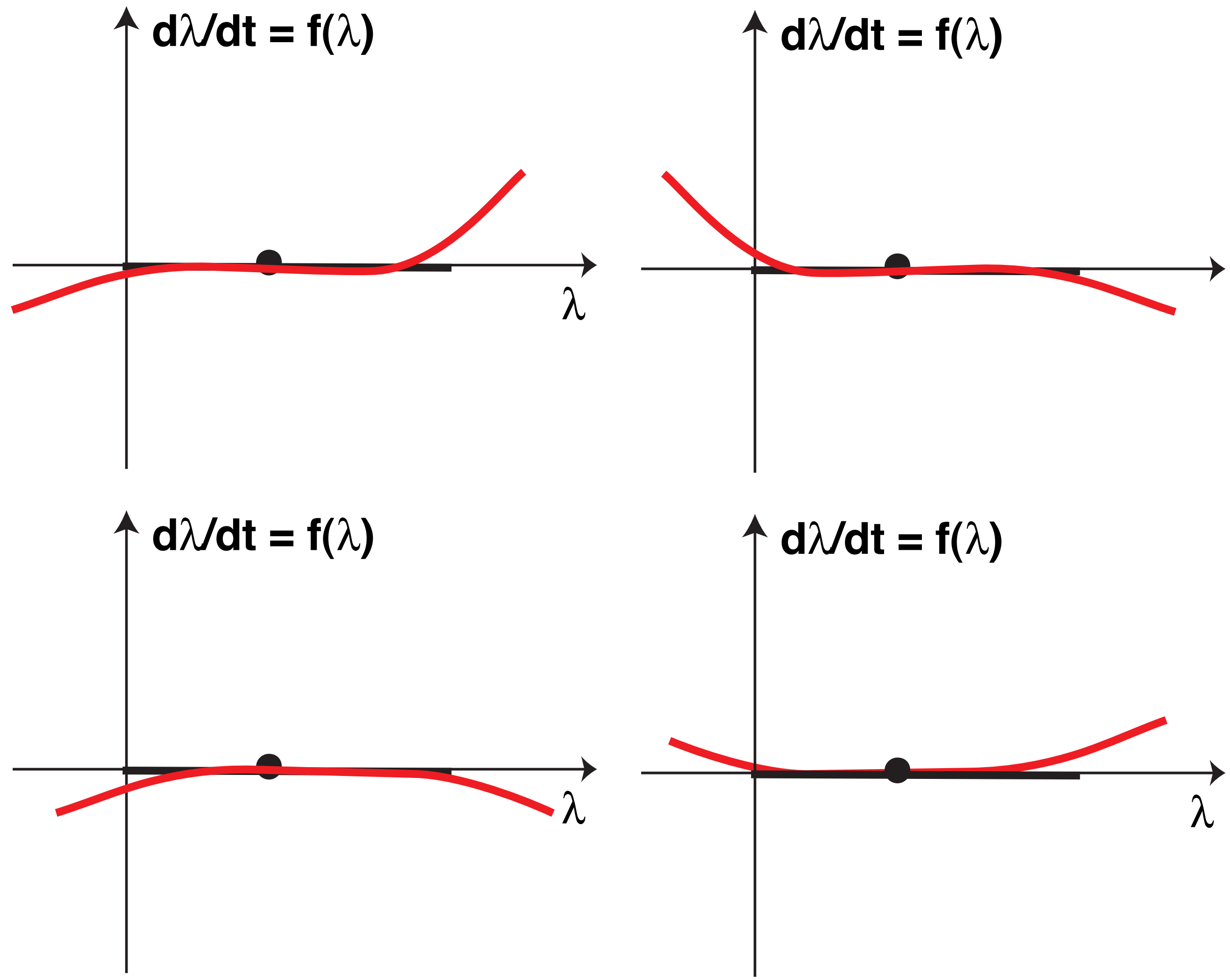


- any real-part positive: unstable



stability in nonlinear systems

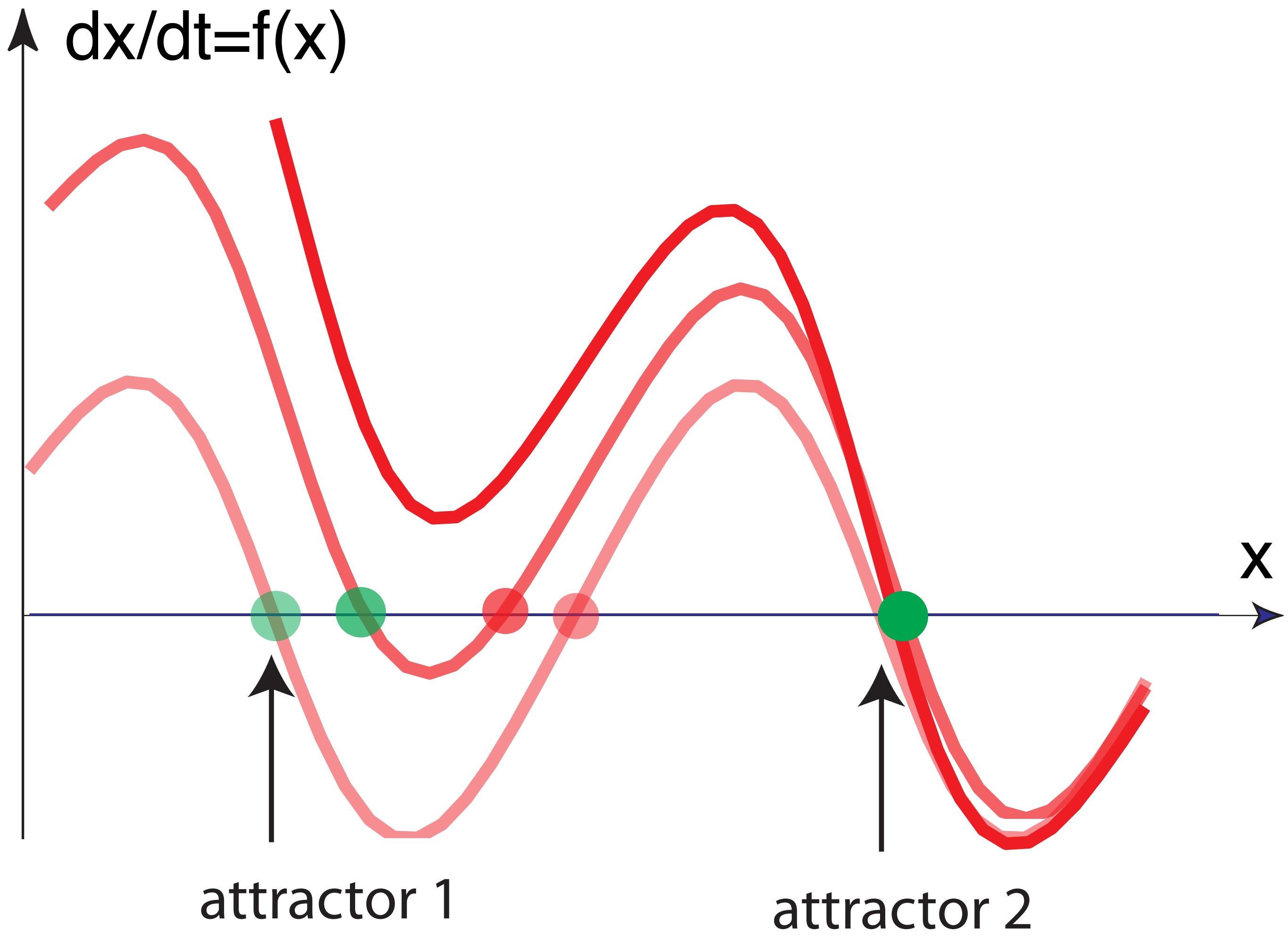
- any real-part zero: undecided: now nonlinearity decides (non-hyberpolic fixed point)



bifurcations

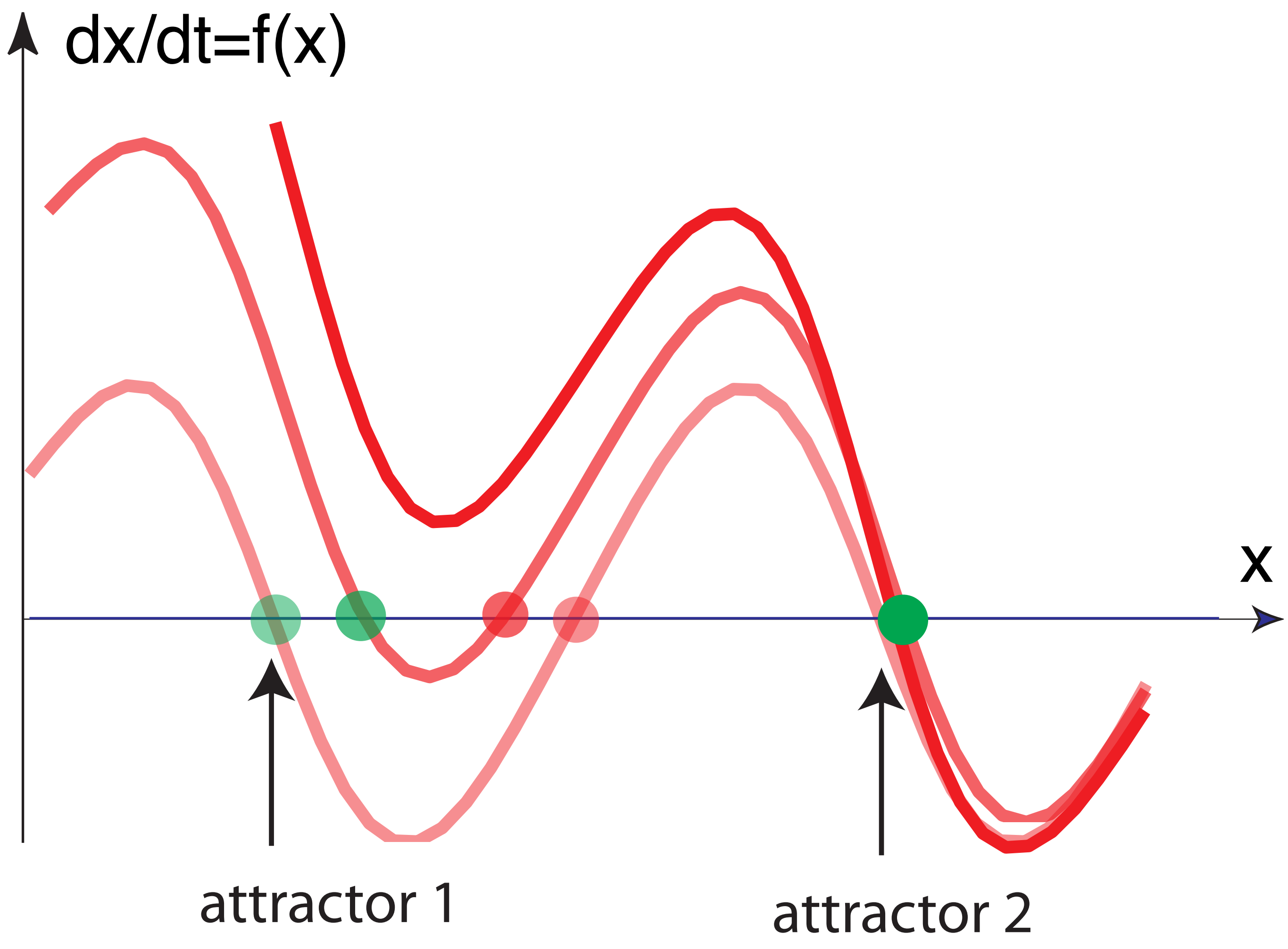
- look now at families of dynamical systems, which depend (smoothly) on parameters
- ask: as the parameters change (smoothly), how do the solutions change (smoothly?)
  - smoothly: topological equivalence of the dynamical systems at neighboring parameter values
  - bifurcation: dynamical systems NOT topological equivalent as parameter changes infinitesimally

bifurcation



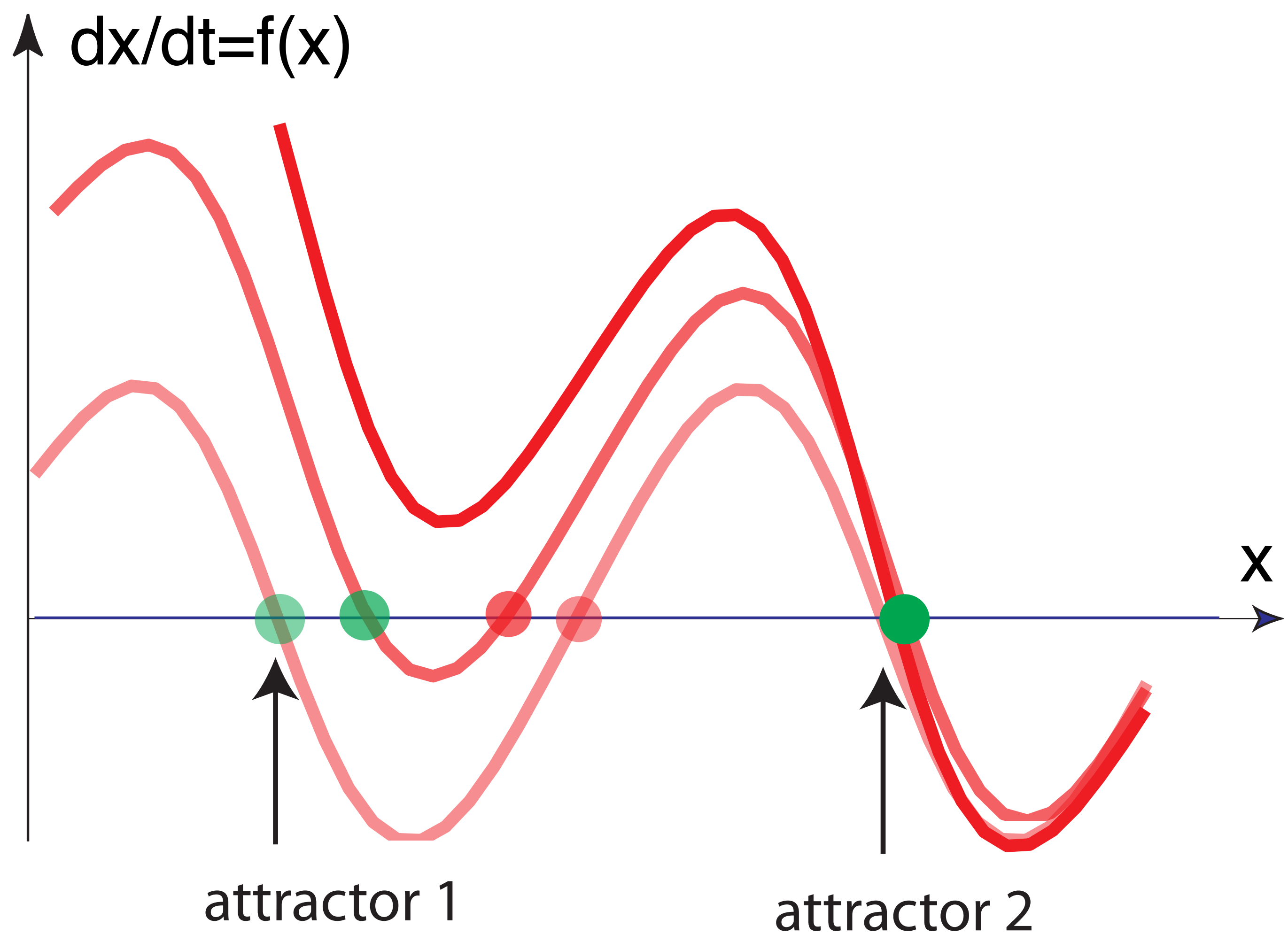
bifurcation

- bifurcation=qualitative change of dynamics (change in number, nature, or stability of fixed points) as the dynamics changes smoothly

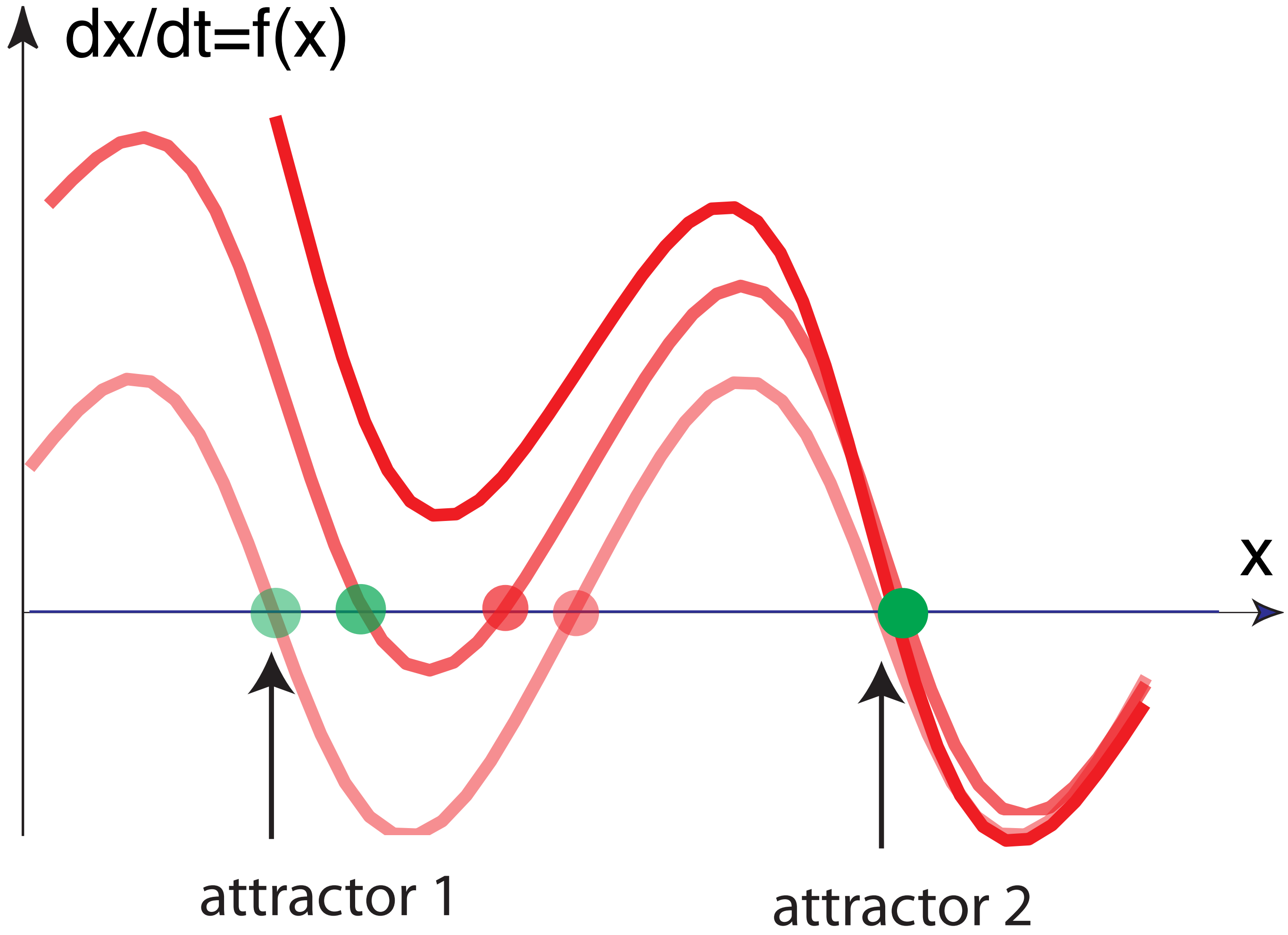


tangent bifurcation

- the simplest bifurcation (co-dimension 0): an attractor collides with a repellor and the two annihilate

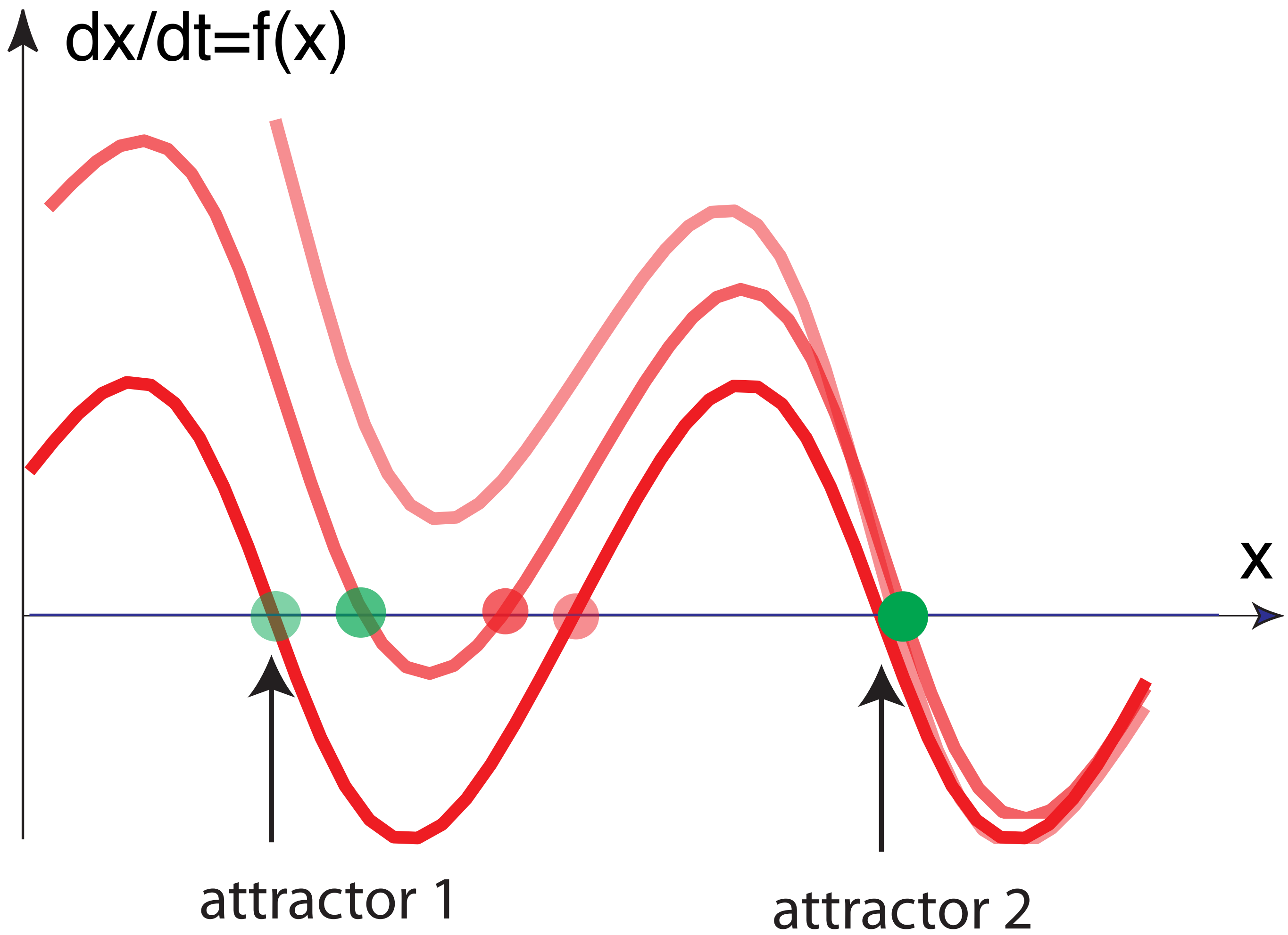


local bifurcation



reverse bifurcation

- changing the dynamics in the opposite direction



bifurcations are instabilities

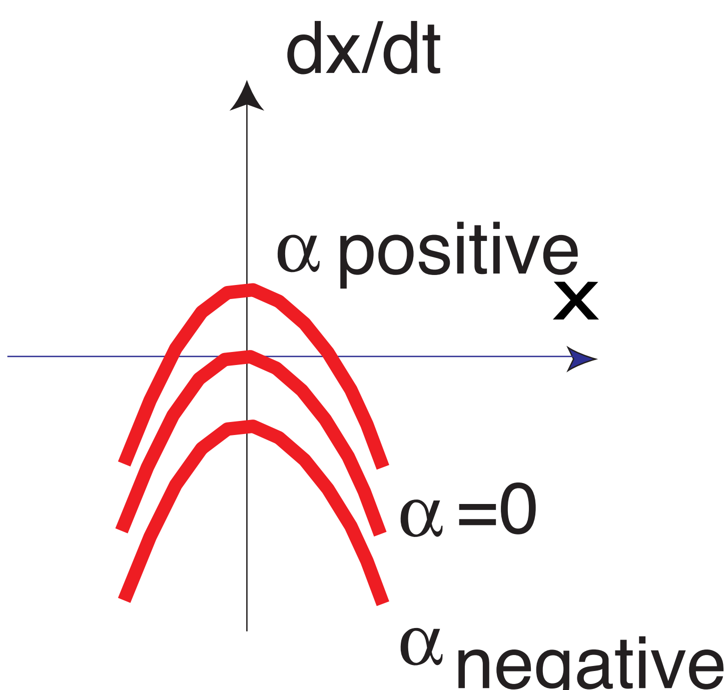
- that is, an attractor becomes unstable before disappearing
- (or the attractor appears with reduced stability)
- formally: a zero-real part is a necessary condition for a bifurcation to occur

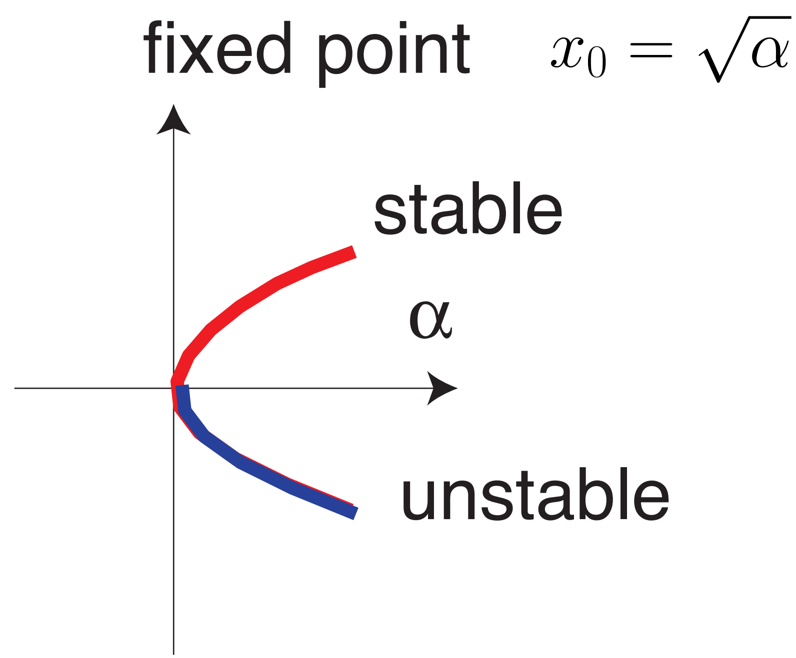
tangent bifurcation

- normal form of tangent bifurcation

$$\dot{x} = \alpha - x^2$$

- (=simplest polynomial equation whose flow is topologically equivalent to the bifurcation)



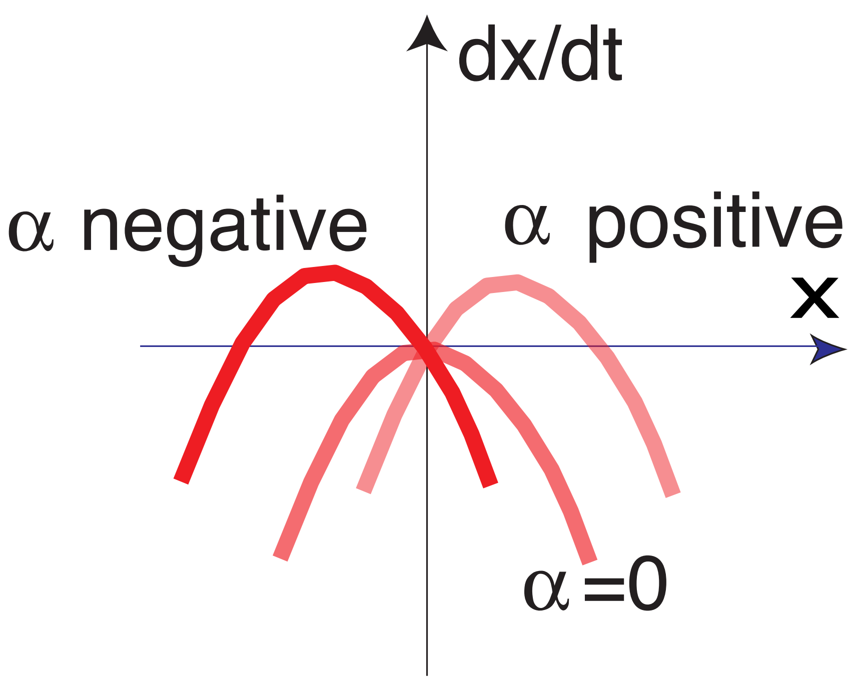


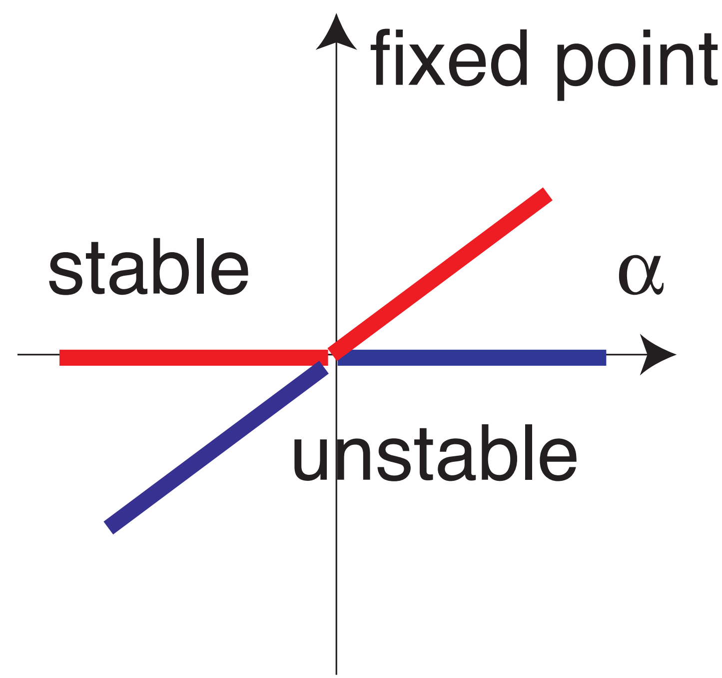
Hopf theorem

- when a single (or pair of complex conjugate) eigenvalue crosses the imaginary axis, one of four bifurcations occur
  - tangent bifurcation
  - transcritical bifurcation
  - pitchfork bifurcation
  - Hopf bifurcation

transcritical bifurcation

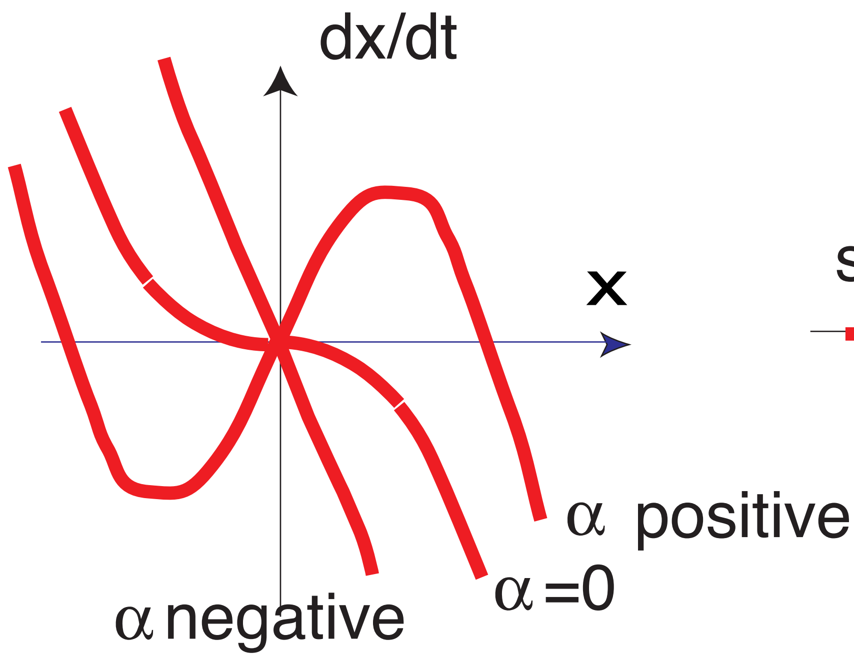
- normal form $\dot{x} = \alpha x - x^2$

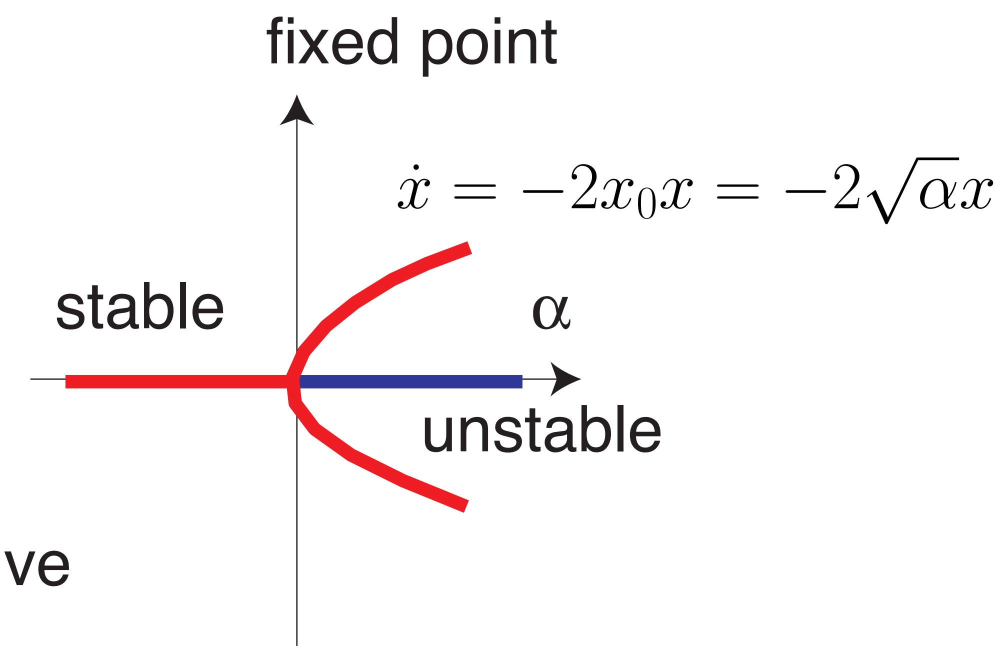




pitchfork bifurcation

- normal form $\dot{x} = \alpha x - x^3$

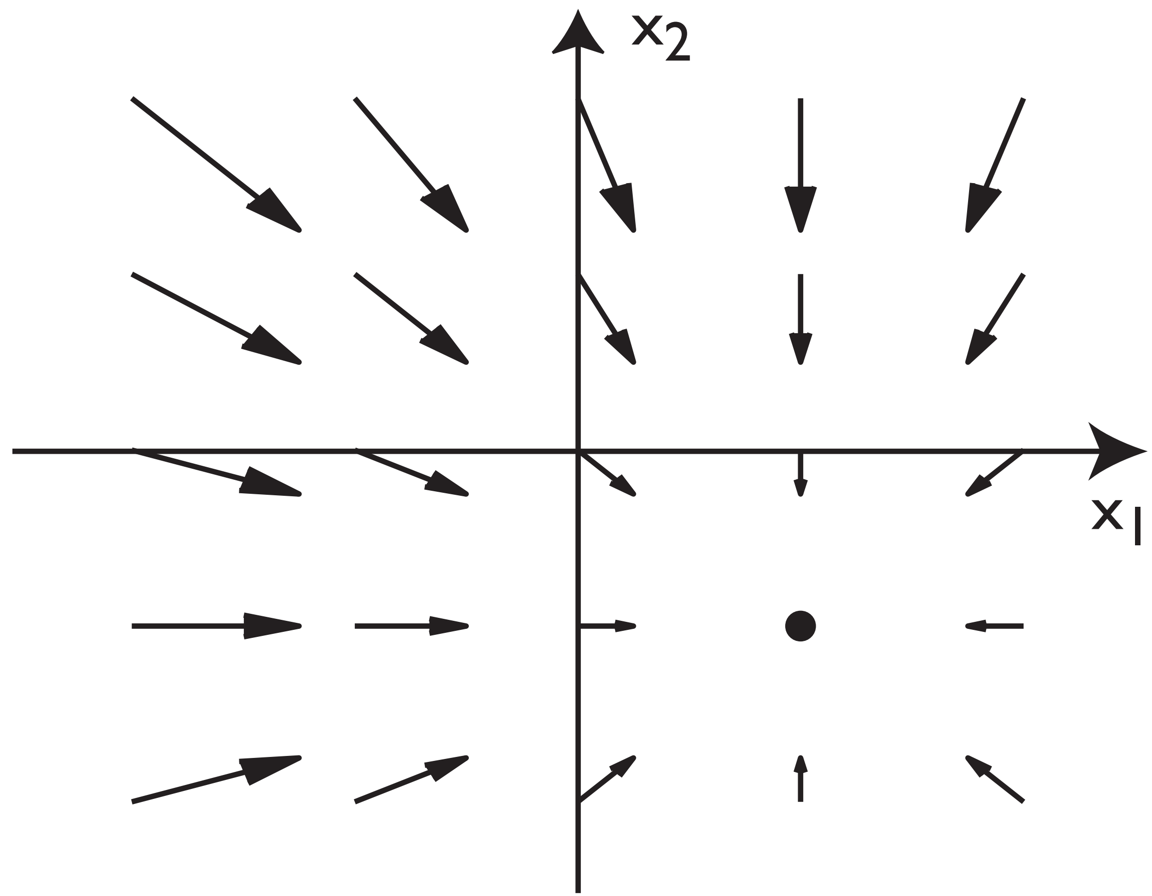




Hopf: need higher dimensions

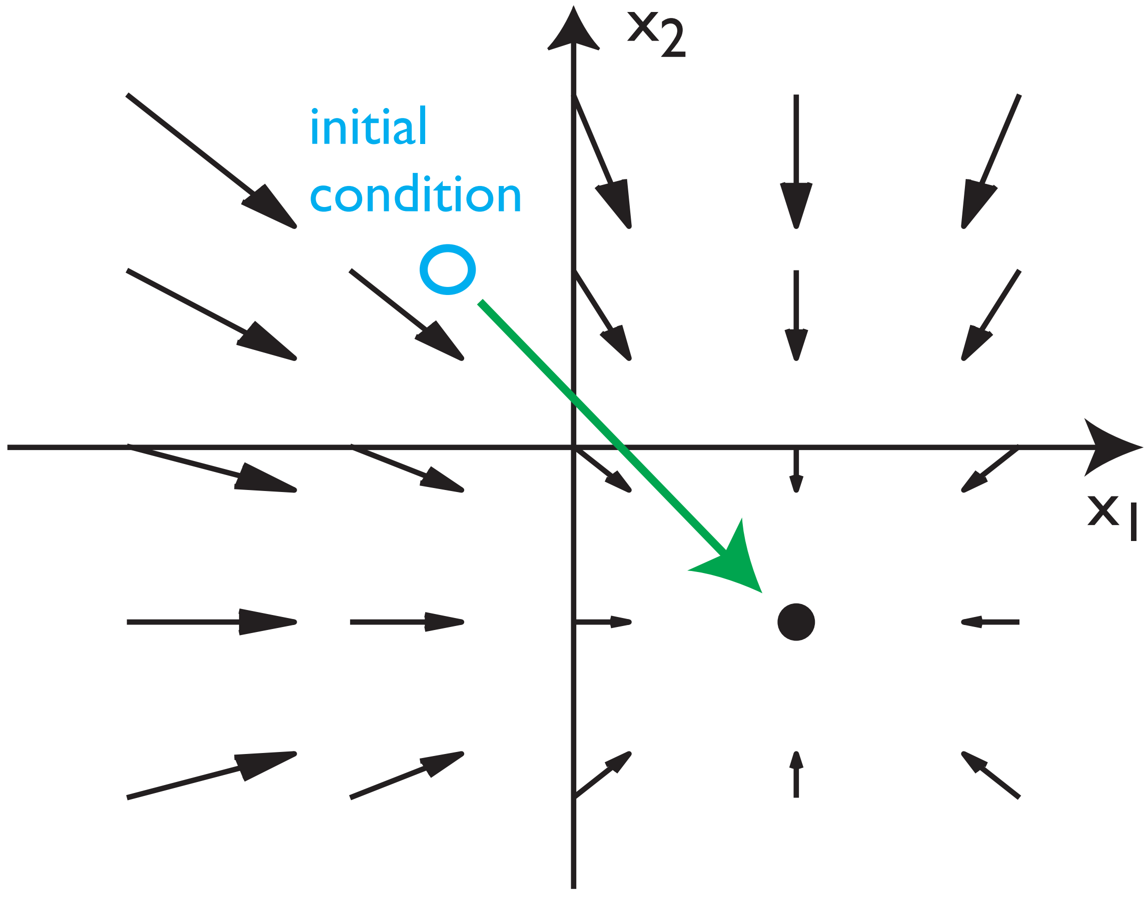
2D dynamical system: vector-field

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$



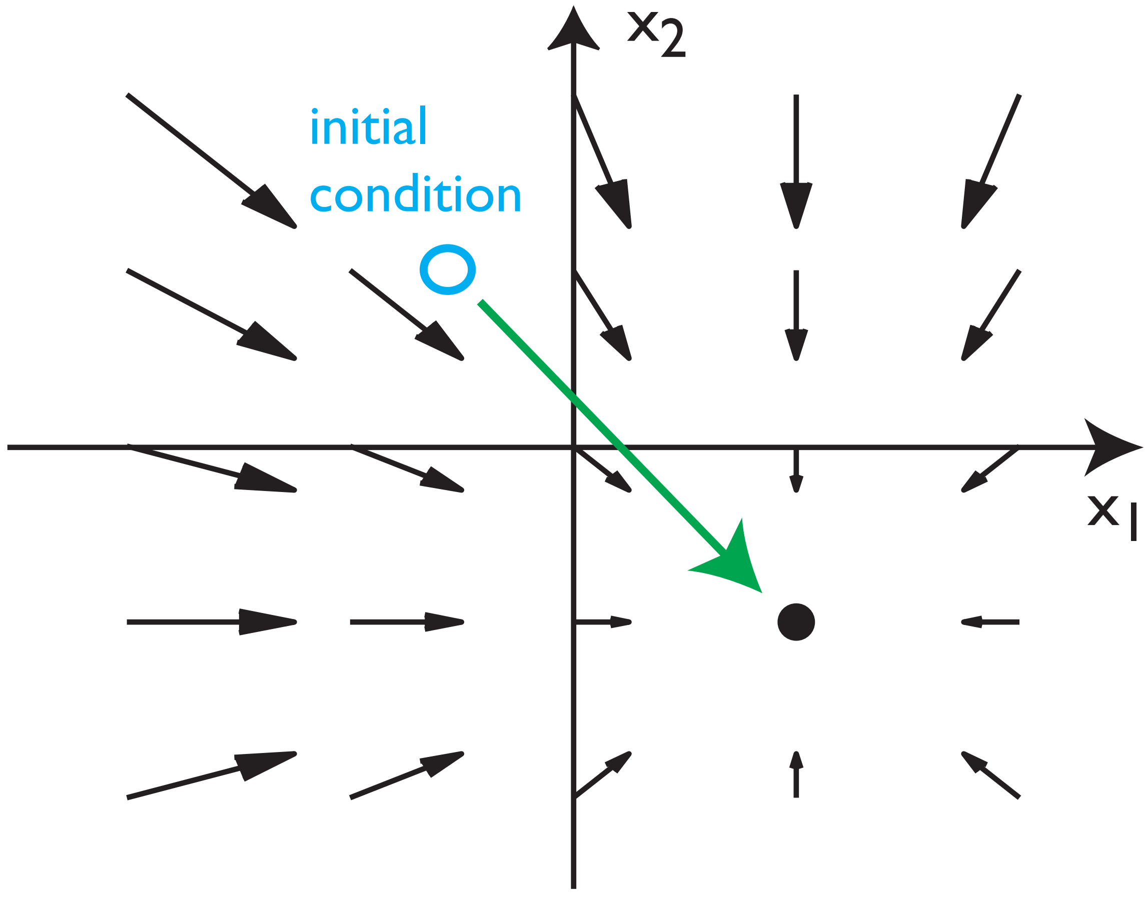
vector-field

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$



fixed point, stability, attractor

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

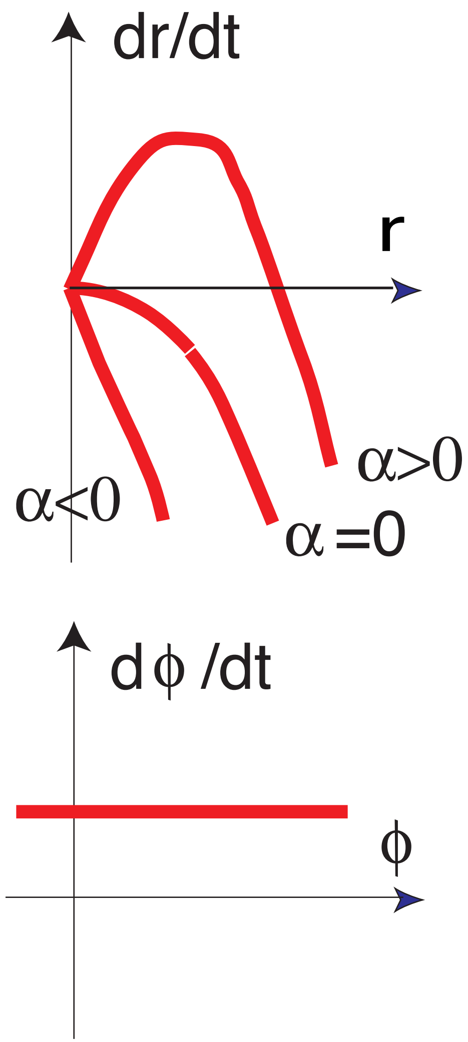


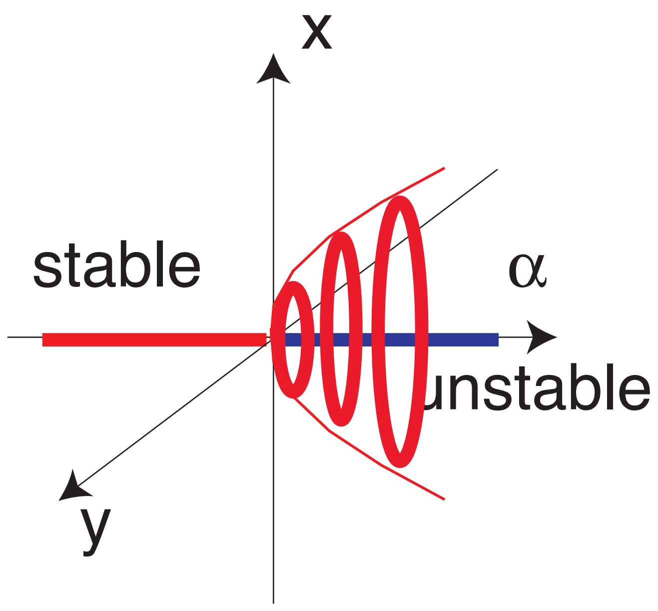
Hopf bifurcation

- normal form

$$\dot{r} = \alpha r - r^3$$

$$\dot{\phi} = \omega$$



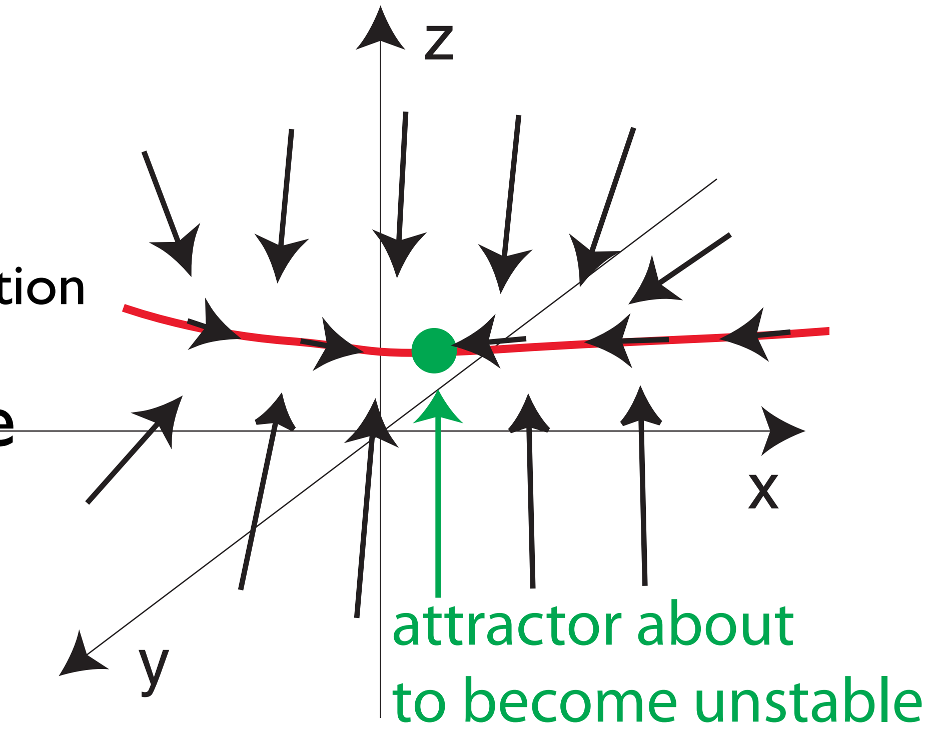


higher bifurcations

- e.g., degenerate (non-linear terms simultaneously zero with real-part of an eigenvalue)
- e.g. higher co-dimension ...

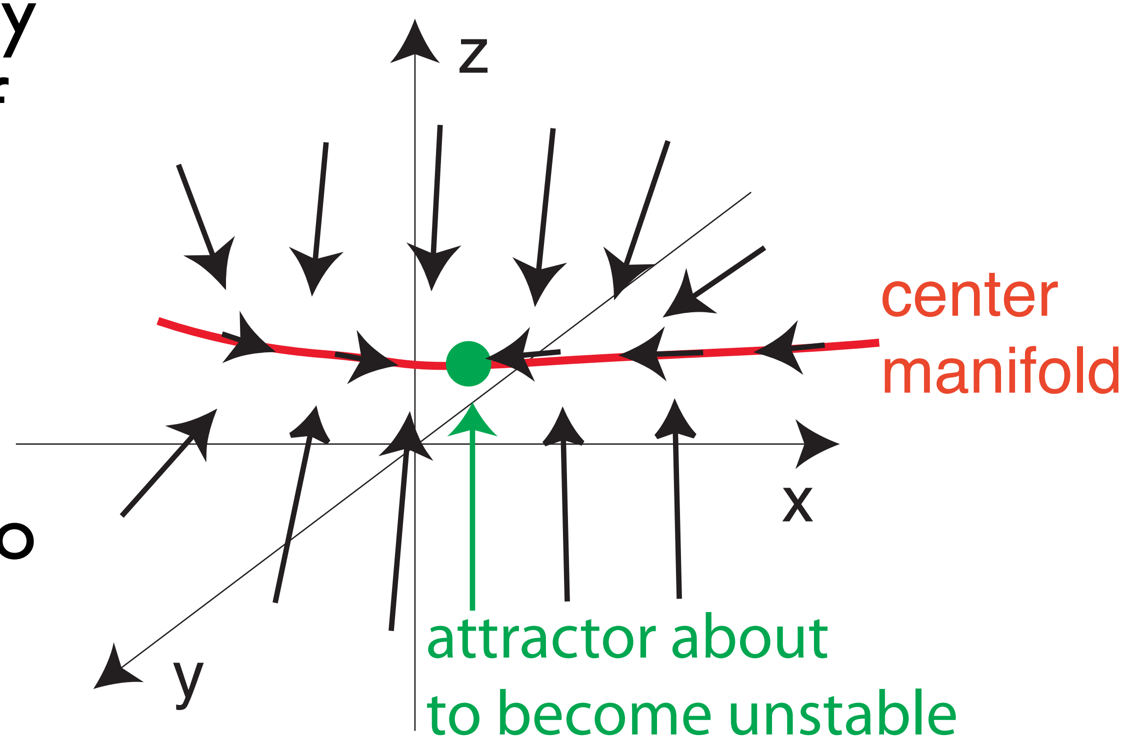
the center manifold

- at bifurcation: real part of one eigenvalue=0
  - corresponding eigenvector: critical direction
- near bifurcation: real part of one
  - eigenvalue close to zero
- => Center manifold theorem



center manifold

- = manifold which is locally tangent to eigenvector of zero real part eigenvalue
- CM theorem: dynamics within the CM is topologically equivalent to original dynamics
- => enormous dimensional reduction



center manifold

- => the 4 elementary bifurcations are “fair” representatives of any dynamical system with the same qualitative flow

forward dynamics

- given known equation, determined fixed points / limit cycles and their stability
  - more generally: determine invariant solutions (stable, unstable and center manifolds)
  - use CMF to simplify dynamics

inverse dynamics

- given classification of solutions (stable states) and their dependence on parameters, determine the dynamical system! That's the modeler's job
- practical approach
  - identify the class of dynamical systems using the 4 elementary bifurcations
  - and use normal form to provide an exemplary representative of the equivalence class of dynamics