Markov Chains

— Lecture Notes —

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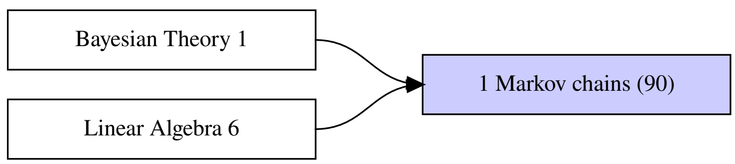
— Summary —



— Lecture 1/1 — → Lecture 1 Exercises, Lecture 1 Solutions

1 Markov chains are models of stochastic processes, here with discrete states and time. They can be conveniently represented by directed graphs with the edges weighted by transition probabilities going from one state to the connected one in one time step. The key simplifying assumption is that the transitions only depend on the current state and not on the history of states. The transition probabilities can be conveniently represented by a transition matrix, and Markov chains can be well analyzed with methods from linear algebra.

Contents

If applicable, core text and formulas are set in dark red, one can repeat the lecture notes quickly by just reading these; ♦ marks important formulas or items worth remembering and learning for an exam; ♢ marks less important formulas or items that I would usually also present in a lecture; + marks sections that I would usually skip in a lecture.
You can also download the teaching material of this topic as zip files and then view them locally on your computer.

LECTURE 1/1

- Lecture 1 Exercises, Lecture 1 Solutions

1 Markov chains

Learning objectives: The learning objective of this unit is that the student can

1. define, give an example of, and decide on simple examples whether it is a
   (a) Markov chain
   (b) graph representation of a Markov chain
   (c) transition matrix
   (d) higher-order transition matrix
   (e) reachable or accessible state
   (f) transient state
   (g) recurrent state
   (h) absorbing state
   (i) communicating class
   (j) reducible Markov chain
   (k) irreducible Markov chain
   (l) stationary distribution
   (m) limiting distribution
   (n) regular transition matrix
2. convert a transition matrix of a Markov chain to its graph representation and *vice versa*
3. reproduce and explain the Chapman-Kolmogorov Theorem
4. explain how to derive a stationary distribution for a given Markov chain
5. explain how to decide whether there is only one or multiple stationary distributions

Learning material:[1]

■ Set of short and quite dense intuitive videos by Normalized Nerd on YouTube

- 9 min : Markov Chains Clearly Explained! Part - 1[2]
  - 1b 0:35 : graphical example of a Markov chain
  - 1a 1:43 : Markov chain
  - 3:20 : determine state probabilities by sampling
  - 1c 2 5:19 : transition matrix
  - 1l 4 6:01 : stationary distribution by linear algebra
  - ~~1m~~ limiting distribution
  - ~~1n~~ regular transition matrix
  - 5 8:16 : detecting multiple stationary distributions
  - 8:44 : end
- 6 min : Markov Chains: Recurrence, Irreducibility, Classes | Part - 2[3]
  - 1f 0:41 : transient states
  - 1g 2:05 : recurrent states
  - ~~1h~~ absorbing state
  - 1e 1j 2:55 : (un)reachable states; reducible Markov chain
  - 1k 3:13 : irreducible Markov chain
  - 1i 4:15 : communicating class
  - 5:56 : end
- 9 min : Markov Chains: n-step Transition Matrix | Part - 3[4]
  - 1d 0:40 : higher-order transition matrix
  - 3 5:01 : Chapman-Kolmogorov Theorem
  - 4 5:54 : stationary distribution from higher-order transition matrix
  - 8:01 : end

+ 13 min : Markov Chains: Generating Sherlock Holmes Stories | Part - 4[5]

+ 18 min : Markov Chains: Simulation in Python — Stationary Distribution Computation | Part - 7[6]

■ Very compact and somewhat mathematical lecture notes from instructor Dimitrios Katselis at University of Illinois at Urbana-Champaign

- 4 pages : Lecture 1: Markov Chains-Part I[7] by Zeyu Zhou and Lucas Buccafusca

  1a 1'20 : Markov chain
  1c 1'75 : transition matrix
  1d 1'95 : higher-order transition matrix
  3 2'15 : Chapman-Kolmogorov Theorem
  1b 2 2'35 : graph representation of a Markov chain
  2'60 : recurrence equation
  1e 3'00 : reachable state
  1i 3'10 : communicating class
  1f 1g 1h 3'20 : transient, recurrent, and absorbing state
  ~~1j~~ reducible Markov chain
  1k 3'60 : irreducible Markov chain
  1l 4 3'65 : stationary distribution
  1m 4'20 : limiting distribution
  1n 4'95 : regular transition matrix
  ~~5~~ multiple stationary distributions
  5'20 : end

+ 6 pages : Lecture 2: Markov Chains-Part II, Steepest and Gradient Descent[8] by Andrew Chen and Zih-Siou Hung

□ More didactic Lecture Notes on Stochastic Processes with Applications in Biology[9] by David F. Anderson

- 52 pages : Chapter 1: Discrete Time Markov Chains

  1a 1'40 : Markov chain
  (1b) 3'20 : examples (and graph representation)
  ~~2~~ convert a transition matrix of a Markov chain to its graph representation
  1c 4'80 : transition matrix
  6'00 : examples
  9'00 : stopping time
  9'80 : strong Markov property
  10'50 : simulation of a Markov chain
  11'80 : example
  1d 12'30 : higher-order transition matrix (1/2)
  3 13'40 : Chapman-Kolmogorov Theorem
  1d 13'70 : higher-order transition matrix (2/2)
  14'00 : examples
  1l 15'40 : stationary distribution
  1j 1k 16'20 : (ir)reducible Markov chain (1/2)
  16'40 : examples
  1e 17'70 : accessible state

17’90 : example

1i 18’40 : communicating class

18’90 : examples

1j 1k 19’30 : (ir)reducible Markov chain (2/2)

1h 19’40 : closed subset and absorbing state

20’00 : examples (of (a)periodic states)

20’50 : (a)periodic state/class/Markov chain

21’20 : examples

1f 1g 22’10 : transient and recurrent state

24’80 : examples

26’50 : a recurrent class is a closed set

26’70 : example

27’05 : a closed set is not generally a recurrent class

27’15 : absorbtion probabilities (1/2)

27’45 : example *Gambler’s ruin*

28’90 : absorbtion probabilities (2/2)

30’50 : example *Gambler’s ruin* special case

30’90 : number of steps until absorbtion

31’80 : example

32’60 : number of steps between two states

33’00 : examples

1l 34’30 : stationary distribution

(1m 5) 35’20 : examples

4 37’50 : irreducible aperiodic Markov chains have a unique stationary distribution (1/2)

38’70 : examples

4 39’70 : irreducible aperiodic Markov chains have a unique stationary distribution (2/2)

41’55 : examples

~~1m~~ regular transition matrix

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\+ 24 pages : Chapter 2: Discrete Time Markov Chains in the Life Sciences

[1] Generic instruction: Consider the (possibly nested) list of resources like a horizontal tree with an invisible root on the very left, and decide from left to right what you want to select to work through. The invisible root node has to be selected. For any selected parent node all children nodes marked with ■ or ● are mandatory and have to be selected. Children nodes marked with □ or ○ are optional and may be selected in addition to get a better understanding of the material. If a parent node has no mandatory child, then at least one optional child has to be selected. Children marked with + provide additional voluntary material that can be safely ignored, typically going beyond the scope of the section. Children of non-selected parents may be ignored. ■ and □ indicate children that cover (almost) the whole material of the section. Missing content might then be indicated by struck through references to the corresponding learning objectives. Items tend to be ordered by precedence and/or recommended temporal order from top to bottom, assuming that you prefer to first watch a video before reading through lecture notes. If a detailed table of content for videos or lecture notes is given, references to learning objectives might be provided in green, 1:30 should be read as 1 min and 30 seconds, and 1’30 should be read as page 1 at about 30% of the page. Video times may be linked directly to the indicated position in the video, but be aware that the video might be downloaded anew each time you click on a time. Resources without author name are usually authored by Laurenz Wiskott and his team.

[2] https://www.youtube.com/watch?v=i3AkTO9HLXo, accessed 2021-04-28

[3] https://www.youtube.com/watch?v=VNHeFp6zXKU, accessed 2021-04-28

[4] https://www.youtube.com/watch?v=Zo3ieESzr4E, accessed 2021-04-28

[5] https://www.youtube.com/watch?v=E4WcBWuQQws, accessed 2021-04-28
[6] https://www.youtube.com/watch?v=G7FIQ9fXl6U, accessed 2021-04-28
[7] https://www.ini.rub.de/PEOPLE/wiskott/Teaching/Mathematics/Extra/Katselis2019-Lecture1.pdf, accessed 2021-08-27 (2021-04-27)
[8] https://www.ini.rub.de/PEOPLE/wiskott/Teaching/Mathematics/Extra/Katselis2019-Lecture2.pdf, accessed 2021-08-27 (2021-04-27)
[9] https://u.math.biu.ac.il/~amirgi/SBA.pdf, accessed 2021-04-27

Quizzes and Exercises: Deepen and test your understanding with the following

- Section 1 Exercises, Section 1 Solutions