Oscillatory Networks: Pattern Recognition without a Superposition Catastrophe

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Abstract

Using an oscillatory network model that combines classical network models with phase dynamics, we demonstrate how the superposition catastrophe of pattern recognition may be avoided in the context of phase models. The model is designed to meet two requirements: on/off states should correspond to high/low phase velocities, and patterns should be retrieved in coherent mode. Non-overlapping patterns can be simultaneously active with mutually different phases. For overlapping patterns, competition can be used to reduce coherence to a subset of patterns. The model thereby solves the superposition problem.

1 Introduction

For a network with two or more active patterns, where activity is solely defined by on/off states, a subsequent stage of information processing may not be able to read out single patterns. This so-called superposition catastrophe has long been recognized as a major challenge for neural network modelling (Rosenblatt, 1961). It is related to the binding problem, see (Roskies, 1999) and (Müller et al., 2001).

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The superposition and binding problems motivated von der Malsburg to propose grouping of neural units based on temporal correlation. This allows several non-overlapping patterns to be active at the same time and still be separable due to different temporal properties (von der Malsburg, 1981; von der Malsburg, 1985). For example, they may synchronize with different phases for different patterns (von der Malsburg and Schneider, 1986). Subsequently, associative memory based on temporal coding has been implemented in oscillatory neural networks. Recognized patterns may then correspond to limit cycles. The first models that implemented temporal coding were based on Wilson-Cowan-like dynamics, where the oscillators were defined in terms of coupled excitatory and inhibitory units (von der Malsburg and Schneider, 1986; Baird, 1986; Freeman et al., 1988; Li and Hopfield, 1989). These approaches have been extended in (Wang et al., 1990; von der Malsburg and Buhmann, 1992). Segmentation is a particular example of the superposition problem. Correspondingly, many applications are concerned with this task, for example (Terman and Wang, 1995; Wang and Terman, 1995; Wang and Terman, 1997) with applications such as medical image segmentation (Shareef et al., 1999).

Oscillatory networks with associative memory were also studied by using a phase description for the oscillators. Whenever such a parametrization is possible, the analysis of oscillatory systems may be simplified significantly, see (Kuramoto, 1984; Winfree, 2001). In this paper, we want to study the avoidance of the superposition catastrophe in the context of phase models. Our discussion will be based on a network model, where the real-valued activity $u_k$ of each unit $k$ is complemented with a phase $\theta_k$, $k = 1, \ldots, N$. The phases $\theta_k$ are supposed to parametrize a temporal structure of the signals. We consider
a generalization of classical neural network dynamics:

\[
\tau_u \frac{du_k}{dt} = -u_k + \frac{1}{N} \sum_{l=1}^{N} w_{kl}(\theta) g(u_l) + I_k, \tag{1a}
\]

\[
\tau_\theta \frac{d\theta_k}{dt} = 2\pi \ g(u_k) \ (1 + S_k(u, \theta)), \tag{1b}
\]

where

\[
w_{kl}(\theta) = g_{kl} \left( \alpha + \beta \cos(\theta_k - \theta_l) \right), \tag{2}
\]

\[
S_k(u, \theta) = \frac{\beta}{N} \sum_{l=1}^{N} g_{kl} \sin(\theta_l - \theta_k) \ g(u_l). \tag{3}
\]

The activation function is \( g(x) = (1 + \tanh(x))/2 \), the \( \tau_u, \tau_\theta > 0 \) are time-scales, and the \( I_k \) are external inputs. The \( w_{kl}(\theta) \) are phase-dependent weights, specified by the \( g_{kl} \) and real-valued parameters \( \alpha, \beta \geq 0 \). In our examples we will use \( \alpha > \beta \). In accordance with the mentioned interpretation of the phases, the model is designed so that the 1/0 limits of \( g(u_k) \) describe on/off states, leading to high/low frequencies of \( \theta_k \). This motivates the factor \( g(u_k) \) on the r.h.s. of Eq. (1b). The \( S_k(u, \theta) \) will correct the phase velocity and imply synchronization. For on-states with \( g(u_k) \to 1 \), the phase velocity will approach \( (2\pi/\tau_\theta)(1 + S_k) \), while for off-states, \( g(u_k) \to 0 \), the phase velocity will vanish. For more detailed discussion and motivation of \( w_{kl}(\theta), S_k(u, \theta) \) see below.

Understanding a possible relevance of Eqs. (1) to biology would be of interest. An interpretation of the units in terms of single biological neurons is not intended. Possibly, an interpretation in terms of populations of neurons may be found. An exact interpretation of variables in biological terms, however, is outside the scope of this paper. For the purpose of this paper, it is sufficient to see the model as a formal framework that allows the implementation of the
benefits that temporal coding should hold for pattern recognition.

Specifying an oscillatory network model requires two choices to be made. Firstly, a model for the single oscillators without coupling has to be chosen. Secondly, the coupling between the oscillators has to specified. A common choice for the oscillator dynamics is the Stuart-Landau model. This assigns a complex-valued dynamics to the oscillator and may be derived from a general ordinary differential equation near the Hopf bifurcation, see (Kuramoto, 1984). A natural choice for the couplings is linear in the complex-valued normal form coordinate (Kuramoto, 1975). The complete system may then be interpreted as a generalized version of the discrete complex-valued Ginzburg-Landau reaction-diffusion system. In the following, we refer to it as the generalized Ginzburg-Landau (GGL) model. Most approaches towards associative memory in the context of models with phases and amplitudes use the GGL model with vanishing shear, see (Hoppensteadt and Izhikevitch, 2003) for a short review. Using pure phase models as an adiabatic approximation, associative memory has been implemented by identifying the coupling strengths with Hebbian weights (Abbott, 1990; Sompolinsky et al., 1990; Baldi and Meir, 1990; Sompolinsky et al., 1991; Kuramoto et al., 1992; Sompolinsky and Tsodyks, 1992; Kuzmina and Surina, 1994; Kuzmina et al., 1995). The corresponding step in the context of phase and amplitude dynamics was taken by using a complexified analog of the Cohen-Grossberg-Hopfield function (Takeda and Kishigami, 1992; Chakravarthy and Gosh, 1994; Aoyagi, 1995; Hoppensteadt and Izhikevitch, 1996). This approach imitated the method that was used for discrete dynamics (Noest, 1988a; Noest, 1988b). Reviews of complex-valued neural networks and their application to pattern recognition may also be found in (Hirose, 2003).

The GGL model is the simplest model for coupling Stuart-Landau oscil-
lators, but not the only possible one. Alternative models have been studied with and without regard to associative memory. For example, Tass and Haken proposed three different models and used one to model synchronization of neural activity in the visual cortex (Tass, 1993; Tass and Haken, 1996a; Tass and Haken, 1996b). In the context of associative memory, the Stuart-Landau oscillator of the GGL model was replaced with an oscillator model that has on/off states (Aoyagi, 1995). A phase-locked loop model was introduced to allow implementations with well-developed circuit technology (Hoppensteadt and Izhikevitch, 2000). Another model was proposed that allowed a smooth transition between fixed point mode and oscillatory mode by appropriately changing a parameter (Chakravarthy and Gosh, 1996). We will find that also the model of Eqs. (1) is not of the GGL type.

In Section 2, we review associative memory based on the GGL model with vanishing shear and identical natural frequencies. We will argue that the phase dynamics of this model is not suited to solve the superposition problem by giving different phases to different patterns and the background. In the context of weakly-coupled GGL models, a solution has been proposed that relates the pooling of neural oscillators to frequency gaps resulting from non-identical natural frequencies (frequency modulation, FM), see (Hoppensteadt and Izhikevitch, 1997, section 5.4.2). In Section 3, we take the different approach of modifying the underlying system and starting from Eqs. (1) instead. This will allow the use of phase gaps to prevent the superposition catastrophe. We will give a detailed comparison of Eqs. (1) and the GGL model, and will also relate our approach to the FM proposal. In Section 4, we present simple examples. In Section 5, we conclude with a summary and outlook.
2 The Generalized Ginzburg-Landau (GGL) Model

We will now review and discuss associative memory based on the GGL model. Notice, however, that we do not discuss the storage of complex-valued patterns. The according changes may be included without difficulty by going to complex-valued Hermitean weights (Noest, 1988a; Noest, 1988b; Takeda and Kishigami, 1992; Chakravarthy and Gosh, 1994; Aoyagi, 1995). We will present the models in a form that will allow a convenient comparison with Eqs. (1) in Section 3.

2.1 The GGL model as gradient system

The discrete complex Ginzburg-Landau (GGL) model may be expressed in terms of complex coordinates (Kuramoto, 1975), see also (Hoppensteadt and Izhikevitch, 1997, sections 10.1-3):

\[
\text{GGL}: \frac{dz_k}{dt} = (\tilde{I}_k + i\omega_k)z_k - (\sigma_k + i\eta_k)z_k |z_k|^2 + \beta \frac{N}{N} \sum_{l=1}^{N} g_{kl} z_l, \tag{4}
\]

where \( \tilde{I}_k, \omega_k, \sigma_k, \eta_k, \) and the weights \( g_{kl} \) are real-valued parameters, \( k = 1, \ldots, N. \)

Associative memory has been introduced for the parameter choices \( \eta_k = 0, \) corresponding to vanishing shear, and \( \omega_k = \Omega, \) see (Hoppensteadt and Izhikevitch, 1997, section 10.4) and the short review in (Hoppensteadt and Izhikevitch, 2003). The latter choice allows us to set \( \omega_k = 0 \) by going to the co-moving frame, \( \theta_k \rightarrow \theta_k + \Omega t. \) We may also set \( \sigma_k = 1. \) With \( z_k = V_k \exp (i\theta_k), \tilde{z}_k = V_k \exp (-i\theta_k), \) where \( V, \theta \) are real, Eq. (4) is then equivalent to

\[
\begin{align*}
\text{GGL}: \quad \frac{dV_k}{dt} &= \tilde{I}_k V_k - V_k^3 + \frac{1}{N} \sum_{l=1}^{N} \tilde{w}_{kl}(\theta) V_l, \tag{5a} \\
\text{GGL}: \quad \frac{d\theta_k}{dt} &= \frac{1}{V_k} \tilde{S}_k(V, \theta). \tag{5b}
\end{align*}
\]
where \( \tilde{w}_{kl} \) is \( w_{kl} \) with \( \alpha = 0 \), as given by Eq. (2), and \( \tilde{S}_k \) is obtained from \( S_k \) by replacing \( g(u_k) \rightarrow V_k \). The \( g_{kl} \sin(\theta_l - \theta_k) V_l \) terms in \( \tilde{S}_k(V, \theta) \) tend to synchronize (desynchronize) unit \( k \) with units \( l \), whenever \( g_{kl} > 0 \) \((g_{kl} < 0)\).

Without couplings, \( g_{kl} = 0 \), it is easily seen, that for \( \tilde{I}_k < 0 \) the origin \( V_k = 0 \) is a stable fixed point, at \( \tilde{I}_k = 0 \) the unit \( k \) experiences a Hopf-bifurcation, generating a stable limit cycle with radius \( \sqrt{\tilde{I}_k} \) for \( \tilde{I}_k > 0 \).

Introducing the Lyapunov function

\[
\tilde{L}(V, \theta) = -\frac{1}{2N} \sum_{k,l} \tilde{w}_{kl} (\theta) V_k V_l + \tilde{P}(V), \quad (6)
\]

with

\[
\tilde{P}(V) = \sum_k \left( -\frac{\tilde{I}_k}{2} V_k^2 + \frac{1}{4} V_k^4 \right), \quad (7)
\]

allows us to express the dynamics of Eqs. (5) as a gradient system (Noest, 1988a; Noest, 1988b; Takeda and Kishigami, 1992; Chakravarthy and Gosh, 1994; Aoyagi, 1995; Hoppensteadt and Izhikevitch, 1996). Assuming that the \( g_{kl} \) are symmetric, one obtains

\[
\text{GGL : } \frac{dV_k}{dt} = -\frac{\partial}{\partial V_k} \tilde{L}(V, \theta), \quad (8a)
\]

\[
\text{GGL : } \frac{d\theta_k}{dt} = -\frac{1}{V_k^2} \frac{\partial}{\partial \theta_k} \tilde{L}(V, \theta). \quad (8b)
\]

The minima of \( \tilde{L}(V, \theta) \) should then correspond to storage of patterns. In Section 2.2, we specify the storage of patterns, and relate \( \tilde{L}(V, \theta) \) to a coherence measure for the patterns.
2.2 Memory, competition, and coherence

We consider $P$ patterns with components $\xi_k^p \in \{0, 1\}$, $k = 1, \ldots, N$. Using these patterns, we specify the weights $g_{kl}$ of Eqs. (2), (3) as

$$
g_{kl} = d_{kl} + \frac{1}{P} \sum_{p} \xi_k^p \xi_l^p + \lambda \frac{-1}{P^2} \sum_{p \neq q} \xi_k^p \xi_l^q.
$$

(9)

The couplings $d$ correspond to a background geometry, the excitatory part $h$ corresponds to Hebbian memory, and the inhibitory part $c$ establishes competition between the patterns with $\lambda \geq 0$. Analogously, any inhibitory part of $d$ establishes competition between units.

The complex coordinates of Eq. (4) may be used to define a coherence measure $C_p$ for each pattern $p$:

$$
Z_p(V, \theta) = \frac{1}{N_p} \sum_k \xi_k^p z_k = C_p \exp(i \Psi_p),
$$

(10)

where $Z_p, C_p, \Psi_p$ depend on $V, \theta$ and $|Z_p| = C_p$, $0 \leq C_p \leq 1$. These correspond to the Kuramoto coherence measure restricted to the active parts of the stored patterns (Kuramoto, 1984). See (Strogatz, 2000) for a review of the Kuramoto model and references to recent results. We call $\Psi_p$ the phase of pattern $p$ and $C_p$ its coherence. $N_p$ denotes the number of non-zero components of pattern $p$.

Using Eq. (9) with vanishing background geometry, i.e. $d = 0$, Eq. (6) may
then be written as

\[ \tilde{\mathcal{L}} = -\frac{\beta}{2N} \sum_{k,l} g_{kl} \cos (\theta_k - \theta_l) V_k V_l + \tilde{\mathcal{P}} \]

\[ = -\frac{\beta}{2N} \Re \sum_{k,l} g_{kl} \exp \{ i (\theta_k - \theta_l) \} V_k V_l + \tilde{\mathcal{P}} \]

\[ = -\frac{\beta N}{2P} \left( \sum_p \rho_p^2 C_p^2 - \frac{\lambda}{P} \sum_{p \neq q} \rho_p \rho_q C_p C_q \cos (\Psi_p - \Psi_q) \right) + \tilde{\mathcal{P}}, \tag{11} \]

where \( \rho_p = N_p/N \) denotes the density of pattern \( p \). Since \( \tilde{\mathcal{L}} \) could be given this form, without competition, i.e. \( \lambda = 0 \), Eqs. (8) imply that the \( \beta \)-dependent terms of Eqs. (5) introduce a tendency to maximize the coherence of all patterns. With \( \lambda > 0 \), this tendency is accompanied by a competition between the patterns, so that any pattern \( p \) with significant coherence \( C_p \) will try to suppress the coherence \( C_q \) of any other pattern \( q \) or will arrange for a phase difference between \( \Psi_p \) and \( \Psi_q \) so that \( \tilde{\mathcal{L}} \) is minimized. For example, assume \( P = 2 \) and both patterns are non-overlapping. The first sum in the last line of Eqs. (11) will imply \( C_1, C_2 \to 1 \), while the second sum introduces a phase difference \( (\Psi_2 - \Psi_1) \to \pi/2 \mod \pi \) in order to minimize \( \tilde{\mathcal{L}} \). In fact, such a dynamics is exactly what we observe in example 3 of subsection 4.4.1.

This tendency towards coherence of a pattern and and decoherence between non-overlapping patterns bears a resemblance to the results obtained for the Wang-Terman model (Terman and Wang, 1995). The Wang-Terman model, however, is based on coupled relaxation oscillators and patterns were not stored via Hebbian connections. Instead, a specific two-dimensional background geometry was assumed for the purpose of image segmentation. Using the two-dimensional background geometry makes the Wang-Terman model rather comparable to lattice models such as the ones studied in (Sakagushi et al., 1987; Sakagushi et al., 1988). The latter, however, were not applied
to image segmentation, do not use inhibition and do not have on/off states. Moreover, differences between the synchronization properties of relaxation and nonrelaxation oscillators have been described in (Somers and Kopell, 1993; Somers and Kopell, 1995; Izhikevitch, 2000).

2.3 Phase dynamics with vanishing amplitudes

The coupling in Eq. (5b) has a remarkable effect on the synchronization properties, see also (Hoppensteadt and Izhikevitch, 1997, page 302). It contains the terms

\[
GGL : \frac{1}{V_k} \sum_{l=1}^{N} g_{kl} \sin (\theta_l - \theta_k) V_l
\]

on the r.h.s of Eq. (5b). Due to the factor $1/V_k$, the units get more susceptible to phase differences as $V_k \to 0$. Moreover, a state $V_k$ approaching the off-state is synchronized more strongly with on-states $V_l$, not with other off-states (we assume $g_{kl} > 0$). Thus, there is a strong tendency towards global synchrony (infinitely strong for off-states, since $1/V_k \to \infty$, as $V_k \to 0$), where on- and off-states all acquire the same phase.

In Section 3.2 we will assume $V_k = g(u_k)$ in order to compare Eqs. (1) with the GGL model. The tendency towards global coherence, resulting from Eq. (12), is then not compatible with our interpretation of the $V_k$, see the introductory remarks. We intend to identify on/off states of units with high/low frequencies $d\theta_k/dt$. Therefore, the phase of a unit that is close to an off state should not be strongly driven towards synchronization with on-state phases. When returning to Eqs. (1) in the next section, we will therefore find that our model differs from the associative memory GGL model not only by introducing the frequency-driving activity $d\theta_k/dt = (2\pi/\tau_\theta) g(u_k) + \ldots$, but also by accompanying this change with an alternative phase coupling that allows off-state units to have phases that differ from on-states phases. This feature is
necessary for using phase differences to avoid the superposition problem.

### 2.4 Weak couplings and temporal coding

The central topic of this paper is the superposition problem and its possible solution based on grouping of neural oscillators due to temporal properties. Therefore, it should be compared to an earlier approach that will now be reviewed in the context of weakly coupled systems.

For the case of weak couplings, the amplitude dynamics in Eqs. (5), $\tilde{I}_k > 0$, $k = 1, \ldots, N$, may be adiabatically eliminated, see (Kuramoto, 1984). This will leave the phase dynamics:

$$
\frac{d\theta_k}{dt} = \omega_k + \frac{\beta}{N} \sum_{l=1}^{N} \tilde{g}_{kl} \sin (\theta_l - \theta_k),
$$

where $0 < \beta \ll 1$ and $\tilde{g}_{kl}$ is obtained from the adiabatic values $V_k \rightarrow V_k^\infty$, $V_l \rightarrow V_l^\infty$, leading to $\tilde{g}_{kl} = (V_l^\infty / V_k^\infty) g_{kl}$. Weak couplings will also imply $V_k^\infty > 0$, so that $\tilde{g}_{kl} < \infty$. In contrast, oscillator death with $V_k^\infty = 0$ (as well as self-ignition, $\tilde{I}_k < 0$) may occur for stronger couplings, see (Aronson et al., 1990).

In Eq. (13), we also re-established non-identical frequencies $\omega_k$. In Eq. (5b), these were set to $\Omega$, and eliminated by going to the co-moving frame, $\theta_k \rightarrow \theta_k + \Omega t$. Non-identical frequencies have been used to propose neural grouping based on different frequency, see (Hoppensteadt and Izhikevitch, 1997, section 5.4.2). This frequency modulation (FM) proposal uses a number of basic frequencies $\Omega_\Gamma$, $\Gamma = 1, \ldots, G$, and assumes that

$$
\omega_k \in \{\Omega_0, \Omega_1, \ldots, \Omega_G\},
$$

with $G \leq N$. Two units $k$ and $l$ will then belong to the same group (ensemble, pool), if their natural frequencies will be the same, $\Omega_{\Gamma(k)} = \Omega_{\Gamma(l)}$, up to terms of
order $\beta$. This approach gets particularly interesting when higher-order phase couplings are included in Eq. (13). The equality condition is then generalized to a resonance condition (Hoppensteadt and Izhikevitch, 1997, section 9.4.3). In section 3.4, we will relate the FM proposal to our approach.

3 Phase Extension of Classical Networks

In this section we return to the model of Eqs. (1). The system is compared with the GGL model, pattern recognition capabilities are discussed, and temporal coding is compared with the FM proposal that was mentioned at the end of the last section.

3.1 The model

The system of Eqs. (1) reads

\[ \tau_u \frac{du_k}{dt} = -u_k + \frac{1}{N} \sum_{l=1}^{N} w_{kl}(\theta) g(u_l) + I_k, \]

\[ \tau_\theta \frac{d\theta_k}{dt} = 2\pi g(u_k) (1 + S_k(u, \theta)), \]

where

\[ w_{kl}(\theta) = g_{kl} (\alpha + \beta \cos(\theta_k - \theta_l)), \]

\[ S_k(u, \theta) = \frac{\beta}{N} \sum_{l=1}^{N} g_{kl} \sin(\theta_l - \theta_k) g(u_l). \]

The weights $w_{kl}$ agree with the $\tilde{w}_{kl}$ of the GGL model, except for the parameter $\alpha$ that was introduced to give the weights also a classical part. The term classical refers to phase-independence, see the discussion in (von der Malsburg, 1999). This difference was motivated by introducing in Eqs. (1) a phase coupling that does not replace but complement classical network dynamics. In
fact, for the examples of Section 4 we will always use \( \alpha > \beta \geq 0 \). The \( S_k(u, \theta) \) are obtained from \( \tilde{S}_k(V, \theta) \) by replacing \( V_i \rightarrow g(u_i) \).

With \( \beta > 0 \), the \( \sin(\theta_i - \theta_k) \) in \( S_k \) introduce a tendency to synchronize (desynchronize) the units \( k \) and \( l \), assuming \( g_{kl} > 0 \) \( (g_{kl} < 0) \), and the \( \cos(\theta_k - \theta_l) \) in \( w_{kl} \) strengthens (weakens) the couplings if the units are in-phase (out-of-phase). These features agree with that of Eqs. (5). Notice, however, the different strengths of phase couplings, given by \( g(\theta_k) \) in Eqs. (1) and by \( 1/V_k \) in Eqs. (5), which lead to the different behaviour of units that are close to off-states. We will return to a comparison between Eqs. (1) and (5) in Section 3.2.

With \( \beta = 0 \), the weights are constant, \( w_{kl}(\theta) = \alpha g_{kl} \), and Eq. (1a) decouples from the phase dynamics, thereby reducing to a classical model, while the phase dynamics reduces to uncoupled oscillations, \( d\theta_k/dt = (2\pi/\tau_g) \ g(u_k) \). In this sense, the complete dynamics of Eqs. (1) is an extension of the classical dynamics.

### 3.2 Gradient dynamics, frequency-driving activity, and comparison with GGL models

In the following, we identify \( V_k \equiv |z_k| = g(u_k) \). Interpreting the activity \( g(u_i) \) as an amplitude allows to compare Eqs. (1) and (5). Notice that we do not restrict the \( V_k \) to be small; the range of the \( V_k \) is \( 0 < V_k < 1 \), due to the definition of \( g \). As \( |u_k| \rightarrow \infty \), the \( V_k \) run into saturation.

The comparison between Eqs. (1) and (5) will get most transparent by relating Eqs. (1) to gradient dynamics, in analogy to Section 2.1. Using the weights \( w_{kl}(\theta) \) of Eq. (2), a natural generalization of the Cohen-Grossberg-Hopfield function (Cohen and Grossberg, 1983; Hopfield, 1984) for amplitudes
and phases reads:

\[ \mathcal{L}(V, \theta) = -\frac{1}{2N} \sum_{k,l} w_{kl}(\theta) V_k V_l + \mathcal{P}(V), \tag{15} \]

where the potential is given by

\[ \mathcal{P}(V) = \sum_k \int_{V_k}^{V_k} \left( g^{-1}(x) - I_k \right) dx = \sum_k \left\{ \frac{1}{2} \left( V_k \ln V_k + (1 - V_k) \ln (1 - V_k) \right) - I_k V_k \right\}. \tag{16} \]

For deriving Eq. (17), we used that the \( u_k \) may be obtained from \( V_k \) through

\[ u_k = g^{-1}(V_k) = \frac{1}{2} \ln \frac{V_k}{1 - V_k}. \tag{18} \]

We applied the integration formula \( \int \ln x \, dx = x \ln x - x \) and set the integration constant in Eq. (17) to zero.

We may now relate Eqs. (1) to \( \mathcal{L} \), just as Eqs. (5) were related to \( \mathcal{L} \). Using \( dg(u)/du = 2V_k(1 - V_k) \), Eqs. (1) may then be written:

\[ \tau_u \frac{dV_k}{dt} = 2V_k(1 - V_k) \left\{ -\frac{1}{2} \ln \frac{V_k}{1 - V_k} + \frac{1}{N} \sum_{l=1}^{N} w_{kl}(\theta) V_l + I_k \right\} \tag{19a} \]

\[ = 2V_k(1 - V_k) \left( -\frac{\partial}{\partial V_k} \right) \mathcal{L}(V, \theta), \tag{19b} \]

\[ \tau_\theta \frac{d\theta_k}{dt} = 2\pi V_k \left( 1 + \tilde{S}_k(V, \theta) \right) \tag{19c} \]

\[ = 2\pi V_k + 2\pi \left( -\frac{\partial}{\partial \theta_k} \right) \mathcal{L}(V, \theta). \tag{19d} \]

The system splits into a gradient part and the frequency-driving activity term \( 2\pi V_k \). For the remainder of this subsection, we compare the system of Eqs. (19)
with the GGL model of Eqs. (5), (8).

The amplitude dynamics of Eq. (19a) is similar to Eq. (5a). The ln-term in Eq. (19a) is a higher-order analog of the \( \tilde{I}_k V_k - V_k^3 \) terms in Eq. (5a). The additional factors \( V_k(1 - V_k) \) correspond to the saturation as \( V_k \to 0, 1 \).

The main difference is in the phase dynamics. Notice first, that the GGL model could be formulated as the pure gradient system of Eqs. (8) only by assuming vanishing shear, i.e. \( \eta_k = 0 \) in Eq. (4). Therefore, the term \( 2\pi V_k \)
Eq. (19d) may be compared to the shear term of the GGL model that is usually set to zero when pattern recognition is implemented, see (Hoppensteadt and Izhikevitch, 1997, section 10.4). In contrast, we consider the non-vanishing shear-like term to be essential for our interpretation of the oscillatory units. Setting the \( 2\pi g(u_k) + \ldots \) term on the r.h.s. of Eq. (1b) to zero would destroy the relation between the activity \( u_k \) and the frequency \( d\theta_k/dt \) of the signals. This realizes the requirement that on/off states should correspond to high/low frequencies of the phases.

Moreover, in order to establish a consistent picture, the effect of the \( S_k(u, \theta) \) should not be growing as \( V_k \to 0 \). In Section 2.3, we mentioned that the coupling \( \tilde{S}_k/V_k \) in Eq. (5b) leads to global synchrony, establishing a strong tendency towards equal phases of on- and off-states. Such a coupling would be in conflict with low frequencies as \( V_k \to 0 \). Therefore, an obvious difference between Eqs. (5) and (19) is that the latter uses the coupling \( V_k S_k \) instead. Thereby, synchronization between units \( k \) and \( l \), assuming \( g_{kl} > 0 \), is enforced only for on-states, i.e. the unit \( k \) gets less susceptible to phase differences as \( V_k \to 0 \). In terms of the gradient system, the difference arises due to the factor \( 1/V_k^2 \) in Eq. (8b) that is absent in Eq. (19d). Therefore, with regard to the \( V_k \) the coupling in Eqs. (19) is of higher order than the couplings of the GGL model.
3.3 Pattern recognition capabilities

Patterns stored according to Eq. (9) are related to minima of $\mathcal{L}$. Correspondingly, one could expect that gradient dynamics in Eqs. (19) tends to retrieve stored patterns by approaching the minima of $\mathcal{L}$. However, due to the term $2\pi V_k$ in Eq. (19d), the system of Eqs. (1) is not a pure gradient system. We find

$$\tau_u \frac{d}{dt} \mathcal{L}(V, \theta) = - \sum_k \left( 2V_k (1 - V_k) \left( \frac{\partial \mathcal{L}}{\partial V_k} \right)^2 \right)$$

$$+ \frac{\tau_u}{\tau_\theta} \frac{\partial \mathcal{L}}{\partial \theta_k} \left( \frac{\partial \mathcal{L}}{\partial \theta_k} - V_k \right).$$

In contrast to GGL models with vanishing shear, the r.h.s. includes terms that may be non-negative. These may imply an increase in the $\mathcal{L}(V, \theta)$ values. Let us write the corresponding terms in the form

$$\sum_k \frac{\partial \mathcal{L}}{\partial \theta_k} V_k = \frac{\beta}{N} \sum_{k,l} V_k - V_l \frac{V_k - V_l}{2} \sin (\theta_l - \theta_k) V_k g_{kl} V_l.$$  \hspace{2cm} (21)

In the following, we will discuss the effect of these terms on the pattern recognition capability of the system.

The terms of Eq. (21) result from the first term $2\pi V_k + \ldots$ on the r.h.s. of Eq. (1b). Why could this term eventually cause an increase of $\mathcal{L}(V, \theta)$? Say, $\theta_k = \theta_l + \delta \theta \pmod{2\pi}$ with $0 < \delta \theta < \pi$. On one hand, the sign of $\sin (\theta_l - \theta_k)$ is such that the $\theta_l$ tend to be fastened and the $\theta_k$ tend to be slowed down in order to reach synchrony (assuming $g_{kl} > 0$ and $\alpha > \beta \geq 0$). With $V_k - V_l < 0$, this dynamics is supported by the frequency-driving activity terms. The corresponding terms in Eq. (21) will be negative and will imply a tendency of $\mathcal{L}(V, \theta)$ to decrease. On the other hand, for $V_k - V_l > 0$, the
activity term may outvote the $\sin (\theta_l - \theta_k)$ tendency: $\theta_k$ could remain faster than $\theta_l$. This corresponds to the positive terms of Eq. (21), which may cause $\mathcal{L}(V, \theta)$ to increase. As a result, however, $\theta_k$ will approach $\theta_l$ from “the other side”, i.e. the situation $\theta_k = \theta_l - \delta \theta \pmod{2\pi}$ is reached. Should $V_k - V_l > 0$ still be the case, the terms of Eq. (21) will then become negative and will imply a tendency of $\mathcal{L}(V, \theta)$ to decrease again. Obviously, without additional inquiry, we may not judge whether the foregoing increase will be cancelled by the following decrease of $\mathcal{L}(V, \theta)$. Therefore, we will now look closer at the $V$ dynamics.

Assuming an adiabatic approximation, $\tau_u \ll \tau_\theta$, we find that the only term that may become negative inside the bracket of Eq. (20) is suppressed by $(\tau_u/\tau_\theta)$. Moreover, due to their (fast) dynamics, the $V_k$ will have approached their on/off states (then $\partial \mathcal{L}/\partial V_k \to 0$), when the terms of order $\tau_u/\tau_\theta$ get relevant. Assuming sufficiently large values of $\alpha$ in Eq. (2), we may arrange the set of indices $k$ so that $V_k = 1 - \epsilon_k$, $k = 1, \ldots, M$, and $V_k = \epsilon_k$, $k = M + 1, \ldots, N$, where $0 < \epsilon_k \ll 1$, for some $M \leq N$. Then

$$\sum_k \frac{\partial \mathcal{L}}{\partial \theta_k} V_k \to \frac{\beta}{N} \sum_{k,l=1}^M \sin (\theta_k - \theta_l) g_{kl} + O(\epsilon) = O(\epsilon),$$

(22)

due to the anti-symmetry of $\sin (\theta_k - \theta_l) g_{kl}$. Combining these aspects, we find that frequency-driving activity terms only imply a $(\tau_u/\tau_\theta) O(\epsilon)$ correction to $d\mathcal{L}/dt$, and we may expect that these terms will not significantly affect the pattern recognition capabilities. This expectation will be confirmed by the examples in Section 4.
3.4 Comparison with the frequency modulation approach to temporal coding

Before giving examples for the pattern recognition behaviour, we want to relate Eqs. (1) to the frequency modulation (FM) approach, described in Section 2.4. Remember the weak coupling limit in Eq. (13):

\[
d_{\theta_k} = \Omega_{\Gamma(k)} + \frac{\beta}{N} \sum_{l=1}^{N} \tilde{g}_{kl} \sin (\theta_l - \theta_k).
\]

Eq. (1b) : \[ \frac{2\pi}{\tau_{\theta}} g(u_k) \quad g(u_k) g_{kl} g(u_l) \]

Here, we have added a comparison with Eq. (1b). We may relate the FM proposal to Eqs. (1) by assuming two basic frequencies: \( \Omega_1 = 0 \) (off-states), \( \Omega_2 = 2\pi/\tau_{\theta} \) (on-states). The difference is that for the FM approach the \( \Omega_\Gamma \) have been external parameters, while in Eqs. (1) they are subject to the dynamics of Eq. (1a). Whether the system approaches \( \Omega_1 \) or \( \Omega_2 \) depends on external inputs and initial values. Notice also, that in the context of the complete dynamics of Eqs. (1) also the couplings are subject to a dynamics. Due to the term \( g(u_k) g_{kl} g(u_l) \), a synchronization is not enforced if one or both of the units \( k, l \) approach an off-state.

In the context of brain dynamics, it has been speculated that the external character of natural frequencies would actually turn dynamical in a more complete setting: “It is reasonable to speculate that the brain has mechanisms to regulate the natural frequencies \( \Omega_\Gamma \) of its neurons so that the neurons can be entrained into different pools at different times simply by adjusting the \( \Omega_\Gamma \)” (Hoppensteadt and Izhikevitch, 1997, section 5.4.2). In the introduction, we mentioned that here we do not aim at a biological interpretation. Nevertheless, it is obvious that Eqs. (1) provide an extension of Eq. (23), where the analog
of the natural frequencies may result from a dynamical process.

Notice, however, a difference between the FM approach and the direction we take. The FM approach uses only frequency gaps for separating the neural pools, labelled by $\Omega_1, \ldots, \Omega_G$. The analogy with Eqs. (1) goes only to separating on-states ($\Omega_1$) and off-states ($\Omega_2$). Among the on-states, we go on with a separation based on phase gaps. In particular, for overlapping patterns, competitive couplings are used to separate their phases. Using competition as the separating mechanism has been proposed already in (von der Malsburg, 1981).

With regard to biological interpretation, it has been suggested that temporal coding based on frequency gaps versus temporal coding based on phase gaps may be complementary (Hoppensteadt and Izhikevitch, 1999), the former being valid in the weak-coupling regime, while the latter may be more suitable for strong couplings (Izhikevitch, 1999). Notice in this respect that the model of Eqs. (1) is indeed applicable with strong couplings, in particular due to the saturation properties of the activation function.

4 Examples

In this section, we illustrate the dynamics of Eqs. (1) in the context of simple examples. We continue to use the short form $V_k = g(u_k)$. In Section 4.1, we comment on input choices. In Section 4.2, we specify the parameters and present the patterns that we use for the examples. In Section 4.3, we consider the weights without competition, i.e. $\lambda = 0$ in Eq. (9). We apply external input to the network and demonstrate how coherent pattern recognition retrieves the stored patterns. As expected, the frequency-driving activity $2\pi V_k$ separates the phases of active units from the background. We compare coherent pattern recognition to the classical dynamics. This will help to understand the limitations of the classical approach and to value the additional features
that arise due to the phase couplings, most notably the avoidance of the super-
position problem. In Section 4.4, we include competition between the patterns,
i.e. $\lambda > 0$, and study the resulting effects.

4.1 Pattern retrieval and input choices

Choosing the activation function $g$ of Eq. (1) so that on/off states of a unit $k$
correspond to 1/0 limits of $V_k$ introduces a subtlety regarding the inputs $I$.
This may be understood when writing Eq. (1) in terms of $h(x) = \tanh(x) =
2g(x) - 1$ instead. For the present discussion it is sufficient to consider $\beta = 0$,
then
\[
\tau_u \frac{du_k}{dt} = -u_k + \frac{\alpha}{2N} \sum_{l=1}^{N} g_{kl} h(u_l) + J_k ,
\]  
(24)

where $J$ is related to $I$ through
\[
I_k = J_k - \frac{\alpha}{2N} \sum_{l=1}^{N} g_{kl} .
\]  
(25)

The subtlety is related to the interpretation of vanishing inputs. Vanishing
inputs should be identified not with vanishing $I$ but with $J = 0$. The related $I$
value is then given by the second term in Eq. (25). Only for such a value of $I$
will $u = 0$ imply $du/dt = 0$ so that vanishing inputs correspond to a neutral or
undecided state at the origin of $u$-space. During this Section, we will specify
the inputs in terms of $J$. 

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4.2 A simple network

We now want to illustrate the dynamics of Eqs. (1) by discussing simple examples. We choose \( N = 6, \ P = 3 \), and store the patterns

\[
\xi^1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \xi^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix},
\]

(26)

with \( N_1 = N_2 = 2, \ N_3 = 3 \). This form allows us to distinguish two cases: overlapping and non-overlapping patterns. Pattern 1 does not overlap with patterns 2 and 3, while patterns 2 and 3 overlap at unit 4. The patterns are illustrated in Figure 1. The Hebbian weights \( h \) and the inhibitory competition terms \( c \) are given by Eq. (9):

\[
h_{kl} = \frac{1}{P} \begin{pmatrix} 110000 \\ 110000 \\ 001100 \\ 001211 \\ 000111 \\ 000111 \end{pmatrix}, \quad c_{kl} = \frac{-1}{P^2} \begin{pmatrix} 001211 \\ 001211 \\ 110111 \\ 221211 \\ 111100 \\ 111100 \end{pmatrix}.
\]

(27)

We set \( d = 0 \) in Eq. (9) since here we are not interested in background effects.

In the following, the global Hebbian parameter \( \alpha \) is set to \( \alpha = 2PN \). This is large enough to establish attractors of the system that realize storage of the patterns. On-states correspond to \( V_k \to 1 \), while off-states correspond to
$V_k \to 0$. The time scales are set to $\tau_u = \tau_\theta/4 = \tau$. This is a mild form of the $\tau_u \ll \tau_\theta$ scenario, underlying the adiabatic approximation in Section 3.3. Simulations are performed using Euler’s method with time step $dt = 0.02\tau$. The initial values, random values for the $\theta$ and small random values for $u_k$ (so that $V_k \approx 1/2$), are the same for all examples in this section. The examples are distinguished by different values for $\beta$, $\lambda$, and inputs $I_k$ (expressed in terms of $J_k$, see Section 4.1).

4.3 Coherent pattern recognition

We begin with two examples for the response to external inputs. In Section 4.1 we argued that vanishing inputs should be identified with $J_k = 0$, where $J$ is related to $I$ by Eq. (25). Therefore, excitatory input should be identified with $J_k > 0$ and inhibitory input with $J_k < 0$.

Both examples compare coherent pattern recognition with the classical case. The classical case corresponds to $\beta = 0$. For the phase coding scenario we choose $\beta = 0.1\alpha$. During this subsection, we do not include competition, i.e. $\lambda = 0$.

4.3.1 Example 1

First, we set $J_1 = -E$, $J_6 = +E$, where $E = 10$, while all other inputs vanish. The resulting dynamics is displayed in Figure 2. We find that the inhibitory input at unit 1 suppresses pattern 1, while the excitatory input at unit 6 excites pattern 3. Moreover, due to its overlap with pattern 3 pattern 2 is also excited. This is true for the classical and the phase coding case. It is the behaviour that should be expected. If phase coding is included, $\beta = 0.1\alpha$, we find that the active units synchronize after a few multiples of the time scale.

The example confirms that we succeeded in constructing a pattern recog-
nition mechanism that retrieves patterns in a coherent mode. The frequency-
driving activity term in Eq. (1b) is essential in separating the active from non-
active units. The frequencies of active units approach $2\pi/\tau_0$, while the frequency
of non-active units are frozen towards zero. The implications for the superpo-
sition problem are illustrated with the next example.

4.3.2 Example 2

The superposition problem may now be demonstrated by choosing both inputs
to be positive, $J_1 = J_6 = +E$. Then pattern 1 is also excited, see Figure 3.
In the classical case, $\beta = 0$, the superposition of active patterns no longer
allows us to distinguish the single patterns. For example, it is no longer pos-
sible to distinguish pattern 1 since all units are active. Consequently, there
is a problem for information processing along the succeeding stages. This is
the superposition catastrophe that plagues the classical approach (Rosenblatt,

Comparing the classical situation to the case $\beta = 0.1\alpha$ makes it obvious
why the inclusion of phase coding helps to avoid the superposition problem.
Now the units do not only carry information about being on or off, related to
high or low frequencies, but also information about the phases. We find that
the units 1 and 2 of pattern 1 synchronize separately from the units 3 to 6 of
patterns 2 and 3. As a result, in the output pattern 1 may be distinguished
from the rest due to its different phase. Whenever the succeeding stage of
information processing is sensitive to this phase difference it will be able to
identify pattern 1 despite the fact that the other units are also on.

4.4 Competition between patterns

While the superposition problem is a severe drawback in the application of
classical networks, another problem are mixed states that correspond to a
common activation of overlapping patterns. This arises when using the Hebbian weights without additional coupling of the units. It is only a problem, of course, if we aim at getting a unique pattern as the output of the network. Then the situation of examples 1 and 2 is not satisfying since pattern 2 and 3 are activated simultaneously due to being connected through Hebbian connections. This is why we now include additional inhibitory weights $c_{kl}$ that introduce competition between the patterns, i.e. we use non-vanishing $\lambda$ in Eq. (9).

### 4.4.1 Example 3

We set $\lambda = 1.2P$. No external inputs are applied. To make the influence of phase coding obvious, we now use a larger value for $\beta$ by choosing $\beta = 0.8\alpha$ for the non-classical case. The resulting dynamics is displayed in Figure 4. It may be understood as follows.

Let us first discuss the classical situation. When analyzing our random initial values for $u_k$, they are found to be slightly positive, so that the $V_k$ are slightly above $1/2$ and the activities tend to run towards the on-states. This, however, invokes the competition process, and for the classical scenario, $\beta = 0$, the units of pattern 3 succeed in suppressing all the other activities, so that finally the units 4 to 6 of pattern 3 approach the on-state attractor, while the other units move towards the off-state attractor. The flat lines in the phase diagram correspond to switched off units 1 to 3. The dominance of pattern 3 is mainly due to $N_3 > N_{2,1}$.

With phase coding the situation is different and allows an interesting observation. We find that although again pattern 3 dominates pattern 2, we now end up with a still active pattern 1. The reason for this may be understood when realizing that the two winning patterns are out of phase. In fact, the phase difference approaches $\pi$. Thus pattern 1 escaped the competition by
desynchronizing with pattern 3 so that the coupling \( w(\theta) \) and thus the competition is weakened. As a result, the system ends up in a situation similar to Figure 3, but now the superposition problem between the overlapping patterns has been solved due to the competition, i.e. unit 3 and thereby pattern 2 is suppressed. Two patterns may therefore be simultaneously active and still be separable. This is another example for avoiding the superposition problem.

Notice that an analogous behaviour occurs when global inhibition is used, i.e. competition between units instead of competition between patterns. A possible example is obtained from \( \lambda = 0 \) and \( d_{kl} = -1 \) for all \( k, l \) in Eq. (9). Again, in the classical case pattern 3 is winning, while in the presence of phase coding the patterns 1 and 3 are active with different phases as illustrated for the case of Figure 4.

5 Summary and Outlook

In this paper, we used the oscillatory network model of Eqs. (1) to study a solution to the superposition catastrophe in the context of phase models. The system of Eqs. (1) was obtained by extending classical neural network models to include phase dynamics. It was designed to meet two basic requirements: on/off states should correspond to high/low frequencies of the phases, and patterns should be retrieved in coherent (synchronized) mode.

Identifying the activity \( g(u_k) \) of Eqs. (1) with an oscillation amplitude \( V_k \) allows us to compare the model with Eqs. (5), a generalized version of the discrete complex Ginzburg-Landau (GGL) model with vanishing shear and identical natural frequencies. This model describes Stuart-Landau oscillators (i.e., oscillators close to a Hopf-bifurcation) coupled to lowest order, a model that is frequently used for implementing associative memory, see (Hoppensteadt and Izhikevitch, 2003). With \( V_k = g(u_k) \), the GGL model does not
obey the first of the two requirements. That phase dynamics leads to different results as oscillatory units \( k \) approach their off-state, \( V_k \to 0 \). We did also compare our approach with an alternative proposal for grouping oscillatory units based on temporal properties, see (Hoppensteadt and Izhikevitch, 1997, section 5.4.2). This frequency modulation (FM) proposal is founded on choosing different natural frequencies for oscillatory units, resulting in the grouping of units with nearly identical natural frequencies. With regard to a separation of on- and off-states, Eqs. (1) may be seen as a dynamical extension of this approach. However, with regard to a grouping among on-states, the FM approach would be based on frequency gaps, while the model of Eqs. (1) uses phase gaps. Correspondingly, we mentioned that (regarding biological systems) a complementarity has been suggested (Izhikevitch, 1999), according to which temporal coding with frequency gaps arises for weakly-coupled systems, while temporal coding based on phase gaps may be more suitable for strong couplings.

In the context of Eqs. (1), the solution of the superposition problem is straightforward. It is in accordance with original proposals of temporal coding (von der Malsburg, 1981; von der Malsburg, 1985), see also (von der Malsburg, 2003). The frequency-driving activity term \( 2\pi g(u_k) \) in Eq. (1b) separates the active units from the background so that active units, \( V_k \to 1 \), approach the frequency \((2\pi/\tau_0)(1 + S_k)\), while non-active units, \( V_k \to 0 \), are frozen to zero frequency. Among the active units, the phases provide an additional labelling of the activities. In case of coherent pattern recognition, due to synchronizing phase couplings, each component may be identified with a single phase. Different patterns may then have the same frequency and may still be distinguished by the phase differences between the patterns. In order to avoid mixed states of overlapping patterns, a competition between the patterns
may be introduced. This was achieved by introducing appropriate inhibitory weights in addition to the excitatory Hebbian weights. The coherence of the patterns may then be reduced to the winning subset.

In this paper, we did not specify mechanisms by which successive stages of information processing can read out the phases of different components. Future work should approach this issue so that a more complete picture arises. Moreover, taking the limit \( N \to \infty \) is essential for obtaining phase transitions between ordered and disordered states, see (Kuramoto, 1984; Strogatz, 2000). Studying the proposed and related models in this limit should be of particular interest. Higher-order phase couplings, and possibly related phenomena such as clustering should also be of interest, see (Tass, 1999). In the context of the FM proposal, these led to resonance conditions for the natural frequencies.

An early version of the Wang-Terman model was studied for patterns that were overlapping at one unit, resembling our example in section 4 (Wang et al., 1990). Presentation of overlapping patterns as input led to common activation of these patterns: the units at the overlap participated in each of the overlapping patterns, see (Wang et al., 1990, Figure 3). This differs from our example 3, where competition led to a winner among the overlapping patterns. In case that a common activation of overlapping patterns should be desirable, different approaches might work for the phase models. For example, the phase model could be extended to the coupling of higher-order modes. Overlapping units could then participate in a higher-frequency mode that may simultaneously synchronize with the lower-frequency non-overlapping parts of the patterns via resonances. This would still allow the patterns to desynchronize in the non-overlapping parts. Modifying the phase model to establish such a feature, however, is beyond the scope of this paper.

We presented simple examples to illuminate the dynamical content of Eqs. (1).
Evidently, to gain importance as a pattern recognition method, it should be
demonstrated that the method stands the test of the more advanced applica-
tions. In case of locally coupled relaxation oscillators, this step was taken by
Wang and Terman, who studied segmentation of real images. They observed
that a straight-forward application of relaxation oscillator networks led to an
unsuitable segmentation of the real image, leading to many tiny fragments, a
problem they called fragmentation. This problem was solved by introducing a
lateral potential for each oscillator (Wang and Terman, 1997), see also (Sha-
reef et al., 1999). It will be of interest to study how the present phase model
should be applied to real images. In case that fragmentation arises, it would
be particularly interesting to study whether some hierarchical processing could
help to combine the fragments, possibly in analogy to hierarchical processing
supported by higher-level cortical areas. Given the relative simplicity of the
phase models in comparison with coupled relaxation oscillator models, one may
hope that necessary steps to include high-level processing could be more easily
analyzed and implemented.

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References


Figure Captions

Figure 1: The stored patterns according to Eq. (26), drawn together with the resulting Hebbian connections $h_{kl}$ of Eq. (27) and numbering of the units. The filled/empty circles correspond to on/off states.

Figure 2: Example 1. The lines connecting the units indicate the Hebbian connections $h_{kl}$ of Eq. (27), see Figure 1. The filled/empty circles correspond to on/off states. We compare the classical case ($\beta = 0$) with phase coding ($\beta > 0$). A: $\beta = 0$, the classical case. The inhibitory input $J_1$ suppresses pattern 1. The excitatory input $J_6$ activates patterns 3 and pattern 2 through unit 4. B: $\beta = 0.1\alpha$, phase coding. Coherent activity is indicated by circles with parallel stripes. The connected active units synchronize within a few multiples of the time-scale. The corresponding patterns reach maximal coherence. (Whenever mod $2\pi$ is applied to $\theta$ the values are connected for the sake of better visualization.)

Figure 3: Example 2. Both external inputs are now excitatory. As a consequence, all units are activated. Again, we compare the classical case ($\beta = 0$) with that of phase coding ($\beta > 0$). A: $\beta = 0$, the classical case. The superposition problem arises. The activation carries no information about single patterns. B: $\beta = 0.1\alpha$, phase coding. Now the units of the two connected components synchronize separately. Different orientations of the stripes correspond to different phases. The superposition problem is avoided. Pattern 1 is distinguishable from the rest. Separating also patterns 2 and 3 requires competition (see Example 3).

Figure 4: Example 3. The network now also includes inhibitory weights $\lambda c_{kl}$, $\lambda = 1.2P$. These introduce competition between patterns. The network
is now completely connected, as illustrated by the additional broken lines. No external input is applied. A: $\beta = 0$, the classical case. Pattern 3 is the winning pattern (it is larger than the others). B: $\beta = 0.8\alpha$, phase coding. Again pattern 3 wins over pattern 2. Pattern 1, however, is now able to survive. Phase coding allows pattern 1 to escape the competition by desynchronizing with pattern 3. This is illustrated by circles with stripes of orthogonal orientation.
Figure 1: Burwick NC ms 3102
Figure 2: Burwick NC ms 3102
Figure 3: Burwick NC ms 3102
Figure 4: Burwick NC ms 3102