Image Representation by Complex Cell Responses

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We present an analysis of the representation of images as the magnitudes of their transform with complex-valued Gabor wavelets. Such a representation is a model for complex cells in the early stage of visual processing and of high technical usefulness for image understanding, because it makes the representation insensitive to small local shifts. We show that if the images are band limited and of zero mean, then reconstruction from the magnitudes is unique up to the sign for almost all images.

1 Introduction

The first stage of processing of visual stimuli in the cortex is constituted by simple and complex cells in V1. Simple cell responses are modeled to considerable accuracy by linear convolution with Gabor functions (Daugman, 1985; Jones & Palmer, 1987). Pairs of cells differing in phase by 90 degrees are frequently found (Pollen & Ronner, 1981). Complex cells differ from simple cells by showing less specificity concerning the position of the stimulus. Their responses are well modeled by the magnitudes of the Gabor filter responses (short Gabor magnitudes) (Pollen & Ronner, 1983). Although the properties of these cells are more complicated, especially concerning temporal behavior, this functional description remains a good approximation to the cell responses.

Besides the biological modeling, convolution with Gabor functions and magnitude building are very useful for technical image processing purposes. The complex cells can be combined to more complicated feature detectors such as corner detectors (Würtz & Lourens, 2000). They have also

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proven useful for higher image understanding tasks such as texture classification (Fogel & Sagi, 1989; Portilla & Simoncelli, 2000), recognition of faces (Lades et al., 1993; Würtz, 1997; Duc, Fischer, & Bigün, 1999; Wiskott, Fellous, Krüger, & von der Malsburg, 1997), vehicles (Wu & Bhanu, 1997), hand gestures (Triesch & von der Malsburg, 1996), and analysis of electron microscopic sections of brain tissue (König, Kayser, Bonin, & Würtz, 2001). Regarding texture classification, in Portilla and Simoncelli (2000), the successful steerable filters have been enhanced to matched filters at the price of redundancy, because the magnitudes capture important properties of natural textures so well.

The deeper reason for this is that the magnitude operation introduces robustness under local shifts in the sense that in the presence of small displacements, the images of the Gabor magnitudes are more robust than the full complex-valued responses because they are much smoother. This robustness is crucial for the registration part of recognition systems, which have to cope with small local deformations. As a practical consequence, similarity landscapes between local features are smoother if magnitudes are used, which makes matching faster and less prone to local maxima (Lades et al., 1993; Würtz, 1997; Wiskott et al., 1997).

If the Gabor functions are arranged into a wavelet transform and the sampling is dense enough (see section 2), then the original image can be recovered from the transform values with arbitrary quality (except for the DC value). Given the useful properties of the magnitudes of the Gabor transform, an important theoretical question is how much image information can be recovered from that. It has been observed experimentally that a recognizable image can be recovered from the Gabor magnitudes (von der Malsburg & Shams, 2002).

In this letter, we present a proof that, given appropriate transform parameters and band limitation, no image information is lost beside the DC value of the image and a global sign. The proof uses techniques from Hayes (1982) and applies to all images except a possible subset of measure zero. The extension to localized Fourier transforms such as Gabor wavelets has not been shown before. Theorems 4 and 5 have been mentioned without details of the proof in Wundrich, von der Malsburg, and Würtz (2002), where we presented an algorithm for image reconstruction from the Gabor amplitudes.

The fact that images can be reconstructed from their Fourier amplitudes or phases alone is not widely known within the computer vision community. There is an extensive discussion on the relative importance of phase and amplitude information, which is reviewed briefly at the beginning of section 4.

The practical importance of reconstruction from Gabor amplitudes seems quite limited. For the visual system, it is clearly not a problem because the simple cell information is readily available. The importance of our result lies in the theoretical analysis of the effects of the nonlinearity introduced by the magnitudes. We demonstrate that, on the one hand, the resulting representation shows invariance under small shifts and, on the other hand, not more information is lost than by subsampling by a factor of two, the global sign and the DC value.

2 Gabor Wavelets

For the analysis of image properties at various scales, the wavelet transform is in wide use. The image is projected onto a family of wavelet functions, which are derived from a single mother wavelet by translation $\vec{x}_0$, rotation with a matrix $Q(\vartheta)$, and scaling by a of the image plane:

$$\mathcal{I}(\vec{x}_0, a, \vartheta) = \int_{\mathbb{R}^2} d^2x\, I(\vec{x}) \frac{1}{a}\, \psi^* \left[\frac{1}{a} \mathrm{Q}(\vartheta)(\vec{x} - \vec{x}_0) \right]. \tag{2.1}$$

The mother wavelet (and, consequently, all wavelets) must satisfy the admissibility condition (Kaiser, 1994), which means that their DC value is zero ($\hat{\psi}(\vec{0}) = 0$).

For modeling biological properties as well as to fulfill admissibility, standard Gabor functions are modified by a term that removes their DC value, turning the real and imaginary part into a strict matched filter pair. Following Murenzi (1989), we let:

$$\psi(\vec{x}) = \frac{1}{\sigma\tau} \exp\left(-\frac{1}{2}\|\mathrm{S}_{\sigma,\tau}\vec{x}\|^2\right)\left(\exp(j\vec{x}^T\vec{e}_1) - \exp\left(-\frac{\sigma^2}{2}\right)\right) \tag{2.2}$$

$$\hat{\psi}(\vec{\omega}) = \exp\left(-\frac{1}{2}\|\mathrm{S}_{\sigma,\tau}^{-1}(\vec{\omega} - \vec{e}_1)\|^2\right) - \exp\left(-\frac{1}{2}(\|\mathrm{S}_{\sigma,\tau}^{-1}\vec{\omega}\|^2 + \sigma^2)\right). \tag{2.3}$$

In these equations, the diagonal matrix $\mathrm{S}_{\sigma,\tau} = Diag(1/\sigma, 1/\tau)$ controls the shape of the elliptical gaussian relative to the wavelength.

We now switch from continuous functions to discretely sampled images of $N_1 \times N_2$ pixels. This lattice is denoted by $\mathbb{S}_{\vec{N}}$, the sampling interval for image and translation space by Δ. To avoid confusion, we use three different symbols for the different Fourier transforms: the continuous one (FT) is denoted by $\hat{I}(\vec{\omega})$, the 2D equivalent of Fourier series (DSFT) by $\check{I}(\vec{\nu})$, and the completely discretized and finite version (DFT) by $\breve{I}(\vec{\rho})$, defined on the same lattice $\mathbb{S}_{\vec{N}}$. For simplification, we also use normalized DFT coordinates $\vec{\tilde{\rho}} = [\rho_1/N_1 \quad \rho_2/N_2]^T$.

The final discretization of our wavelet families in both spatial and frequency domain takes the form

$$\psi_{\vec{n}_0,m,l}(\vec{n}) = a_{\min}^{-1} a_0^{-m} \psi \left[a_{\min}^{-1} a_0^{-m} \mathrm{Q}\left(\frac{2\pi l}{L}\right) \Delta(\vec{n} - \vec{n}_0) \right], \tag{2.4}$$

$$\breve{\psi}_{\vec{n}_0,m,l}(\vec{\rho}) = \frac{2\pi a_{\min} a_0^m}{\sqrt{N_1 N_2}\Delta^2} \hat{\psi} \left[a_{\min} a_0^m \mathrm{Q}\left(\frac{2\pi l}{L}\right) \frac{2\pi}{\Delta} \vec{\tilde{\rho}} \right] \exp(-j2\pi \vec{n}_0^T \vec{\tilde{\rho}}). \tag{2.5}$$

Now, the discrete Gabor wavelet transform can be computed in either domain by the inner product,

$$\mathcal{I}(\vec{n}_0, m, l) = \sum_{\vec{n}\in\mathbb{S}_{\vec{N}}} I(\vec{n})\, \psi^*_{\vec{n}_0,m,l}(\vec{n}) = \sum_{\vec{\rho}\in\mathbb{S}_{\vec{N}}} \breve{I}(\vec{\rho})\, \breve{\psi}^*_{\vec{n}_0,m,l}(\vec{\rho}), \tag{2.6}$$

and the inverse transform becomes

$$\breve{I}(\vec{\rho}) = \frac{\Delta^4}{4\pi^2} Y_0(\vec{\rho})^{-1} \sum_{\vec{n}_0\in\mathbb{S}_{\vec{N}}} \sum_{m=0}^{M-1} \sum_{l=0}^{L-1} a_{\min}^{-2} a_0^{-2m} \mathcal{I}(\vec{n}_0, m, l) \breve{\psi}_{\vec{n}_0,m,l}(\vec{\rho}), \tag{2.7}$$

with

$$Y_0(\vec{\rho}) = \sum_{m=0}^{M-1} \sum_{l=0}^{L-1} \left| \hat{\psi} \left[a_{\min} a_0^m \mathrm{Q}\left(\frac{2\pi l}{L}\right) \frac{2\pi}{\Delta} \tilde{\vec{\rho}} \right] \right|^2, \tag{2.8}$$

where $Y_0(\vec{\rho})$ is regularized for inversion by assigning $Y_0^{-1}(\vec{\rho}) = 0$ where $Y_0(\vec{\rho})$ drops below an appropriate threshold. The transform constitutes a frame for a given image class $\mathcal{M}$ if $\forall I \in \mathcal{M}$: $(Y_0(\vec{\rho}) = 0 \Longrightarrow \breve{I}(\vec{\rho}) = 0)$. For full details of the Gabor wavelet framework, see Lee, 1996.

3 Shift Insensitivity of Gabor Magnitudes

We now argue that the magnitude of the Gabor wavelet transform is less sensitive to shift than the transform itself. Generally, the amount of shift that can be tolerated by a representation is directly related to the frequency content of the underlying signal. For signals without band limitation (like Gabor-filtered images), the frequency content can be measured by the normalized frequency moments introduced by Gabor (1946), who called them mean frequencies. The first one,

$$\vec{F}_1(f) = \frac{\int d^2\omega\, \vec{\omega} |\hat{f}(\vec{\omega})|^2}{\|f\|^2}, \tag{3.1}$$

can be interpreted as the center of signal energy in the frequency space. For the Gabor function from equation 2.3 it is easily checked that $\vec{F}_1(\psi) \approx \vec{e}_1$, that is, any Gabor function is centered at the frequency that appears in the exponent of the first gaussian in equation 2.3 (due to the admissibility correction, this holds only approximately). For nonpathological images, $\vec{F}_1(I * \psi) \approx \vec{e}_1$ will also hold, and for any real nonzero image, $\vec{F}_1(I * \psi)$ must be different from $\vec{0}$. On the other hand, the Fourier transform of $|I * \psi|^2$ is the autocorrelation of $\hat{I} \cdot \hat{\psi}$. Any autocorrelation is symmetric around the origin, and therefore $\vec{F}_1(|I * \psi|^2) = \vec{0}$. This argument shows that the magnitudes consist of lower frequencies and consequently show slower variation than the Gabor bands themselves. It can be refined by looking at the second moment—the variance in frequency domain as well.

4 Reconstruction from Fourier Magnitudes

Due to their translation invariance, Fourier magnitudes have been used for pattern recognition (Gardenier, McCallum, & Bates, 1986). The fact that the inverse DFT applied to a modified transform with all magnitudes set to 1 and original phases preserves essential image properties (Oppenheim & Lim, 1981) is frequently interpreted as saying that the Fourier magnitudes contain "less" image information than the phases. However, analytical results and existing phase retrieval algorithms provide hints that the situation is not as simple. Tadmor and Tolhurst (1993) show that the assumption of the global Fourier amplitudes being irrelevant for the image contents is too simple, although the amplitude spectrum averaged over orientations does not vary too much for natural images. As a consequence, the distribution of image energy across orientations is an important distinguishing image property, and strong orientations can be conveyed by the amplitude spectrum alone. Lohmann, Mendlovic, and Shabtay (1997) show that the relative importance of Fourier phase and amplitude can be reversed by modification of a single pixel. These examples are rejected by Millane and Hsiao (2003), because of remaining correlations of the final phases with the original phases. So far, the discussion is not concluded, because generally accepted notions of relative importance and of images are lacking.

The possibility of reconstructing recognizable images from phase or amplitude information alone is not a contradiction to the above results. It only shows that it is very hard to trace the image information in the presence of the simplest of nonlinearities.

In the following we review two articles (Hayes, 1982; Hayes & McClellan, 1982) that show that almost all images can be reconstructed from their Fourier magnitudes. The argument identifies unique reconstructability with the reducibility of a polynomial in D variables, where D is the signal dimension.

4.1 Polynomials in One or More Dimensions.

The set $\mathcal{P}(n, D)$ of all polynomials with complex coefficients and total degree n in D variables is a vector space over $\mathbb{C}$ of dimension $\alpha(n, D)$. In the case $D = 1$, all polynomials are reducible according to the fundamental theorem of algebra, that is, they can be factored into polynomials of lower degree. For polynomials in two or more variables, the situation is different. The following is a modified version of a theorem from Hayes and McClellan (1982) and shows that in a certain sense, reducible polynomials are very uncommon in more than one variable.

Theorem 1. *The subset of polynomials in $\mathcal{P}(n, D)$ that are reducible over the complex numbers corresponds to a set of measure zero in $\mathbb{R}^{2\alpha(n,D)}$, provided $D > 1$ and $n > 1$.*

In the following sections, we will make full use of this result for 2D signal processing. The idea is that only images on a finite support are taken into account and that the many wrong phase functions lead to reconstructed images with nonvanishing values outside this support. Throughout this article, we mean by the *support* of a function of two variables the smallest rectangle with edges parallel to the coordinate axes, which contains all nonzero pixels. This should probably bear a different name, but we do not expect any confusion by this simple terminology.

4.2 The Hayes Theorem and Extensions. Hayes's theorem identifies the 2D z-transform,

$$\check{I}(\vec{z}) = \frac{1}{2\pi} \sum_{\vec{n} \in \mathbb{S}_{\vec{N}}} I(\vec{n}) z_1^{-n_1} z_2^{-n_2}, \tag{4.1}$$

and the 2D discrete space Fourier transform (DSFT) on a compact support, with polynomials in two variables, to which theorem 1 applies.

Theorem 2 (Hayes, 1982). *Let I_1, I_2 be 2D real sequences with support $\mathbb{S}_{\vec{N}}$ and let Ω a set of $|\Omega|$ distinct points in $[-\pi, \pi[^2$ arranged on a lattice $\mathcal{L}(\Omega)$ with $|\Omega| \geq (2N_1 - 1)(2N_2 - 1)$. If $\check{I}_1(\vec{z})$ has at most one irreducible nonsymmetric factor and*

$$|\check{I}_1(\vec{\nu})| = |\check{I}_2(\vec{\nu})| \; \forall \vec{\nu} \in \mathcal{L}(\Omega), \tag{4.2}$$

then

$$I_1(\vec{n}) \in \{I_2(\vec{n}),\; I_2(\vec{N} - \vec{n} - \vec{1}), \\ - I_2(\vec{n}),\; -I_2(\vec{N} - \vec{n} - \vec{1})\}. \tag{4.3}$$

Theorem 2 states that DSFT magnitudes-only reconstruction yields either the original, or a negated, point reflected, or a negated point reflected version of the input signal. Together with the statement of theorem 1 that the set of all reducible polynomials $\check{I}(\vec{z})$ is of measure zero, the technicality about the irreducible nonsymmetric factors can be omitted, and we generalize theorem 2 to complex-valued sequences as follows (the proof is in appendix A):

Theorem 3. *Let I_1, I_2 be complex sequences defined on the compact support $\mathbb{S}_{\vec{N}}$, and let $\check{I}_1(\vec{\nu})$ and $\check{I}_2(\vec{\nu})$ be only trivially reducible (i.e., have factors only of the form $z_1^{p_1} z_2^{p_2}$), and*

$$|\check{I}_1(\vec{\nu})| = |\check{I}_2(\vec{\nu})| \ \forall \vec{\nu} \in \mathcal{L}(\Omega), \tag{4.4}$$

with $\mathcal{L}(\Omega)$, $|\Omega|$ as in theorem 2. Then

$$I_1(\vec{n}) \in \left\{\exp(j\eta) I_2(\vec{n}),\ \exp(j\eta) I_2^*(\vec{N} - \vec{n} - \vec{1}) \mid \eta \in [0, 2\pi[\right\}. \tag{4.5}$$

5 Extension to Gabor Magnitudes

Theorem 3 is a theorem about arbitrary polynomials; therefore, it can be applied equally well to the magnitudes of a complex spatial image signal for the reconstruction of the discrete Fourier transform. Thus, the following is a consequence of theorem 3:

Theorem 4. *Let I_1, I_2 be complex sequences defined on the compact support $\mathbb{T}_{\vec{K}} = \{-(K_1 - 1)/2, \ldots, (K_1 - 1)/2\} \times \{-(K_2 - 1)/2, \ldots, (K_2 - 1)/2\}$ with K_1, K_2 odd numbers, $N_1 \geq 2K_1 - 1$, $N_2 \geq 2K_2 - 1$, and let $I_1(\vec{n})$ and $I_2(\vec{n})$ be only trivially reducible, and*

$$|I_1(\vec{n})| = |I_2(\vec{n})| \qquad \forall \vec{n} \in \mathbb{T}_{\vec{K}}. \tag{5.1}$$

Then

$$\check{I}_1(\vec{\rho}) \in \left\{\exp(j\eta)\check{I}_2(\vec{\rho}),\ \exp(j\eta)\check{I}_2^*(-\vec{\rho}) \mid \eta \in [0, 2\pi[\right\},$$

and consequently

$$I_1(\vec{n}) \in \left\{\exp(j\eta) I_2(\vec{n}),\ \exp(j\eta) I_2^*(\vec{n}) \mid \eta \in [0, 2\pi[\right\}. \tag{5.2}$$

To complete the argument, we now relate the reconstructability from the Gabor magnitudes to the reconstructability from the Gabor transform itself.

Theorem 5 (Gabor magnitude theorem). *Let $\mathcal{B}(N_1, N_2)$ be the space of all zero-mean band-limited functions on $\mathbb{S}_{\vec{N}}$ such that $\check{I}(\vec{\rho}) = 0$ for $|\rho_1| > \frac{N_1}{4}$, $|\rho_2| > \frac{N_2}{4}$, and $\check{I}(\vec{0}) = 0$, and let the wavelet family $\psi_{\vec{n}_0,m,l}(\vec{n})$ constitute a frame in $\mathcal{B}(N_1, N_2)$.*

For all $I_1, I_2 \in \mathcal{B}(N_1, N_2)$ such that $\mathcal{I}_1(\vec{n}_0, m, l) = \langle I_1, \psi_{\vec{n}_0,m,l}\rangle$ and $\mathcal{I}_2(\vec{n}_0, m, l) = \langle I_2, \psi_{\vec{n}_0,m,l}\rangle$ are only trivially reducible polynomials and

$$|\mathcal{I}_1(\vec{n}_0, m, l)| = |\mathcal{I}_2(\vec{n}_0, m, l)| \qquad \forall \vec{n}_0, m, l, \tag{5.3}$$

it follows that

$$I_1(\vec{n}) = \pm I_2(\vec{n}). \tag{5.4}$$

The proof of theorem 5 is presented in appendix C. Theorem 1 states that almost all images fulfill the irreducibility condition. It follows that images that fulfill the band limitation condition of theorem 5 and have zero mean can be reconstructed from their Gabor amplitudes up to the sign. The band limitation implies that the images are sampled at twice the necessary density in each dimension.

6 Discussion

We have shown that almost all images sampled at twice their critical rate can be recovered from their Gabor magnitudes, DC value, and sign. This means that images are still represented well on the complex cell level. The higher sampling rate looks plausible in neuronal terms, if one considers a single complex number to be represented by four positive real numbers (because cell activities cannot be negative). Four simple cells, which code for the linear wavelet coefficient, must be replaced by four complex cells at slightly different positions in order to convey the same information. Thus, the functional relevance of complex cells seems to be the local translational invariance, and no further information seems to be lost. There is, of course, a residual possibility for some natural images to fall into the subset of measure zero of those images not represented uniquely by their Gabor magnitudes.

We have treated only Gabor wavelets, because we are interested in their properties from the point of view of biological models as well as their technical applications. The reconstructability will also hold for other wavelet types. The conditions on the wavelet should be that their Fourier transform is real, localized in frequency space (to satisfy lemma 1; see appendix B), and without too many zeros in frequency space, so that common nonzero points between subbands can always be found.

Another important question that arose is the stability of the uniqueness result in the presence of noise on the amplitudes. We were not able to find a satisfactory theoretical answer to this. We have, however, developed a reconstruction algorithm that is based on the principles of the proof presented here and are in the process of tackling this issue numerically. A brief description of the algorithm can be found in Wundrich et al. (2002).

Appendix A: Proof of Theorem 3

From the equality of the squared moduli, it follows that

$$\check{I}_1(\vec{\nu})\check{I}_1^*(\vec{\nu}) = \check{I}_2(\vec{\nu})\check{I}_2^*(\vec{\nu}). \tag{A.1}$$

The factorization of $\check{I}_1$ and $\check{I}_2$ is

$$\begin{aligned}\check{I}_1(\vec{\nu}) &= |\alpha| \exp(j\beta) \exp(-j\nu_1 p_1) \exp(-j\nu_2 p_2)\check{I}_{10}(\vec{\nu}),\\ \check{I}_2(\vec{\nu}) &= |\gamma| \exp(j\zeta) \exp(-j\nu_1 q_1) \exp(-j\nu_2 q_2)\check{I}_{20}(\vec{\nu}),\end{aligned} \tag{A.2}$$

where $\check{I}_{10}$, $\check{I}_{20}$ are irreducible and normalized. Substituting equation A.2 in A.1, and simplifying yields

$$|\alpha|^2 \check{I}_{10}(\vec{v})\check{I}^*_{10}(\vec{v}) = |\gamma|^2 \check{I}_{20}(\vec{v})\check{I}^*_{20}(\vec{v}). \tag{A.3}$$

Since the polynomials on both sides of equation A.3 are normalized, it follows that $|\gamma| = |\alpha|$. Because of the irreducibility, one polynomial on the left has to be equal to one on the right of equation A.3. This results in the following two cases:

1. $\check{I}_{20}(\vec{v}) = \check{I}_{10}(\vec{v})$
$$\check{I}_2(\vec{v}) = \check{I}_1(\vec{v}) \exp(j(\zeta - \beta)) \exp(-jv_1(q_1 - p_1)) \exp(-jv_2(q_2 - p_2))$$
$$I_2(\vec{n}) = \exp(j(\zeta - \beta)) I_1(n_1 - q_1 + p_1, n_2 - q_2 + p_2), \tag{A.4}$$

2. $\check{I}_{20}(\vec{v}) = \check{I}^*_{10}(\vec{v})$
$$\check{I}_2(\vec{v}) = \check{I}^*_1(\vec{v}) \exp(j(\zeta + \beta)) \exp(-jv_1(q_1 + p_1)) \exp(-jv_2(q_2 + p_2))$$
$$I_2(\vec{n}) = \exp(j(\zeta + \beta)) I^*_1(-n_1 + q_1 + p_1, -n_2 + q_2 + p_2). \tag{A.5}$$

That these two possibilities are compatible with each other can be checked by simple calculations. From the fact that I_1 and I_2 have the same support, the shifts can be determined, and after substitution of $\eta = \zeta \pm \beta$, we end up with the two cases

1. $$I_2(\vec{n}) = \exp(j\eta) I_1(\vec{n}), \tag{A.6}$$
2. $$I_2(\vec{n}) = \exp(j\eta) I^*_1(\vec{N} - \vec{n} - \vec{1}), \tag{A.7}$$

which concludes the proof.

Appendix B: Lemma About Gabor Function

The following lemma is a technicality required for the proof of theorem 5 and allows distinguishing a subband from its point-inflected version.

Lemma 1.

$$\vec{\omega}^T \vec{e}(\vartheta) > 0 \;\Rightarrow\; |\hat{\psi}_{\vec{x}_0,a,\vartheta}(\vec{\omega})|^2 > |\hat{\psi}_{\vec{x}_0,a,\vartheta}(-\vec{\omega})|^2. \tag{B.1}$$

Proof. Because of the wavelet property, it suffices to prove the proposition for the mother wavelet: $\vec{x}_0 = \vec{0}, \theta = 0$ and $a = 1$. Then the $|\cdot|$ can be removed because the functions are real in frequency space. $\vec{\omega}^T \vec{e}(\vartheta)$ reduces to ω_1. After substituting equation 2.3, simplification, and removal of common positive factors, it remains to show that

$$\exp(2\sigma^2\omega_1) - 2\exp(\sigma^2\omega_1) > \exp(-2\sigma^2\omega_1) - 2\exp(-\sigma^2\omega_1), \tag{B.2}$$

which is equivalent to

$$4\sinh(\sigma^2\omega_1)\exp(-\sigma^2\omega_1) > 0 \tag{B.3}$$

by elementary manipulations. Equation B.3 holds for all $\omega_1 > 0$ and all $\sigma \neq 0$.

Appendix C: Proof of Theorem 5

Due to Plancherel's theorem, $\mathcal{I}(\vec{n}_0, m, l)$ is a polynomial:

$$\mathcal{I}_1(\vec{n}_0, m, l) = \sum_{\vec{\rho}\in\mathbb{S}_{\vec{N}}} \breve{I}_1(\vec{\rho})\frac{2\pi a_0^m}{\sqrt{N_1N_2}\Delta^2}\hat{\psi}\left[a_{\min}a_0^m \mathrm{Q}\left(\frac{2\pi l}{L}\right)\frac{2\pi}{\Delta}\vec{\tilde{\rho}}\right]\exp(j2\pi\vec{n}_0^T\vec{\tilde{\rho}}). \tag{C.1}$$

I_1 and I_2 are defined on $\mathbb{S}_{\vec{N}}$, and their Gabor wavelet transforms are only trivially reducible polynomials in each subband (m, l). For the frequency support argument, we shift the DFT frequency box so that $\vec{\rho} = \vec{0}$ is located in the middle of it. Then the support of both $\breve{I}_1$ and $\breve{I}_2$ becomes $\mathbb{T}_{\vec{K}} = \{-(K_1 - 1)/2, \ldots, (K_1-1)/2\}\times\{-(K_2-1)/2, \ldots, (K_2-1)/2\}$. Since the images are real, K_1 and K_2 must be odd numbers, and the support is symmetrical around $\vec{0}$. (With support, we imply that K_1 and K_2 are the smallest numbers to include all nonzero elements of the images.) Furthermore, we define $\breve{\mathcal{I}}^s(\vec{\rho}, m, l)$ and $\breve{I}^s(\vec{\rho})$ as the restriction of $\breve{\mathcal{I}}(\vec{\rho}, m, l)$ and $\breve{I}(\vec{\rho})$ on $\mathbb{T}_{\vec{K}}$.

From the condition of the theorem, $N_1 \geq 2K_1 - 1$, $N_2 \geq 2K_2 - 1$. Thus, theorem 4 can be applied and leads to the following:

$$\begin{aligned}|\mathcal{I}_2(\vec{n}_0, m, l)| = |\mathcal{I}_1(\vec{n}_0, m, l)| \quad \forall\vec{n}_0 \in \mathbb{S}_{\vec{N}},\ \forall m, l \Rightarrow\ & \breve{\mathcal{I}}_2^s(\vec{\rho}, m, l)\\ \in \big\{&\exp(j\eta(m, l))\breve{\mathcal{I}}_1^s(\vec{\rho}, m, l),\\ &\exp(j\eta(m, l))\breve{\mathcal{I}}_1^{s*}(-\vec{\rho}, m, l) \,|\, \eta(m, l) \in [0, 2\pi[\big\}.\end{aligned} \tag{C.2}$$

That is, both Gabor transforms must be equal up to a phase and a possible point reflection, both of which may depend on the subband. In the following steps, we remove the ambiguities exploiting known inter- and intrasubband structure in the frequency domain.

First, we show that the magnitudes must be equal and the point-reflected case cannot happen. Suppose

$$|\breve{\mathcal{I}}_2^s(\vec{\rho}, m, l)| = |\breve{\mathcal{I}}_1^{s*}(-\vec{\rho}, m, l)| \tag{C.3}$$

or, equivalently,

$$|\breve{I}_2^s(\vec{\rho})||\breve{\psi}_{\vec{0},m,l}(\vec{\rho})| = |\breve{I}_1^s(\vec{\rho})||\breve{\psi}_{\vec{0},m,l}(-\vec{\rho})|. \tag{C.4}$$

We pick a $\vec{\rho}_0 \neq \vec{0}$, such that $\breve{I}_1^s(\vec{\rho}_0) \neq 0$ and such that the angle between $\vec{\tilde{\rho}}_0$ and the kernel's center frequency is less than $\pi/2$. From lemma 1, it can be concluded that

$$|\breve{\psi}_{\vec{0},m,l}(\vec{\rho}_0)| > |\breve{\psi}_{\vec{0},m,l}(-\vec{\rho}_0)|. \tag{C.5}$$

Now, if we substitute $\vec{\rho}_0$ into equation C.4, then equation C.5 can be satisfied only if $|\breve{I}_1^s(\vec{\rho}_0)| > |\breve{I}_2^s(\vec{\rho}_0)|$, while substituting $-\vec{\rho}_0$ into equation C.4 makes $|\breve{I}_1^s(\vec{\rho}_0)| < |\breve{I}_2^s(\vec{\rho}_0)|$ necessary. This is a contradiction.

Second, we consider the Fourier phases in each subband:

$$\arg \breve{\mathcal{I}}_2^s(\vec{\rho}, m, l) = \eta(m, l) + \arg \breve{\mathcal{I}}_1^s(\vec{\rho}, m, l). \tag{C.6}$$

Because the phases of the Gabor functions are equal for both images, it follows that

$$\arg \breve{I}_2^s(\vec{\rho}) = \eta(m, l) + \arg \breve{I}_1^s(\vec{\rho}), \tag{C.7}$$

except at the points where the Gabor function itself is zero, which are the grid points on the axis through $\vec{0}$ with the angle $\frac{2\pi}{L}(l + \frac{L}{2})$.

Applying equation C.7 to a $\vec{\rho}_0$ and $-\vec{\rho}_0$ outside that axis and the zeros of $\breve{I}_2^s$ yields, together with the fact that the images are real,

$$\eta(m, l) = 0 \ \vee \ \eta(m, l) = \pi. \tag{C.8}$$

Choosing any two combinations of m, l, there is always some point that lies on neither of the exceptional axes and where the image is nonzero. Thus, $\eta(m, l)$ must be equal for those two levels, and consequently for all. This means that all subbands have the correct Fourier phase, or the phase function of all subbands has an offset by π, and we conclude that

$$\breve{I}_2^s(\vec{\rho}) = \pm \breve{I}_1^s(\vec{\rho}). \tag{C.9}$$

From the band limitation in the conditions of the theorem, $\breve{I}_1$ and $\breve{I}_2$ are zero outside $\mathbb{T}_{\vec{K}}$, and therefore equation C.9 also holds for them. Finally, the inverse DFT yields

$$I_2(\vec{n}) = \pm I_1(\vec{n}), \tag{C.10}$$

which concludes the proof of the theorem.

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